ENDOFUNCTORS OF SINGULARITY CATEGORIES CHARACTERIZING GORENSTEIN RINGS

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ABSTRACT. In this paper, we prove that certain contravariant endofunctors of singularity categories characterize Gorenstein rings.

Let Λ be a noetherian ring. Denote by $\mathsf{D_{sg}}(\Lambda)$ the *singularity category* of Λ, that is, the Verdier quotient of the bounded derived category $\mathsf{D^b}(\Lambda)$ of finitely generated (right) Λ-modules by the full subcategory consisting of bounded complexes of finitely generated projective Λ-modules. We are interested in the following question.

Question 1. What contravariant endofunctor of $\mathsf{D_{sg}}(\Lambda)$ characterizes the Iwanaga–Gorenstein property of Λ?

In this paper we shall consider this question in the case where Λ is commutative and Cohen–Macaulay.

Let R be a commutative Cohen–Macaulay local ring of Krull dimension d. Denote by $\mathsf{CM}(R)$ the category of (maximal) Cohen–Macaulay R-modules and by $\underline{\mathsf{CM}}(R)$ its *stable category*: the objects of $\underline{\mathsf{CM}}(R)$ are the Cohen–Macaulay R-modules, and the hom-set $\mathsf{Hom}_{\underline{\mathsf{CM}}(R)}(M,N)$ is defined as $\underline{\mathsf{Hom}}_R(M,N)$, the quotient module of $\mathsf{Hom}_R(M,N)$ by the submodule consisting homomorphisms factoring through finitely generated projective (or equivalently, free) R-modules. The natural full embedding functor $\mathsf{CM}(R) \to \mathsf{D^b}(R)$ induces an additive covariant functor

$$\eta : \underline{\mathsf{CM}}(R) \to \mathsf{D_{sg}}(R).$$

Furthermore, the assignment $M \mapsto \Omega^d \mathsf{Tr} M$, where Ω and Tr stand for the syzygy and transpose functors respectively (see [1, Chapter 2, §1] for details of the functors Ω and Tr), makes an additive contravariant functor

$$\lambda : \underline{\mathsf{CM}}(R) \to \underline{\mathsf{CM}}(R).$$

The following result gives a partial answer to Question 1.

Theorem 2. *The following are equivalent.*

(1) *The ring R is Gorenstein.*
(2) *The functor η is an equivalence (i.e. η is full, faithful and dense).*

2010 *Mathematics Subject Classification.* 13D09, 13H10, 18E30.

Key words and phrases. singularity category, stable category of Cohen–Macaulay modules, Gorenstein ring, Cohen–Macaulay ring.

The second author was partially supported by JSPS Grant-in-Aid for Scientific Research (C) 25400038.

(3) *There exists a functor* $\phi : \mathsf{D_{sg}}(R) \to \mathsf{D_{sg}}(R)$ *such that the diagram*

$$\begin{tikzcd} \mathsf{D_{sg}}(R) \arrow[r, "\phi"] & \mathsf{D_{sg}}(R) \\ \underline{\mathsf{CM}}(R) \arrow[u, "\eta"] \arrow[r, "\lambda"'] & \underline{\mathsf{CM}}(R) \arrow[u, "\eta"'] \end{tikzcd}$$

of functors commutes up to isomorphism.

Proof. (1) $\Rightarrow$ (2): If R is Gorenstein, then a celebrated theorem of Buchweitz [2, Theorem 4.4.1] implies that the functor η is an equivalence.

(2) $\Rightarrow$ (3): When η is an equivalence, we have a contravariant endofunctor

$$\phi = \eta\lambda\rho : \mathsf{D_{sg}}(R) \to \mathsf{D_{sg}}(R).$$

of $\mathsf{D_{sg}}(R)$, where ρ stands for a quasi-inverse of η. Condition (3) holds for this functor ϕ.

(3) $\Rightarrow$ (1): In the remainder of the proof, we will omit writing free summands. Let

$$\pi : \mathsf{mod}\, R \to \underline{\mathsf{mod}}\, R$$

be the canonical functor from the category of finitely generated R-modules to its stable category, that is, the objects of $\underline{\mathsf{mod}}\, R$ are the finitely generated R-modules and the hom-set $\mathsf{Hom}_{\underline{\mathsf{mod}}\, R}(M,N)$ is defined as $\underline{\mathsf{Hom}}_R(M,N)$.

Assume that there are a contravariant functor $\phi : \mathsf{D_{sg}}(R) \to \mathsf{D_{sg}}(R)$ and an isomorphism

$$\Delta : \phi\eta \to \eta\lambda$$

of functors from $\underline{\mathsf{CM}}(R)$ to $\mathsf{D_{sg}}(R)$. Take a Cohen–Macaulay R-module M. It follows from [1, Proposition (2.21)] that there exists an exact sequence

$$0 \to F \to \mathsf{Tr}\Omega\mathsf{Tr}\Omega M \xrightarrow{f} M \to 0 \tag{2.1}$$

of finitely generated R-modules with F free. The map f induces a morphism

$$\Theta : \mathsf{Tr}\Omega\mathsf{Tr}\Omega \to \mathsf{I}$$

of functors from $\underline{\mathsf{CM}}(R)$ to $\underline{\mathsf{CM}}(R)$, where I stands for the identity functor. Applying the R-dual functor $(-)^* = \mathsf{Hom}_R(-,R)$ to (2.1) gives an exact sequence

$$0 \to M^* \xrightarrow{f^*} (\mathsf{Tr}\Omega\mathsf{Tr}\Omega M)^* \xrightarrow{g} F^* \to \mathsf{Tr}M \xrightarrow{h} \mathsf{Tr}(\mathsf{Tr}\Omega\mathsf{Tr}\Omega M) \to \mathsf{Tr}F \to 0$$

with $\pi(h) = \mathsf{Tr}\pi(f)$; see [1, Lemma (3.9)]. Note that there is also an exact sequence

$$0 \to M^* \xrightarrow{f^*} (\mathsf{Tr}\Omega\mathsf{Tr}\Omega M)^* \xrightarrow{g} F^* \to \mathsf{Ext}^1_R(M,R) \to \mathsf{Ext}^1_R(\mathsf{Tr}\Omega\mathsf{Tr}\Omega M, R).$$

Since $\mathsf{Ext}^1_R(\mathsf{Tr}\Omega\mathsf{Tr}\Omega M, R) = 0$ by [1, Theorem (2.17)] and since $\mathsf{Tr}F$ is free, we obtain an exact sequence

$$0 \to \mathsf{Ext}^1_R(M,R) \to \mathsf{Tr}M \xrightarrow{h'} \mathsf{Tr}(\mathsf{Tr}\Omega\mathsf{Tr}\Omega M) \to 0$$

such that $\pi(h') = \pi(h)$. Taking the d-th syzygies of $\mathsf{Ext}^1_R(M,R)$ and $\mathsf{Tr}(\mathsf{Tr}\Omega\mathsf{Tr}\Omega M)$ and using the horseshoe lemma, we get an exact sequence of Cohen–Macaulay R-modules

$$0 \to \Omega^d\mathsf{Ext}^1_R(M,R) \to \lambda M \xrightarrow{\ell} \lambda(\mathsf{Tr}\Omega\mathsf{Tr}\Omega M) \to 0 \tag{2.2}$$

with $\pi(\ell) = \lambda\pi(f)$. Note that for each short exact sequence $\sigma : 0 \to X \xrightarrow{\alpha} Y \xrightarrow{\beta} Z \to 0$ of Cohen–Macaulay R-modules, the image of σ by the canonical functor π is sent by η to an exact triangle $X \xrightarrow{\alpha} Y \xrightarrow{\beta} Z \rightsquigarrow$ in $\mathsf{D_{sg}}(R)$. Hence, η sends (2.2) to an exact triangle

$$\eta\Omega^d \mathsf{Ext}^1_R(M,R) \to \eta\lambda M \xrightarrow{\eta\lambda(\Theta M)} \eta\lambda(\mathsf{Tr}\Omega\mathsf{Tr}\Omega M) \rightsquigarrow$$

in $\mathsf{D_{sg}}(R)$. We have a commutative diagram

$$\begin{tikzcd}
\eta\lambda M \arrow[r, "\eta\lambda(\Theta M)"] & \eta\lambda(\mathsf{Tr}\Omega\mathsf{Tr}\Omega M) \\
\phi\eta M \arrow[u, "\Delta M", "\cong"'] \arrow[r, "\phi\eta(\Theta M)"'] & \phi\eta(\mathsf{Tr}\Omega\mathsf{Tr}\Omega M) \arrow[u, "\cong", "\Delta(\mathsf{Tr}\Omega\mathsf{Tr}\Omega M)"']
\end{tikzcd}$$

of morphisms in $\mathsf{D_{sg}}(R)$, and the exact sequence (2.1) induces an isomorphism

$$\eta(\Theta M) : \eta(\mathsf{Tr}\Omega\mathsf{Tr}\Omega M) \to \eta M$$

in $\mathsf{D_{sg}}(R)$. Therefore $\eta\Omega^d\mathsf{Ext}^1_R(M,R)$ is isomorphic to 0 in $\mathsf{D_{sg}}(R)$, which means that the R-module $\mathsf{Ext}^1_R(M,R)$ has finite projective dimension. Thus, letting $M := \Omega^d k$, where k denotes the residue field of R, shows that $\mathsf{Ext}^{d+1}_R(k,R)$ has finite projective dimension. If $\mathsf{Ext}^{d+1}_R(k,R) = 0$, then R is Gorenstein. If $\mathsf{Ext}^{d+1}_R(k,R) \neq 0$, then the R-module k has finite projective dimension, which implies that R is regular, so that $\mathsf{Ext}^{d+1}_R(k,R) = 0$, a contradiction. Consequently, in either case R is a Gorenstein ring.

Now the proof of the theorem is completed. ∎

Remark 3. In the proof of the theorem, the assumption that the ring R is commutative is used to deduce the Gorensteinness of R from the fact that the R-module $\mathsf{Ext}^{d+1}_R(k,R)$ has finite projective dimension. For a noncommutative ring Λ with Jacobson radical J and an integer n, the n-th Ext group $\mathsf{Ext}^n_\Lambda(\Lambda/J,\Lambda)$ of the right Λ-modules Λ/J and Λ is not necessarily semisimple as a left Λ-module.

We end this paper by stating a direct consequence of the theorem.

Corollary 4. *Suppose that R is artinian. Then R is Gorenstein if and only if the transpose functor $\mathsf{Tr} : \underline{\mathsf{mod}}\, R \to \underline{\mathsf{mod}}\, R$ extends to the singularity category $\mathsf{D_{sg}}(R)$.*

References

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