CLASSIFICATION OF DOMINANT RESOLVING SUBCATEGORIES BY MODERATE FUNCTIONS

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Abstract. Let R be a commutative noetherian ring. Denote by $\operatorname{mod} R$ the category of finitely generated R-modules. In this paper we study dominant resolving subcategories of $\operatorname{mod} R$. We introduce a new $\mathbb{N}$-valued function on $\operatorname{Spec} R$ which we call a moderate function. Under an acceptable assumption, we construct explicit bijections between the set of dominant resolving subcategories of $\operatorname{mod} R$ and the set of moderate functions on $\operatorname{Spec} R$.

Contents

1. Introduction

Let R be a noetherian ring. Denote by $\operatorname{mod} R$ the category of finitely generated R-modules. A *resolving* subcategory of $\operatorname{mod} R$ is by definition a full subcategory closed under direct summands, extensions and syzygies. Auslander and Bridger [3] introduce this, and prove that the modules of Gorenstein dimension zero form a resolving subcategory. Auslander and Reiten [4] classify the contravariantly finite resolving subcategories of $\operatorname{mod} R$ when R is an artin algebra of finite global dimension by cotilting R-modules, which detects a strong relationship between the notion of resolving subcategories with tilting theory.

Let R be a commutative noetherian ring. Takahashi [26, 27] classifies the contravariantly finite resolving subcategories of $\operatorname{mod} R$ when R is a Gorenstein complete local ring, and the resolving subcategories of $\operatorname{mod} R$ consisting of maximal Cohen–Macaulay modules when R is a hypersurface singularity. Several other classifications of resolving subcategories of $\operatorname{mod} R$ consisting of maximal Cohen–Macaulay modules are provided for a Cohen–Macaulay ring R by Dao, Kobayashi, Nasseh and Takahashi [9, 20, 28]. The resolving subcategories of $\operatorname{mod} R$ consisting of modules of finite projective dimension are classified by Dao and Takahashi [11], the essentially same classification as which is independently given by Angeleri Hügel, Pospíšil, Šťovíček and Trlifaj [2]. A close connection of the resolving subcategories of $\operatorname{mod} R$ with the thick subcategories of the singularity category of R is recognized and studied in [12, 13, 29, 30]. Dao and Takahashi [11] obtain a complete classification of the resolving subcategories of $\operatorname{mod} R$ for a complete intersection R by using a classification theorem of thick subcategories of the singularity category of R due to Stevenson [24]. The structure of resolving subcategories is explored in [1, 10, 13, 21, 25] as well.

A full subcategory $\mathcal{X}$ of $\operatorname{mod} R$ is called *dominant* if $\mathcal{X}$ locally contains the residue field up to direct sums, summands and syzygies. Dao and Takahashi [11] introduce this notion, and classify the dominant resolving subcategories of $\operatorname{mod} R$ in the case where R is a Cohen–Macaulay ring by *grade-consistent functions* on $\operatorname{Spec} R$, which are defined as order-preserving maps $f : \operatorname{Spec} R \to \mathbb{N}$ such that $f(\mathfrak{p}) \leqslant \operatorname{grade} \mathfrak{p}$

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for every prime ideal $\mathfrak{p}$ of R. The main purpose of this paper is to study the structure of a dominant resolving subcategory of $\operatorname{mod} R$ for an arbitrary commutative noetherian ring R.

Our first main result is Theorem 4.3, where we prove that the assertion of [11, Theorem 4.5] remains valid even if we remove the assumption that the base ring is Cohen–Macaulay. In particular, we have the following numerical characterization of the modules belonging to a dominant resolving subcategory.

Theorem 1.1 (Corollary 4.4)**.** *Let R be a commutative noetherian ring. Let $\mathcal{X}$ be a dominant resolving subcategory of $\operatorname{mod} R$. Then for each finitely generated R-module M there is an equivalence*

$$M \in \mathcal{X} \iff \operatorname{depth} M_{\mathfrak{p}} \geqslant \inf_{X \in \mathcal{X}}\{\operatorname{depth} X_{\mathfrak{p}}\} \textit{ for all } \mathfrak{p} \in \operatorname{Spec} R.$$

Applying this theorem and using a homological dimension called *large restricted flat dimension*, we obtain Corollary 4.6, which yields the following corollary. The first assertion of the corollary below is a refinement of a result of Sanders [22], who proves the same assertion under the additional assumption that R admits a canonical module (recall that the existence of a canonical module of a Cohen–Macaulay ring R forces the ring R to have finite Krull dimension).

Corollary 1.2 (Corollaries 4.6(1)⇔(5) and 4.7)**.** *The following statements hold true.*

(1) *Let R be a Cohen–Macaulay ring. Then a resolving subcategory of* $\operatorname{mod} R$ *is dominant if and only if it contains the maximal Cohen–Macaulay R-modules.*
(2) *Let R be a commutative noetherian ring of finite Krull dimension d. Then a resolving subcategory of* $\operatorname{mod} R$ *is dominant if and only if it contains the d-th syzygies of R-modules.*

Next we consider classifying the dominant resolving subcategories of $\operatorname{mod} R$ for an arbitrary commutative noetherian ring R. For this, we introduce a new $\mathbb{N}$-valued function on $\operatorname{Spec} R$. We say that a map $f : \operatorname{Spec} R \to \mathbb{N}$ is a *moderate function* on $\operatorname{Spec} R$ if

(i) $f(\mathfrak{p}) \leqslant \operatorname{depth} R_{\mathfrak{p}}$ for all prime ideals $\mathfrak{p}$ of R, and
(ii) $\operatorname{depth} R_{\mathfrak{p}} - f(\mathfrak{p}) \geqslant \operatorname{depth} R_{\mathfrak{q}} - f(\mathfrak{q}) - \operatorname{ht} \mathfrak{q}/\mathfrak{p}$ for all prime ideals $\mathfrak{p}, \mathfrak{q}$ of R with $\mathfrak{p} \subseteq \mathfrak{q}$.

The moderate functions on $\operatorname{Spec} R$ yield a complete classification of the dominant resolving subcategories of $\operatorname{mod} R$ under an assumption on the existence of maximal Cohen–Macaulay modules.

Theorem 1.3 (Theorem 6.9)**.** *Let R be a commutative noetherian ring. Assume that $R/\mathfrak{p}$ possesses a nonzero maximal Cohen–Macaulay module for each minimal prime ideal $\mathfrak{p}$ of R. Then one has mutually inverse order-preserving bijections*

$$\left\{\begin{matrix}\textit{dominant resolving subcategories}\\ \textit{of } \operatorname{mod} R\end{matrix}\right\} \underset{\psi}{\overset{\phi}{\rightleftarrows}} \left\{\begin{matrix}\textit{moderate functions}\\ \textit{on } \operatorname{Spec} R\end{matrix}\right\},$$

where ϕ, ψ are defined by

$$\phi(\mathcal{X}) = \left[\operatorname{Spec} R \ni \mathfrak{p} \longmapsto \operatorname{depth} R_{\mathfrak{p}} - \inf_{X \in \mathcal{X}}\{\operatorname{depth} X_{\mathfrak{p}}\}\right],$$
$$\psi(f) = \{M \in \operatorname{mod} R \mid \operatorname{depth} R_{\mathfrak{p}} - \operatorname{depth} M_{\mathfrak{p}} \leqslant f(\mathfrak{p}) \textit{ for all } \mathfrak{p} \in \operatorname{Spec} R\}.$$

The assumption of this theorem means that $R/\mathfrak{p}$ satisfies the so-called *small Cohen–Macaulay modules conjecture* for each minimal prime $\mathfrak{p}$. It has been an open problem for almost half a century whether the small Cohen–Macaulay modules conjecture holds for any complete noetherian local ring; see [15, 16]. The assumption of the theorem can be verified easily for a concrete example of R, because every commutative noetherian ring has only finitely many minimal prime ideals. We demonstrate this in Example 6.11.

A more general but more complicated result than the above theorem is proved in Theorem 5.4, the combination of which with some observations on grade-consistent functions and moderate functions recovers the classification theorem of Dao and Takahashi [11] stated above, whose precise statement is:

Corollary 1.4 (Corollary 6.5)**.** *Let R be a Cohen–Macaulay ring. Then the maps ϕ and ψ defined as in Theorem 1.3 induce mutually inverse order-preserving bijections*

$$\left\{\begin{matrix}\textit{dominant resolving subcategories}\\ \textit{of } \operatorname{mod} R\end{matrix}\right\} \underset{\psi}{\overset{\phi}{\rightleftarrows}} \left\{\begin{matrix}\textit{grade-consistent functions}\\ \textit{on } \operatorname{Spec} R\end{matrix}\right\}.$$

The organization of this paper is as follows. Section 2 is devoted to preliminaries for the later sections. In Section 3, we study the structure of the resolving closures of modules over a local ring, which forms a basement of the discussions developed in this paper. In Section 4, we consider when a module belongs to a dominant resolving subcategory, and prove Theorem 1.1 and Corollary 1.2. In Section 5, we explore the maps ϕ and ψ appearing in Theorem 1.3, and give a complicated version of a classification of the dominant resolving subcategories. In Section 6, we introduce and investigate our new functions called moderate functions, and obtain Theorem 1.3 and Corollary 1.4.

2. Preliminaries

In this section, we give several definitions and their properties; in later sections we will use them basically tacitly. We begin with the convention of this paper.

Convention 2.1. All rings are commutative noetherian rings with identity. All modules are finitely generated. All subcategories are full. Let R be a ring. The symbol $\mathbb{N}$ stands for the set of nonnegative integers. We often omit subscripts and superscripts unless there is a danger of confusion.

We give some notation and recall some fundamental notions.

Definition 2.2. (1) Denote by $\operatorname{Spec} R$ (resp. $\operatorname{Max} R$, $\operatorname{Min} R$) the set of prime ideals (resp. maximal ideals, minimal prime ideals) of R. Set $\operatorname{Spec}_0 R = \operatorname{Spec} R \setminus \operatorname{Max} R$. This is called the *punctured spectrum* of R, if R is local. For $\mathfrak{p} \in \operatorname{Spec} R$, let $\kappa(\mathfrak{p})$ be the residue field of $\mathfrak{p}$, that is, $\kappa(\mathfrak{p}) = R_\mathfrak{p}/\mathfrak{p}R_\mathfrak{p}$.

(2) Let M be an R-module. The *nonfree locus* of M is defined as the set of prime ideals $\mathfrak{p}$ of R such that $M_\mathfrak{p}$ is nonfree as an $R_\mathfrak{p}$-module, and denoted by $\operatorname{NF}(M)$. This is a closed subset of $\operatorname{Spec} R$ in the Zariski topology; we refer the reader to [25, §2] for the details of nonfree loci.

(3) Denote by $\operatorname{mod} R$ the category of R-modules, and by $\operatorname{mod}_0 R$ the subcategory of $\operatorname{mod} R$ consisting of modules that are locally free on $\operatorname{Spec}_0 R$. Recall that an R-module M is called *maximal Cohen–Macaulay* if $\operatorname{depth} M_\mathfrak{p} \geqslant \dim R_\mathfrak{p}$ for all $\mathfrak{p} \in \operatorname{Spec} R$. Note that $\operatorname{depth} 0 = \infty$ by definition, and hence M is maximal Cohen–Macaulay if and only if $\operatorname{depth} M_\mathfrak{p} = \dim R_\mathfrak{p}$ for all $\mathfrak{p} \in \operatorname{Supp} M$. Denote by $\operatorname{MCM}(R)$ the subcategory of $\operatorname{mod} R$ consisting of maximal Cohen–Macaulay R-modules.

(4) Let $\mathcal{X}$ be a subcategory of $\operatorname{mod} R$. The *additive closure* $\operatorname{add}_R \mathcal{X}$ of $\mathcal{X}$ is defined to be the subcategory of $\operatorname{mod} R$ consisting of direct summands of finite direct sums of modules in $\mathcal{X}$. For a prime ideal $\mathfrak{p}$ of R, we denote by $\mathcal{X}_\mathfrak{p}$ the subcategory of $\operatorname{mod} R_\mathfrak{p}$ consisting of modules of the form $X_\mathfrak{p}$ with $X \in \mathcal{X}$.

Next we recall the definitions of syzygies and transposes. For the details, we refer the reader to [3].

Definition 2.3. (1) Let M be an R-module, and $n \geqslant 0$ an integer. If there is an exact sequence $0 \to N \to P_{n-1} \to \cdots \to P_1 \to P_0 \to M \to 0$ of R-modules with P_i projective for all $0 \leqslant i \leqslant n-1$, then the R-module N is called an *n-th syzygy* of M and denoted by $\Omega_R^n M$. This is uniquely determined up to projective summands. For a subcategory $\mathcal{X}$ of $\operatorname{mod} R$, we denote by $\Omega_R^n \mathcal{X}$ the subcategory of $\operatorname{mod} R$ consisting of n-th syzygies of R-modules. By definition (or convention), one has $\Omega_R^0 M = M$ and $\Omega_R^0 \mathcal{X} = \mathcal{X}$. Note that $\Omega_R^n \mathcal{X}$ contains the projective R-modules if $n > 0$. We set $\Omega_R = \Omega_R^1$.

(2) Let M be an R-module. If there exists an exact sequence $P_1 \xrightarrow{\partial} P_0 \to M \to 0$ of R-modules, then the *(Auslander) transpose* of M is defined as the cokernel of the dual map $\operatorname{Hom}_R(\partial, R) : \operatorname{Hom}_R(P_0, R) \to \operatorname{Hom}_R(P_1, R)$, and denoted by $\operatorname{Tr}_R M$. This is uniquely determined up to projective summands.

Now we recall the definition of a resolving subcategory, which plays a central role in this paper.

Definition 2.4. A subcategory $\mathcal{X}$ of $\operatorname{mod} R$ is called *resolving* if it satisfies the following four conditions.

(i) $\mathcal{X}$ contains the projective R-modules.

(ii) $\mathcal{X}$ is closed under direct summands, that is, if M is an R-module belonging to $\mathcal{X}$ and N is a direct summand of M, then N also belongs to $\mathcal{X}$.

(iii) $\mathcal{X}$ is closed under extensions, that is, for an exact sequence $0 \to L \to M \to N \to 0$ of R-modules, if L and N belong to $\mathcal{X}$, then M also belongs to $\mathcal{X}$.

(iv) $\mathcal{X}$ is closed under kernels of epimorphisms, that is, for an exact sequence $0 \to L \to M \to N \to 0$ of R-modules, if M and N belong to $\mathcal{X}$, then L also belongs to $\mathcal{X}$.

One can replace (i) with the condition that $\mathcal{X}$ contains R. Also, one can replace (iv) with the condition that $\mathcal{X}$ is closed under syzygies, that is, if M is an R-module belonging to $\mathcal{X}$, then ΩM also belongs to

$\mathcal{X}$. Consequently, a subcategory of $\operatorname{mod} R$ is resolving if and only if it contains R and is closed under direct summands, extensions and syzygies.

Here are a couple of examples of a resolving subcategory.

Example 2.5. (1) Basic properties of maximal Cohen–Macaulay modules show that $\mathrm{MCM}(R)$ is a resolving subcategory of $\operatorname{mod} R$ if (and only if) R is a Cohen–Macaulay ring.
(2) It can be directly verified that $\operatorname{mod}_0 R$ is a resolving subcategory of $\operatorname{mod} R$. More generally, for a subset Φ of $\operatorname{Spec} R$, the subcategory of $\operatorname{mod} R$ consisting of modules M with $\mathrm{NF}(M) \subseteq \Phi$ is resolving.

Finally, we recall the definition of the resolving closure of a module, and some of its properties.

Definition 2.6. Let M be an R-module. The *resolving closure* of M is defined as the smallest resolving subcategory of $\operatorname{mod} R$ containing M, and denoted by $\operatorname{res}_R M$.

Remark 2.7. (1) If $M \in \operatorname{res}_R X$, then $\Omega_R M \in \operatorname{res}_R(\Omega_R X)$ and $M_\mathfrak{p} \in \operatorname{res}_{R_\mathfrak{p}}(X_\mathfrak{p})$ for all $\mathfrak{p} \in \operatorname{Spec} R$.
(2) Let R be a local ring with residue field k. Then $\operatorname{mod}_0 R = \operatorname{res}_R k$. See [30, Corollary 4.3(3)].

3. Resolving closures over a local ring

In this section, we study the structure of the resolving closures of modules over a local ring. The results obtained in this section will form a basis in the next sections.

We begin with the following lemma, which is the same as the statement of [11, Lemma 4.1] except that the latter result has the assumption that the base local ring is Cohen–Macaulay. The same proof as that of [11, Lemma 4.1] works, if the reference [26, Theorem 2.4] in it is replaced with Remark 2.7(2).

Lemma 3.1 (cf. [11, Lemma 4.1])**.** *Let $(R, \mathfrak{m}, k)$ be a local ring. Let $\mathcal{X}$ be a resolving subcategory of $\operatorname{mod} R$ in which there is a module of depth 0. If $\Omega^n k$ belongs to $\mathcal{X}$ for some $n \geqslant 0$, then k belongs to $\mathcal{X}$.*

The proposition below, which is a generalization of Lemma 3.1, guarantees that [11, Proposition 4.2] holds true without assuming that the base local ring is Cohen–Macaulay. The proof of [11, Proposition 4.2] heavily uses the Cohen–Macaulay assumption and does not work without this assumption. We give here a rather different and simpler proof.

Proposition 3.2 (cf. [11, Proposition 4.2])**.** *Let $(R, \mathfrak{m}, k)$ be a local ring. Let M be an R-module, and put $t = \operatorname{depth} M$. Then the containment $\Omega^t k \in \operatorname{res}(M \oplus \Omega^i k)$ holds for all nonnegative integers i.*

Proof. If $i \leqslant t$, then $t - i \geqslant 0$ and $\Omega^t k = \Omega^{t-i}(\Omega^i k) \in \operatorname{res} \Omega^i k \subseteq \operatorname{res}(M \oplus \Omega^i k)$. Let us prove that $\Omega^t k \in \operatorname{res}(M \oplus \Omega^i k)$ for all $i \geqslant t+1$ by induction on t. When $t = 0$, the module M has depth 0, and Lemma 3.1 implies that $\Omega^t k = k$ belongs to $\operatorname{res}(M \oplus \Omega^i k)$. Let $t \geqslant 1$. Then there exists an M-regular element $x \in \mathfrak{m}$. The induction hypothesis applied to M/xM shows $\Omega^{t-1} k \in \operatorname{res}(M/xM \oplus \Omega^j k)$ for all $j \geqslant (t-1)+1 = t$. Applying Ω, we see that $\Omega^t k \in \operatorname{res}(\Omega(M/xM) \oplus \Omega^i k)$ for all $i \geqslant t+1$. There is an exact sequence $0 \to M \xrightarrow{x} M \to M/xM \to 0$, which induces an exact sequence $0 \to \Omega(M/xM) \to M \oplus R^{\oplus m} \to M \to 0$; see [11, Proposition 2.2(1)]. Hence $\Omega(M/xM) \in \operatorname{res} M$, and thus $\Omega^t k \in \operatorname{res}(M \oplus \Omega^i k)$ for all $i \geqslant t+1$. ■

We record a direct consequence of the above proposition.

Corollary 3.3. *Let $(R, \mathfrak{m}, k)$ be a local ring of depth t.*
(1) *Let $\mathcal{X}$ be a resolving subcategory of $\operatorname{mod} R$. If $\Omega^n k$ is in $\mathcal{X}$ for some integer $n \geqslant 0$, then $\Omega^t k$ is in $\mathcal{X}$.*
(2) *If $t = 0$, then it holds that $\operatorname{res} \Omega^i k = \operatorname{mod}_0 R$ for every integer $i \geqslant 0$.*

Proof. (1) Letting $M := R$ in Proposition 3.2 deduces the assertion.

(2) Set $\mathcal{X} = \operatorname{res} \Omega^i k$. It is easy to see that $\mathcal{X}$ is contained in $\operatorname{mod}_0 R$. As $\mathcal{X}$ contains R, which has depth 0, the application of (1) (or Lemma 3.1) to $\mathcal{X}$ gives $k \in \mathcal{X}$. Remark 2.7(2) implies that $\mathcal{X}$ contains $\operatorname{mod}_0 R$. ■

Our next purpose is to remove from [11, Corollary 4.3] the Cohen–Macaulay assumption of the base local ring. For this purpose, we establish a proposition.

Proposition 3.4. *Let $(R, \mathfrak{m}, k)$ be a local ring, and let n be a nonnegative integer. Let M be an R-module which belongs to $\operatorname{mod}_0 R$ and satisfies the inequality $\operatorname{depth} M \geqslant n$. Then M belongs to $\operatorname{res} \Omega^n k$.*

Proof. Since M is locally free on the punctured spectrum of R, the R-module $\operatorname{Ext}_R^1(M, \Omega M)$ has finite length, or in other words, $\mathfrak{a} = \operatorname{ann} \operatorname{Ext}_R^1(M, \Omega M)$ is an $\mathfrak{m}$-primary ideal of R. Hence we find an M-regular sequence $\boldsymbol{x} = x_1, \dots, x_n$ in $\mathfrak{a}$. By [17, Lemma 2.14], the ideal $\mathfrak{a}$ annihilates $\operatorname{Ext}_R^i(M, X)$ for all integers $i > 0$ and all R-modules X. It follows from [30, Corollary 3.2(1)] that there is an isomorphism

$$\Omega^n(M/\boldsymbol{x}M) \cong \textstyle\bigoplus_{j=0}^n (\Omega^j M)^{\oplus \binom{n}{j}}. \tag{3.4.1}$$

Put $N = \operatorname{Tr} \Omega^n \operatorname{Tr} \Omega^n(M/\boldsymbol{x}M)$. Then $N \cong \bigoplus_{j=0}^n \operatorname{Tr} \Omega^n \operatorname{Tr}(\Omega^j M)^{\oplus \binom{n}{j}}$ by (3.4.1), from which we see that N is locally free on the punctured spectrum of R. Hence N is in $\operatorname{mod}_0 R = \operatorname{res} k$, and $\Omega^n N$ is in $\operatorname{res} \Omega^n k$. It follows from [3, Propositions (2.6) and (2.20)] and (3.4.1) that, up to free summands one has

$$\Omega^n N \cong (\Omega^n \operatorname{Tr})^3(\operatorname{Tr}(M/\boldsymbol{x}M)) \gtrdot (\Omega^n \operatorname{Tr})(\operatorname{Tr}(M/\boldsymbol{x}M)) \cong \Omega^n(M/\boldsymbol{x}M) \gtrdot M,$$

where $A \gtrdot B$ means that B is a direct summand of A. Therefore, M belongs to $\operatorname{res} \Omega^n k$. ■

Now we can achieve the purpose, that is, we obtain a proof of [11, Corollary 4.3] without assuming that the base local ring is Cohen–Macaulay. In fact, it is immediate from Propositions 3.2 and 3.4.

Corollary 3.5 (cf. [11, Corollary 4.3])**.** *Let R be a local ring with residue field k. Let $M \in \operatorname{mod}_0 R$ with* $\operatorname{depth} M = t$. *Then there is an equality* $\operatorname{res}(M \oplus \Omega^i k) = \operatorname{res}(\Omega^t k \oplus \Omega^i k)$ *for all nonnegative integers i.*

4. Dominance of resolving subcategories

In this section, we consider the structure of dominant resolving subcategories. More precisely, we give characterizations of the modules belonging to a given dominant resolving subcategory.

First of all, we recall the definition of a dominant subcategory. It is a subcategory locally containing the residue field up to direct sums, summands and syzygies.

Definition 4.1. Let Φ be a subset of $\operatorname{Spec} R$. Let $\mathcal{X}$ be a subcategory of $\operatorname{mod} R$. We say that $\mathcal{X}$ is *dominant* on Φ if for all prime ideals $\mathfrak{p} \in \Phi$ there exists an integer $n \geqslant 0$ such that $\Omega^n \kappa(\mathfrak{p}) \in \operatorname{add} \mathcal{X}_\mathfrak{p}$. We simply say that $\mathcal{X}$ is *dominant* if it is dominant on $\operatorname{Spec} R$.

Remark 4.2. Let $\mathcal{X}$ be a dominant subcategory of $\operatorname{mod} R$. Then $\operatorname{add} \mathcal{X}_\mathfrak{p}$ is a dominant subcategory of $\operatorname{mod} R_\mathfrak{p}$ for every prime ideal $\mathfrak{p}$ of R.

In fact, take any prime ideal P of $R_\mathfrak{p}$. Setting $\mathfrak{q} = P \cap R$, we have $P = \mathfrak{q}R_\mathfrak{p}$. Note that $\kappa(P) \cong \kappa(\mathfrak{q})$. Since $\mathcal{X}$ is dominant, there exists a nonnegative integer n such that $\Omega_{R_\mathfrak{q}}^n \kappa(\mathfrak{q})$ belongs to $\operatorname{add}_{R_\mathfrak{q}} \mathcal{X}_\mathfrak{q}$. We have $\Omega_{(R_\mathfrak{p})_P}^n \kappa(P) \in \operatorname{add}_{(R_\mathfrak{p})_P} (\mathcal{X}_\mathfrak{p})_P \subseteq \operatorname{add}_{(R_\mathfrak{p})_P} (\operatorname{add}_{R_\mathfrak{p}} \mathcal{X}_\mathfrak{p})_P$, which shows that $\operatorname{add}_{R_\mathfrak{p}} \mathcal{X}_\mathfrak{p}$ is dominant.

The following theorem says that the assertion of [11, Theorem 4.5] holds true even if one removes the assumption that the base ring is Cohen–Macaulay.

Theorem 4.3 (cf. [11, Theorem 4.5])**.** *Let $\mathcal{X}$ be a resolving subcategory of* $\operatorname{mod} R$. *Let M be an R-module such that $\mathcal{X}$ is dominant on* $\operatorname{NF}(M)$. *Then there is an equivalence*

$$M \in \mathcal{X} \iff \operatorname{depth} M_\mathfrak{p} \geqslant \inf_{X \in \mathcal{X}}\{\operatorname{depth} X_\mathfrak{p}\} \textit{ for all } \mathfrak{p} \in \operatorname{NF}(M).$$

Proof. In the proof of [11, Theorem 4.5], replace [11, Proposition 4.2 and Corollary 4.3] with our Proposition 3.2 and Corollary 3.5, respectively (see also Remark 4.2). Then the argument does work. ■

We record an immediate consequence of Theorem 4.3, which contains no statement on nonfree loci.

Corollary 4.4. *Let $\mathcal{X}$ be a dominant resolving subcategory of* $\operatorname{mod} R$. *Then an R-module M belongs to $\mathcal{X}$ if and only if* $\operatorname{depth} M_\mathfrak{p} \geqslant \inf_{X \in \mathcal{X}}\{\operatorname{depth} X_\mathfrak{p}\}$ *for all* $\mathfrak{p} \in \operatorname{Spec} R$.

To give our next results, we need to recall the definition of the large restricted flat dimension of a module, together with some of its basic properties.

Definition 4.5. The *large restricted flat dimension* of an R-module M is defined by

$$\operatorname{Rfd}_R M = \sup_{\mathfrak{p} \in \operatorname{Spec} R}\{\operatorname{depth} R_\mathfrak{p} - \operatorname{depth} M_\mathfrak{p}\}.$$

One has $\operatorname{Rfd}_R M \in \mathbb{N} \cup \{-\infty\}$, and $\operatorname{Rfd}_R M = -\infty$ if and only if $M = 0$. Also, $\operatorname{Rfd}_R M \leqslant \dim R$. We refer the reader to [5, Theorem 1.1] and [8, Proposition (2.2) and Theorem (2.4)].

Applying Corollary 4.4, we obtain the following criterion for a resolving subcategory to be dominant. The equivalence (1) $\Leftrightarrow$ (4) (resp. (1) $\Leftrightarrow$ (3)) is shown in [11, Corollary 4.6] under the additional assumption that R is a Cohen–Macaulay ring (resp. that R is a Cohen–Macaulay ring of finite Krull dimension).

Corollary 4.6. *Let $\mathcal{X}$ be a resolving subcategory of* $\operatorname{mod} R$. *The following three conditions are equivalent.*

(1) *The subcategory $\mathcal{X}$ is dominant.*
(2) *For each nonzero R-module M one has $\Omega^r M \in \mathcal{X}$, where $r := \operatorname{Rfd}_R M \in \mathbb{N}$.*
(3) *For every prime ideal $\mathfrak{p}$ of R there exists a nonnegative integer n such that $\Omega^n(R/\mathfrak{p}) \in \mathcal{X}$.*

If $\dim R = d < \infty$, then the above three conditions are also equivalent to the following three conditions.

(4) *For every prime ideal $\mathfrak{p}$ of R one has $\Omega^d(R/\mathfrak{p}) \in \mathcal{X}$.*
(5) *One has $\Omega^d(\operatorname{mod} R) \subseteq \mathcal{X}$, that is, $\mathcal{X}$ contains all the d-th syzygies.*
(6) *One has $\Omega^n(\operatorname{mod} R) \subseteq \mathcal{X}$ for some $n \geqslant 0$.*

Proof. (2) $\Rightarrow$ (3): Put $n = \operatorname{Rfd}_R(R/\mathfrak{p}) \in \mathbb{N}$. The module $\Omega^n(R/\mathfrak{p})$ belongs to $\mathcal{X}$.

(3) $\Rightarrow$ (1): Let $\mathfrak{p}$ be a prime ideal of R. Then $\Omega^n(R/\mathfrak{p})$ belongs to $\mathcal{X}$ for some $n \geqslant 0$. Localization at $\mathfrak{p}$ shows that $\Omega^n \kappa(\mathfrak{p})$ belongs to $\mathcal{X}_\mathfrak{p}$, and hence it belongs to $\operatorname{add} \mathcal{X}_\mathfrak{p}$. Therefore, $\mathcal{X}$ is dominant.

(1) $\Rightarrow$ (2): Fix $\mathfrak{p} \in \operatorname{Spec} R$. There are inequalities $\operatorname{depth}(\Omega^r M)_\mathfrak{p} \geqslant \inf\{\operatorname{depth} R_\mathfrak{p}, \operatorname{depth} M_\mathfrak{p} + r\} \geqslant \operatorname{depth} R_\mathfrak{p}$, where the first inequality follows from the depth lemma, while the second holds since $r \geqslant \operatorname{depth} R_\mathfrak{p} - \operatorname{depth} M_\mathfrak{p}$. As $R \in \mathcal{X}$, we have $\operatorname{depth} R_\mathfrak{p} \geqslant \inf_{X \in \mathcal{X}}\{\operatorname{depth} X_\mathfrak{p}\}$. Corollary 4.4 yields $\Omega^r M \in \mathcal{X}$.

Thus, the conditions (1)–(3) are equivalent. Now we assume that R has finite Krull dimension d.

(3) $\Leftarrow$ (6) $\Leftarrow$ (5) $\Rightarrow$ (4) $\Rightarrow$ (3): The implications are evident.

(2) $\Rightarrow$ (5): Let M be a nonzero R-module, and put $r = \operatorname{Rfd}_R M \in \mathbb{N}$. The syzygy $\Omega^r M$ belongs to $\mathcal{X}$. Since $r \leqslant d$, we have $d - r \geqslant 0$ and $\Omega^d M = \Omega^{d-r}(\Omega^r M)$ belongs to $\mathcal{X}$. It follows that $\Omega^d(\operatorname{mod} R) \subseteq \mathcal{X}$. ■

Using the above corollary, we can refine a result of Sanders [22, Corollary 8.6]; the following corollary removes from [22, Corollary 8.6] the assumption that R has a canonical module, and in particular, the corollary removes from it the assumption that R has finite Krull dimension. (The existence of a canonical module of a Cohen–Macaulay ring implies the ring being of finite Krull dimension; see [18, Corollary 1.4].)

Corollary 4.7. *Let R be a Cohen–Macaulay ring. A resolving subcategory of* $\operatorname{mod} R$ *is dominant if and only if it contains* $\operatorname{MCM}(R)$.

Proof. The "if" part: Fix $\mathfrak{p} \in \operatorname{Spec} R$ and put $r = \operatorname{Rfd}_R(R/\mathfrak{p}) \in \mathbb{N}$. As $\mathcal{X}$ contains $\operatorname{MCM}(R)$, it suffices to prove that $M = \Omega_R^r(R/\mathfrak{p})$ is a maximal Cohen–Macaulay R-module. Let $\mathfrak{q} \in \operatorname{Spec} R$. The $R_\mathfrak{q}$-module $M_\mathfrak{q}$ is isomorphic to $N = \Omega_{R_\mathfrak{q}}^r((R/\mathfrak{p})_\mathfrak{q})$ up to free summands. As $r \geqslant \operatorname{depth} R_\mathfrak{q} - \operatorname{depth}(R/\mathfrak{p})_\mathfrak{q}$, the $R_\mathfrak{q}$-module N is maximal Cohen–Macaulay, and so is $M_\mathfrak{q}$. Therefore, M is a maximal Cohen–Macaulay R-module.

The "only if" part: Let M be a nonzero maximal Cohen–Macaulay R-module, and set $r = \operatorname{Rfd}_R M \in \mathbb{N}$. Corollary 4.6 implies that $\Omega^r M$ belongs to $\mathcal{X}$. Since $\operatorname{depth} R_\mathfrak{p} - \operatorname{depth} M_\mathfrak{p} = \dim R_\mathfrak{p} - \operatorname{depth} M_\mathfrak{p}$ is equal to 0 (resp. $-\infty$) if $\mathfrak{p} \in \operatorname{Supp} M$ (resp. $\mathfrak{p} \notin \operatorname{Supp} M$), we have $r = 0$. Therefore, M belongs to $\mathcal{X}$. ■

Another application of Corollary 4.6 is a local-to-global principle for dominant resolving subcategories.

Corollary 4.8. *For a resolving subcategory $\mathcal{X}$ of* $\operatorname{mod} R$ *the following equivalences hold true.*

$$\begin{aligned} \mathcal{X} \textit{ is dominant} &\iff \operatorname{add} \mathcal{X}_\mathfrak{p} \textit{ is dominant for every } \mathfrak{p} \in \operatorname{Spec} R \\ &\iff \operatorname{add} \mathcal{X}_\mathfrak{m} \textit{ is dominant for every } \mathfrak{m} \in \operatorname{Max} R. \end{aligned}$$

Proof. In view of Remark 4.2, it is enough to show that the third condition implies the first. Let $\mathfrak{p}$ be a prime ideal of R. Choose a maximal ideal $\mathfrak{m}$ of R that contains $\mathfrak{p}$. Put $r = \operatorname{Rfd}_R(R/\mathfrak{p}) \in \mathbb{N}$ and $s = \operatorname{Rfd}_{R_\mathfrak{m}}(R/\mathfrak{p})_\mathfrak{m} \in \mathbb{N}$. We then have $r \geqslant s$ by [8, Proposition (2.3)]. The subcategory $\operatorname{add} \mathcal{X}_\mathfrak{m}$ of $\operatorname{mod} R_\mathfrak{m}$ is resolving by [11, Lemma 3.2(1)]. Suppose that $\operatorname{add} \mathcal{X}_\mathfrak{m}$ is dominant. Then $M = \Omega_{R_\mathfrak{m}}^s((R/\mathfrak{p})_\mathfrak{m})$ is in $\operatorname{add} \mathcal{X}_\mathfrak{m}$ by Corollary 4.6. We have $r - s \geqslant 0$ and $N = \Omega_{R_\mathfrak{m}}^r((R/\mathfrak{p})_\mathfrak{m}) = \Omega_{R_\mathfrak{m}}^{r-s} M$ is also in $\operatorname{add} \mathcal{X}_\mathfrak{m}$. Since $(\Omega_R^r(R/\mathfrak{p}))_\mathfrak{m}$ is isomorphic to N up to free summand, we see that $(\Omega_R^r(R/\mathfrak{p}))_\mathfrak{m}$ belongs to $\operatorname{add} \mathcal{X}_\mathfrak{m}$ for any $\mathfrak{m} \in \operatorname{Max} R$ with $\mathfrak{p} \subseteq \mathfrak{m}$. Note that this holds true even if $\mathfrak{p} \nsubseteq \mathfrak{m}$, since in this case $(\Omega_R^r(R/\mathfrak{p}))_\mathfrak{m}$ is free. Now, the application of [11, Proposition 3.3] yields $\Omega_R^r(R/\mathfrak{p}) \in \mathcal{X}$. It follows that $\mathcal{X}$ is dominant. ■

5. The maps ϕ and ψ

In this section, we introduce a pair of maps ϕ and ψ between a set of subcategories of $\operatorname{mod} R$ and a set of functions on $\operatorname{Spec} R$, and investigate them. These maps will play an essential role in the next section.

Definition 5.1. (1) Let $\mathcal{C}$ be a subcategory of $\operatorname{mod} R$. We denote by $\mathbb{F}(\mathcal{C})$ the set of maps $f : \operatorname{Spec} R \to \mathbb{N}$ such that for all $\mathfrak{p} \in \operatorname{Spec} R$ there exists $E \in \mathcal{C}$ satisfying

$$\operatorname{depth} R_{\mathfrak{p}} - \operatorname{depth} E_{\mathfrak{p}} = f(\mathfrak{p}), \text{ and } \operatorname{depth} R_{\mathfrak{q}} - \operatorname{depth} E_{\mathfrak{q}} \leqslant f(\mathfrak{q}) \text{ for all } \mathfrak{q} \in \operatorname{Spec} R.$$

(2) Let $\mathcal{X}$ be a subcategory of $\operatorname{mod} R$, and let $\mathfrak{p}$ be a prime ideal of R. We set

$$\phi(\mathcal{X})(\mathfrak{p}) := \operatorname{depth} R_{\mathfrak{p}} - \inf_{X \in \mathcal{X}}\{\operatorname{depth} X_{\mathfrak{p}}\} = \sup_{X \in \mathcal{X}}\{\operatorname{depth} R_{\mathfrak{p}} - \operatorname{depth} X_{\mathfrak{p}}\}.$$

Let $f : \operatorname{Spec} R \to \mathbb{N}$ be a map. We define a subcategory $\psi(f)$ of $\operatorname{mod} R$ by

$$\psi(f) := \{M \in \operatorname{mod} R \mid \operatorname{depth} R_{\mathfrak{p}} - \operatorname{depth} M_{\mathfrak{p}} \leqslant f(\mathfrak{p}) \text{ for all } \mathfrak{p} \in \operatorname{Spec} R\}.$$

(3) For $f, g \in \operatorname{Map}(\operatorname{Spec} R, \mathbb{N})$ we write $f \leqslant g$ if $f(\mathfrak{p}) \leqslant g(\mathfrak{p})$ for all $\mathfrak{p} \in \operatorname{Spec} R$. Note that $\operatorname{Map}(\operatorname{Spec} R, \mathbb{N})$ is a partially ordered set with respect to the relation $\leqslant$. The set of subcategories of $\operatorname{mod} R$ is also a partially ordered set with respect to the inclusion relation $\subseteq$.

The following proposition gives rise to our order-preserving maps ϕ and ψ.

Proposition 5.2. (1) *The assignment $\mathcal{X} \mapsto \phi(\mathcal{X})$ gives an order-preserving map from the set of subcategories of $\operatorname{mod} R$ containing R to $\mathbb{F}(\operatorname{mod} R)$.*
(2) *The assignment $f \mapsto \psi(f)$ gives an order-preserving map from $\operatorname{Map}(\operatorname{Spec} R, \mathbb{N})$ to the set of dominant resolving subcategories of $\operatorname{mod} R$.*

Proof. (1) Let $\mathcal{X}$ be a subcategory of $\operatorname{mod} R$ containing R, and let $\mathfrak{p} \in \operatorname{Spec} R$. Then $\inf_{X \in \mathcal{X}}\{\operatorname{depth} X_{\mathfrak{p}}\} \leqslant \operatorname{depth} R_{\mathfrak{p}}$, so that $\phi(\mathcal{X})(\mathfrak{p}) \in \mathbb{N}$. It follows that $\phi(\mathcal{X}) \in \operatorname{Map}(\operatorname{Spec} R, \mathbb{N})$. Choosing a module $E \in \mathcal{X}$ such that $\inf_{X \in \mathcal{X}}\{\operatorname{depth} X_{\mathfrak{p}}\} = \operatorname{depth} E_{\mathfrak{p}}$, we have $\phi(\mathcal{X})(\mathfrak{p}) = \operatorname{depth} R_{\mathfrak{p}} - \operatorname{depth} E_{\mathfrak{p}}$. For each $\mathfrak{q} \in \operatorname{Spec} R$ we have $\inf_{X \in \mathcal{X}}\{\operatorname{depth} X_{\mathfrak{q}}\} \leqslant \operatorname{depth} E_{\mathfrak{q}}$, and $\phi(\mathcal{X})(\mathfrak{q}) \geqslant \operatorname{depth} R_{\mathfrak{q}} - \operatorname{depth} E_{\mathfrak{q}}$. Therefore $\phi(\mathcal{X}) \in \mathbb{F}(\operatorname{mod} R)$. It is straightforward that $\phi(\mathcal{Y}) \leqslant \phi(\mathcal{Z})$ for subcategories $\mathcal{Y}, \mathcal{Z}$ of $\operatorname{mod} R$ containing R and with $\mathcal{Y} \subseteq \mathcal{Z}$.

(2) Let $f : \operatorname{Spec} R \to \mathbb{N}$ be a map, and fix a prime ideal $\mathfrak{p}$ of R. The inequality $\operatorname{depth} R_{\mathfrak{p}} - \operatorname{depth} R_{\mathfrak{p}} = 0 \leqslant f(\mathfrak{p})$ shows $R \in \psi(f)$. If X is an R-module and Y is a direct summand of X, then $\operatorname{depth} Y_{\mathfrak{p}} \geqslant \operatorname{depth} X_{\mathfrak{p}}$. This shows that if X belongs to $\psi(f)$, then so does Y. Let $0 \to L \to M \to N \to 0$ be an exact sequence of R-modules. The depth lemma gives $\operatorname{depth} M_{\mathfrak{p}} \geqslant \inf\{\operatorname{depth} L_{\mathfrak{p}}, \operatorname{depth} N_{\mathfrak{p}}\}$ and $\operatorname{depth} L_{\mathfrak{p}} \geqslant \inf\{\operatorname{depth} M_{\mathfrak{p}}, \operatorname{depth} N_{\mathfrak{p}} + 1\}$. Using these inequalities, we see that if L, N (resp. M, N) are in $\psi(f)$, then so is M (resp. L). It follows that $\psi(f)$ is a resolving subcategory of $\operatorname{mod} R$. Put $r = \operatorname{Rfd}_R(R/\mathfrak{p}) \in \mathbb{N}$ and $M = \Omega^r(R/\mathfrak{p})$. Fix $\mathfrak{q} \in \operatorname{Spec} R$. We have $r \geqslant \operatorname{depth} R_{\mathfrak{q}} - \operatorname{depth}(R/\mathfrak{p})_{\mathfrak{q}}$, and $\operatorname{depth} M_{\mathfrak{q}} \geqslant \operatorname{depth} R_{\mathfrak{q}}$ by the depth lemma (this is the same argument as in the proof of (1) $\Rightarrow$ (2) in Corollary 4.6). Hence $\operatorname{depth} R_{\mathfrak{q}} - \operatorname{depth} M_{\mathfrak{q}} \leqslant 0 \leqslant f(\mathfrak{q})$, which shows $M \in \psi(f)$. Thus $\psi(f)$ is dominant (by Corollary 4.6). It is straightforward that $\psi(f) \subseteq \psi(g)$ for $f, g \in \operatorname{Map}(\operatorname{Spec} R, \mathbb{N})$ with $f \leqslant g$. ■

We investigate the structure of the compositions of the maps ϕ and ψ.

Proposition 5.3. (1) *Let $\mathcal{X}$ be a subcategory of $\operatorname{mod} R$ containing R. Then $\psi\phi(\mathcal{X})$ is the smallest dominant resolving subcategory of $\operatorname{mod} R$ containing $\mathcal{X}$.*
(2) *Let $f : \operatorname{Spec} R \to \mathbb{N}$ be a map. One then has the equality $\phi\psi(f) = \max\{g \in \mathbb{F}(\operatorname{mod} R) \mid g \leqslant f\}$.*

Proof. (1) Proposition 5.2 implies that $\psi\phi(\mathcal{X})$ is a dominant resolving subcategory of $\operatorname{mod} R$. We have

$$\psi\phi(\mathcal{X}) = \{M \in \operatorname{mod} R \mid \operatorname{depth} M_{\mathfrak{p}} \geqslant \inf_{X \in \mathcal{X}}\{\operatorname{depth} X_{\mathfrak{p}}\} \text{ for all } \mathfrak{p} \in \operatorname{Spec} R\}. \tag{5.3.1}$$

It is seen from (5.3.1) that $\psi\phi(\mathcal{X})$ contains $\mathcal{X}$. Let $\mathcal{Y}$ be a dominant resolving subcategory of $\operatorname{mod} R$ containing $\mathcal{X}$. Since ϕ and ψ are order-preserving by Proposition 5.2, it holds that $\psi\phi(\mathcal{X}) \subseteq \psi\phi(\mathcal{Y})$. Applying (5.3.1) to $\mathcal{Y}$ and using Corollary 4.4, we observe $\psi\phi(\mathcal{Y}) = \mathcal{Y}$. Now the assertion follows.

(2) Proposition 5.2 implies $\phi\psi(f) \in \mathbb{F}(\operatorname{mod} R)$. For each prime ideal $\mathfrak{p}$ of R we have

$$\phi\psi(f)(\mathfrak{p}) = \sup\left\{\operatorname{depth} R_{\mathfrak{p}} - \operatorname{depth} X_{\mathfrak{p}} \;\middle|\; \begin{array}{c} X \text{ is an } R\text{-module such that for all prime ideals} \\ \mathfrak{q} \text{ of } R \text{ one has } \operatorname{depth} R_{\mathfrak{q}} - \operatorname{depth} X_{\mathfrak{q}} \leqslant f(\mathfrak{q}) \end{array}\right\}. \tag{5.3.2}$$

It is seen from (5.3.2) that $\phi\psi(f) \leqslant f$. Let g be an element of $\mathbb{F}(\operatorname{mod} R)$ with $g \leqslant f$. Then, for each prime ideal $\mathfrak{p}$ of R there exists an R-module Z such that $\operatorname{depth} R_{\mathfrak{p}} - \operatorname{depth} Z_{\mathfrak{p}} = g(\mathfrak{p})$ and $\operatorname{depth} R_{\mathfrak{q}} - \operatorname{depth} Z_{\mathfrak{q}} \leqslant g(\mathfrak{q})$ for all $\mathfrak{q} \in \operatorname{Spec} R$. It is observed from (5.3.2) that $\phi\psi(f)(\mathfrak{p}) \geqslant g(\mathfrak{p})$. Thus the assertion follows. ■

Now we state and prove the main result of this section. The following theorem provides a classification of the dominant resolving subcategories of $\operatorname{mod} R$ by the set $\mathbb{F}(\operatorname{mod} R)$, using the maps ϕ, ψ.

Theorem 5.4. *The maps ϕ and ψ induce mutually inverse order-preserving bijections*

$$\phi : \{\textit{dominant resolving subcategories of } \operatorname{mod} R\} \rightleftarrows \mathbb{F}(\operatorname{mod} R) : \psi.$$

Proof. Fix a dominant resolving subcategory $\mathcal{X}$ of $\operatorname{mod} R$ and an element $f \in \mathbb{F}(\operatorname{mod} R)$. Proposition 5.2 shows that $\phi(\mathcal{X})$ belongs to $\mathbb{F}(\operatorname{mod} R)$ and $\psi(f)$ is a dominant resolving subcategory of $\operatorname{mod} R$. By Proposition 5.3 one has $\psi\phi(\mathcal{X}) = \mathcal{X}$ and $\phi\psi(f) = f$. Thus the theorem follows. ∎

We introduce two homological dimensions, and subcategories of $\operatorname{mod} R$ which the dimensions define.

Definition 5.5. (1) We denote by $\operatorname{pd}_R M$ (resp. $\operatorname{CMdim}_R M$) the *projective dimension* (resp. *Cohen–Macaulay dimension* of M. For the details (including the definition) of Cohen–Macaulay dimension, we refer the reader to [14, §3]. Basic properties of these dimensions are listed in [11, Lemma 5.2].

(2) We denote by $\operatorname{fpd} R$ (resp. $\operatorname{fcd} R$) the subcategory of $\operatorname{mod} R$ consisting of modules M such that $M_\mathfrak{p}$ has finite projective (resp. Cohen–Macaulay) dimension as an $R_\mathfrak{p}$-module. Note that $\operatorname{fpd} R$ consists of the R-modules of finite projective dimension (as R-modules); see [6, Lemma 4.5].

We record one more property of the maps ϕ and ψ, which we will not use in this paper. This is a corollary of [11, Theorem 3.5].

Proposition 5.6. *Let $\mathcal{X}$ be a resolving subcategory of $\operatorname{mod} R$. If $\mathcal{X}$ is contained in $\operatorname{fpd} R$, then*

$$\mathcal{X} = \psi\phi(\mathcal{X}) \cap \operatorname{fpd} R.$$

Proof. Let $M \in \operatorname{fpd} R$. Then M is in $\psi\phi(\mathcal{X})$ if and only if $\operatorname{depth} R_\mathfrak{p} - \operatorname{depth} M_\mathfrak{p} \leqslant \sup_{X\in\mathcal{X}}\{\operatorname{depth} R_\mathfrak{p} - \operatorname{depth} X_\mathfrak{p}\}$ for all prime ideals $\mathfrak{p}$. The Auslander–Buchsbaum formula says that this is equivalent to saying that $\operatorname{pd}_{R_\mathfrak{p}} M_\mathfrak{p} \leqslant \sup_{X\in\mathcal{X}}\{\operatorname{pd}_{R_\mathfrak{p}} X_\mathfrak{p}\}$, which is equivalent to saying that $M \in \mathcal{X}$ by [11, Theorem 3.5]. ∎

6. Moderate functions on $\operatorname{Spec} R$

In Theorem 5.4 we obtained a complete classification of the dominant resolving subcategories, but it contains the defect that the definition of the set $\mathbb{F}(\operatorname{mod} R)$ which parametrizes the dominant resolving subcategories involves some information on modules. In this section, under some acceptable assumption, we give a complete classification of the dominant resolving subcategories without this defect.

We begin with recalling the definition of a grade-consistent function, and making the definition of our new function which we call a moderate function.

Definition 6.1. Let $f : \operatorname{Spec} R \to \mathbb{N}$ be a map.

(1) We say that f is a *grade-consistent function* on $\operatorname{Spec} R$ if $f(\mathfrak{p}) \leqslant \operatorname{grade} \mathfrak{p}$ for all $\mathfrak{p} \in \operatorname{Spec} R$, and $f(\mathfrak{p}) \leqslant f(\mathfrak{q})$ for all $\mathfrak{p}, \mathfrak{q} \in \operatorname{Spec} R$ with $\mathfrak{p} \subseteq \mathfrak{q}$, that is to say, f is order-preserving.

(2) We say that f is a *moderate function* on $\operatorname{Spec} R$ if $f(\mathfrak{p}) \leqslant \operatorname{depth} R_\mathfrak{p}$ for all $\mathfrak{p} \in \operatorname{Spec} R$ and

$$\operatorname{depth} R_\mathfrak{p} - f(\mathfrak{p}) \geqslant \operatorname{depth} R_\mathfrak{q} - f(\mathfrak{q}) - \operatorname{ht} \mathfrak{q}/\mathfrak{p} \tag{6.1.1}$$

for all $\mathfrak{p}, \mathfrak{q} \in \operatorname{Spec} R$ with $\mathfrak{p} \subseteq \mathfrak{q}$.

As a trivial example, the zero map is both a grade-consistent function and a moderate function.

We establish a lemma on the relationship between the depths of localized modules.

Lemma 6.2. *Let $\mathfrak{p}, \mathfrak{q} \in \operatorname{Spec} R$ with $\mathfrak{p} \subseteq \mathfrak{q}$. Let $M \in \operatorname{mod} R$. Then $\operatorname{depth}_{R_\mathfrak{p}} M_\mathfrak{p} \geqslant \operatorname{depth}_{R_\mathfrak{q}} M_\mathfrak{q} - \operatorname{ht} \mathfrak{q}/\mathfrak{p}$.*

Proof. Replacing R with $R_\mathfrak{q}$, we may assume that R is local. What we want to show is that $\operatorname{depth} M_\mathfrak{p} \geqslant \operatorname{depth} M - \dim R/\mathfrak{p}$. In view of [7, Proposition 1.2.10(a)], it is enough to show $\operatorname{grade}(\mathfrak{p}, M) \geqslant \operatorname{depth} M - \dim R/\mathfrak{p}$. This inequality is indeed a module version of [19, Exercise 17.5(i)]. Putting $g = \operatorname{grade}(\mathfrak{p}, M)$, we find an M-regular sequence $\boldsymbol{x} = x_1, \dots, x_g$ in $\mathfrak{p}$. Then $\operatorname{grade}(\mathfrak{p}, M/\boldsymbol{x}M) = 0$ by [7, Proposition 1.2.10(d)], and $\operatorname{depth}(M/\boldsymbol{x}M)_\mathfrak{q} = 0$ for some $\mathfrak{q} \in \operatorname{Spec} R$ with $\mathfrak{p} \subseteq \mathfrak{q}$ by [7, Proposition 1.2.10(a)]. We have $\mathfrak{q} \in \operatorname{Ass} M/\boldsymbol{x}M$, and obtain $\operatorname{depth} M - g = \operatorname{depth} M/\boldsymbol{x}M \leqslant \dim R/\mathfrak{q} \leqslant \dim R/\mathfrak{p}$, where the first inequality follows from [7, Proposition 1.2.13]. Thus $g \geqslant \operatorname{depth} M - \dim R/\mathfrak{p}$. ∎

Recall that for a local ring R the *Cohen–Macaulay defect* $\operatorname{cmd} R$ of R is defined by $\operatorname{cmd} R = \dim R - \operatorname{depth} R$. We investigate the condition (6.1.1) in the definition of a moderate function, which concludes that over a Cohen–Macaulay ring the moderate functions are the same as the grade-consistent functions.

Proposition 6.3. *Let $f : \operatorname{Spec} R \to \mathbb{N}$ be a map.*

(1) *If f is order-preserving, then it satisfies the inequality* (6.1.1)*.*
(2) *If f is a grade-consistent function, then it is a moderate function.*
(3) *Suppose that R_P is catenary and equidimensional for all $P \in \operatorname{Spec} R$. Then the inequality* (6.1.1) *is equivalent to the inequality*
$$f(\mathfrak{p}) + \operatorname{cmd} R_{\mathfrak{p}} \leqslant f(\mathfrak{q}) + \operatorname{cmd} R_{\mathfrak{q}}.$$
(4) *Assume that R is Cohen–Macaulay. The function f is moderate if and only if it is grade-consistent.*

Proof. (1) Let $\mathfrak{p}, \mathfrak{q}$ be prime ideals of R such that $\mathfrak{p} \subseteq \mathfrak{q}$. If f is order-preserving, then $f(\mathfrak{q}) - f(\mathfrak{p}) \geqslant 0 \geqslant \operatorname{depth} R_{\mathfrak{q}} - \operatorname{ht} \mathfrak{q}/\mathfrak{p} - \operatorname{depth} R_{\mathfrak{p}}$ by Lemma 6.2, and hence $\operatorname{depth} R_{\mathfrak{p}} - f(\mathfrak{p}) \geqslant \operatorname{depth} R_{\mathfrak{q}} - f(\mathfrak{q}) - \operatorname{ht} \mathfrak{q}/\mathfrak{p}$.

(2) The assertion is an easy consequence of (1) and [7, Proposition 1.2.10(a)].

(3) The inequality (6.1.1) is equivalent to the inequality $f(\mathfrak{p}) + \operatorname{cmd} R_{\mathfrak{p}} \leqslant f(\mathfrak{q}) + \operatorname{cmd} R_{\mathfrak{q}} - (\operatorname{ht} \mathfrak{q} - \operatorname{ht} \mathfrak{q}/\mathfrak{p} - \operatorname{ht} \mathfrak{p})$. Since the local ring $R_{\mathfrak{q}}$ is catenary and equidimensional by assumption, there is an equality $\dim R_{\mathfrak{q}} = \dim R_{\mathfrak{q}}/\mathfrak{p}R_{\mathfrak{q}} + \operatorname{ht} \mathfrak{p}R_{\mathfrak{q}}$, which is equivalent to saying that $\operatorname{ht} \mathfrak{q} - \operatorname{ht} \mathfrak{q}/\mathfrak{p} - \operatorname{ht} \mathfrak{p} = 0$.

(4) Let P be a prime ideal of R. Since R is a Cohen–Macaulay ring, there is an equality $\operatorname{grade} P = \operatorname{depth} R_P$ by [7, Theorem 2.1.3(b)]. Also, R_P is catenary and equidimensional and $\operatorname{cmd} R_P = 0$, as R_P is a Cohen–Macaulay local ring. It is now observed from (3) that the assertion holds true. ∎

We investigate the relationship among the set of grade-consistent functions, the set of moderate functions and the sets $\mathbb{F}(\mathcal{C})$ with $\mathcal{C} \in \{\operatorname{fpd} R, \operatorname{fcd} R, \operatorname{mod} R\}$. In particular, it turns out that a grade-consistent function is always a moderate function.

Proposition 6.4. (1) *The set $\mathbb{F}(\operatorname{fpd} R)$ (resp. $\mathbb{F}(\operatorname{fcd} R)$) consists of the maps $f : \operatorname{Spec} R \to \mathbb{N}$ such that for all $\mathfrak{p} \in \operatorname{Spec} R$ there exists $E \in \operatorname{mod} R$ with $\operatorname{pd}_{R_{\mathfrak{p}}} E_{\mathfrak{p}} = f(\mathfrak{p})$ (resp. $\operatorname{CMdim}_{R_{\mathfrak{p}}} E_{\mathfrak{p}} = f(\mathfrak{p})$) and $\operatorname{pd}_{R_{\mathfrak{q}}} E_{\mathfrak{q}} \leqslant f(\mathfrak{q})$ (resp. $\operatorname{CMdim}_{R_{\mathfrak{q}}} E_{\mathfrak{q}} \leqslant f(\mathfrak{q})$) for all $\mathfrak{q} \in \operatorname{Spec} R$.*

(2) *One has the following equalities and inclusions of subsets of* $\operatorname{Map}(\operatorname{Spec} R, \mathbb{N})$.
$$\left\{\begin{matrix} \text{grade-consistent function} \\ \text{on } \operatorname{Spec} R \end{matrix}\right\} = \mathbb{F}(\operatorname{fpd} R) = \mathbb{F}(\operatorname{fcd} R) \subseteq \mathbb{F}(\operatorname{mod} R) \subseteq \left\{\begin{matrix} \text{moderate function} \\ \text{on } \operatorname{Spec} R \end{matrix}\right\}.$$

Proof. (1) The assertion is a consequence of the Auslander–Buchsbaum formula and [14, Theorem 3.8].

(2) The inclusions $\operatorname{fpd} R \subseteq \operatorname{fcd} R \subseteq \operatorname{mod} R$ induce the inclusions $\mathbb{F}(\operatorname{fpd} R) \subseteq \mathbb{F}(\operatorname{fcd} R) \subseteq \mathbb{F}(\operatorname{mod} R)$. Let f be a grade-consistent function on $\operatorname{Spec} R$. It is seen from [11, Lemma 5.3] and (1) that $f \in \mathbb{F}(\operatorname{fpd} R)$. Conversely, take any element $g \in \mathbb{F}(\operatorname{fcd} R)$. By (1), for each prime ideal $\mathfrak{p}$ of R there exists an R-module $E(\mathfrak{p})$ with $\operatorname{CMdim}_{R_{\mathfrak{p}}} E(\mathfrak{p})_{\mathfrak{p}} = g(\mathfrak{p})$ and $\operatorname{CMdim}_{R_{\mathfrak{q}}} E(\mathfrak{p})_{\mathfrak{q}} \leqslant g(\mathfrak{q})$ for all prime ideals $\mathfrak{q}$ of R. If $\mathfrak{p} \subseteq \mathfrak{q}$, then $g(\mathfrak{p}) = \operatorname{CMdim}_{R_{\mathfrak{p}}} E(\mathfrak{p})_{\mathfrak{p}} \leqslant \operatorname{CMdim}_{R_{\mathfrak{q}}} E(\mathfrak{p})_{\mathfrak{q}} \leqslant g(\mathfrak{q})$ by [14, Proposition 3.10]. We find a prime ideal $\mathfrak{r}$ of R with $\mathfrak{p} \subseteq \mathfrak{r}$ and $\operatorname{grade} \mathfrak{p} = \operatorname{depth} R_{\mathfrak{r}}$; see [7, Proposition 1.2.10(a)]. Hence $g(\mathfrak{p}) \leqslant g(\mathfrak{r}) = \operatorname{CMdim}_{R_{\mathfrak{r}}} E(\mathfrak{r})_{\mathfrak{r}} \leqslant \operatorname{depth} R_{\mathfrak{r}} = \operatorname{grade} \mathfrak{p}$, where the second inequality follows from [14, Theorem 3.8]. Consequently, g is a grade-consistent function on $\operatorname{Spec} R$, and the two equalities in the assertion follow.

It remains to show that each element $h \in \mathbb{F}(\operatorname{mod} R)$ is a moderate function on $\operatorname{Spec} R$. For each $\mathfrak{p} \in \operatorname{Spec} R$ there exists $Z \in \operatorname{mod} R$ such that $\operatorname{depth} R_{\mathfrak{p}} - \operatorname{depth} Z_{\mathfrak{p}} = h(\mathfrak{p})$ and $\operatorname{depth} R_{\mathfrak{q}} - \operatorname{depth} Z_{\mathfrak{q}} \leqslant h(\mathfrak{q})$ for all $\mathfrak{q} \in \operatorname{Spec} R$. Hence $h(\mathfrak{p}) \leqslant \operatorname{depth} R_{\mathfrak{p}}$. If $\mathfrak{p} \subseteq \mathfrak{q}$, then $\operatorname{depth} R_{\mathfrak{p}} - h(\mathfrak{p}) = \operatorname{depth} Z_{\mathfrak{p}} \geqslant \operatorname{depth} Z_{\mathfrak{q}} - \operatorname{ht} \mathfrak{q}/\mathfrak{p} \geqslant \operatorname{depth} R_{\mathfrak{q}} - h(\mathfrak{q}) - \operatorname{ht} \mathfrak{q}/\mathfrak{p}$, where the first inequality follows from Lemma 6.2. Thus h is moderate. ∎

We recover [11, Theorem 1.4], which is one of the main results of the paper [11]. It is indeed an immediate consequence of the combination of Theorem 5.4 and Propositions 6.3(4), 6.4(2).

Corollary 6.5. *If R is a Cohen–Macaulay ring, then one has mutually inverse order-preserving bijections*
$$\phi : \{\textit{dominant resolving subcategories of } \operatorname{mod} R\} \rightleftarrows \{\textit{grade-consistent functions on } \operatorname{Spec} R\} : \psi.$$

We establish a lemma which is used in the proof of our theorem below.

Lemma 6.6. (1) *Let R be a domain. Let M be a nonzero maximal Cohen–Macaulay R-module. Then the equality $\operatorname{Supp} M = \operatorname{Spec} R$ holds true.*

(2) *Let M be an R-module. Let $\boldsymbol{x} = x_1, \dots, x_n$ be a sequence of elements of R such that $n \geqslant 1$ and that x_1 is M-regular. Then there is an inclusion* $\operatorname{Supp} M \cap \mathrm{V}(\boldsymbol{x}) \subseteq \mathrm{NF}(M/\boldsymbol{x}M)$.

(3) *Let R be a local ring. Let M be an R-module of depth 0. Let n be an integer with $0 \leqslant n \leqslant \operatorname{depth} R$. Let N be an n-th syzygy of M. One then has* $\operatorname{depth} N = n$. *(In particular, $N \neq 0$.)*

Proof. (1) Since $M \neq 0$, there exists $\mathfrak{p} \in \operatorname{Ass} M$. Then $\mathfrak{p} \in \operatorname{Supp} M$. We have $0 = \operatorname{depth} M_{\mathfrak{p}} \geqslant \dim R_{\mathfrak{p}}$, and hence $\mathfrak{p} \in \operatorname{Min} R$. As R is a domain, $\mathfrak{p} = 0$. Therefore $0 \in \operatorname{Supp} M$, and thus $\operatorname{Supp} M = \operatorname{Spec} R$.

(2) Suppose that there exists a prime ideal $\mathfrak{p}$ of R such that $\mathfrak{p} \in \operatorname{Supp} M \cap \mathrm{V}(\boldsymbol{x})$ and $\mathfrak{p} \notin \mathrm{NF}(M/\boldsymbol{x}M)$. Then $M_{\mathfrak{p}} \neq 0$ and there is an isomorphism $M_{\mathfrak{p}}/\boldsymbol{x}M_{\mathfrak{p}} \cong R_{\mathfrak{p}}^{\oplus m}$ for some $m \geqslant 0$. Since $\boldsymbol{x}R_{\mathfrak{p}}$ is contained in $\mathfrak{p}R_{\mathfrak{p}}$, Nakayama's lemma implies $m \neq 0$, that is to say, $m \geqslant 1$. As $M_{\mathfrak{p}}/\boldsymbol{x}M_{\mathfrak{p}}$ is annihilated by $\boldsymbol{x}$, we must have $\boldsymbol{x}R_{\mathfrak{p}} = 0$, which implies $x_1 R_{\mathfrak{p}} = 0$. By assumption, the sequence $0 \to M \xrightarrow{x_1} M$ is exact, and so is the localized sequence $0 \to M_{\mathfrak{p}} \xrightarrow{x_1} M_{\mathfrak{p}}$. The equality $x_1 R_{\mathfrak{p}} = 0$ implies that the map $M_{\mathfrak{p}} \xrightarrow{x_1} M_{\mathfrak{p}}$ is zero, and hence we get $M_{\mathfrak{p}} = 0$. This contradiction shows that $\operatorname{Supp} M \cap \mathrm{V}(\boldsymbol{x})$ is contained in $\mathrm{NF}(M/\boldsymbol{x}M)$.

(3) The assertion is easily shown by using induction on n and the depth lemma. ■

To state our theorem below, we need to recall one of the well-known homological conjectures.

Definition 6.7. We say that R satisfies the *small Cohen–Macaulay modules conjecture*, (SCM) for short, if there exists a nonzero maximal Cohen–Macaulay R-module.

Remark 6.8. Hochster [15] (see also [16]) conjectures that any complete local ring satisfies (SCM). If R is a complete (or more generally, Nagata) local ring with $\dim R \leqslant 2$, then R satisfies (SCM). In fact, for $\mathfrak{p} \in \operatorname{Spec} R$ with $\dim R/\mathfrak{p} = \dim R$ the integral closure of $R/\mathfrak{p}$ is a maximal Cohen–Macaulay R-module.

The following theorem is the main result of this section. Note that the assumption of the theorem requires only finitely many verifications, since a ring has only finitely many minimal primes.

Theorem 6.9. *Assume that $R/\mathfrak{p}$ satisfies* (SCM) *for each $\mathfrak{p} \in \operatorname{Min} R$. Then $\mathbb{F}(\operatorname{mod} R)$ coincides with the set of moderate functions on $\operatorname{Spec} R$. Therefore ϕ, ψ induce mutually inverse order-preserving bijections*

$$\phi : \{\textit{dominant resolving subcategories of } \operatorname{mod} R\} \rightleftarrows \{\textit{moderate functions on } \operatorname{Spec} R\} : \psi.$$

Proof. The latter assertion follows from the former and Theorem 5.4. Let us show the former assertion. By Proposition 6.4(2), it is enough to verify that any moderate function f on $\operatorname{Spec} R$ belongs to $\mathbb{F}(\operatorname{mod} R)$. Fix a prime ideal $\mathfrak{p}$ of R and put $n = \operatorname{ht} \mathfrak{p}$. We want to prove that there exists an R-module E such that $\operatorname{depth} R_{\mathfrak{p}} - \operatorname{depth} E_{\mathfrak{p}} = f(\mathfrak{p})$ and $\operatorname{depth} R_{\mathfrak{q}} - \operatorname{depth} E_{\mathfrak{q}} \leqslant f(\mathfrak{q})$ for all prime ideals $\mathfrak{q}$ of R.

Let $n = 0$. Then the prime ideal $\mathfrak{p}$ is minimal, and hence $\operatorname{depth} R_{\mathfrak{p}} = 0$. Since $0 \leqslant f(\mathfrak{p}) \leqslant \operatorname{depth} R_{\mathfrak{p}} = 0$, we get $f(\mathfrak{p}) = 0$. Setting $E = R$, we have $\operatorname{depth} R_{\mathfrak{p}} - \operatorname{depth} E_{\mathfrak{p}} = 0 = f(\mathfrak{p})$ and $\operatorname{depth} R_{\mathfrak{q}} - \operatorname{depth} E_{\mathfrak{q}} = 0 \leqslant f(\mathfrak{q})$ for all prime ideals $\mathfrak{q}$ of R. Thus the case $n = 0$ is done, and we let $n \geqslant 1$.

Choose a minimal prime ideal $\mathfrak{p}_0$ of R contained in $\mathfrak{p}$ such that $\operatorname{ht} \mathfrak{p}/\mathfrak{p}_0 = \operatorname{ht} \mathfrak{p} = n$. By the assumption of the theorem, there exists a nonzero maximal Cohen–Macaulay $R/\mathfrak{p}_0$-module M. Lemma 6.6(1) especially says $\mathfrak{p}/\mathfrak{p}_0 \in \operatorname{Supp}_{R/\mathfrak{p}_0} M$, which implies $\mathfrak{p} \in \operatorname{Supp}_R M$. There are equalities

$$\operatorname{grade}(\mathfrak{p}/\mathfrak{p}_0, M) = \operatorname{depth} M_{\mathfrak{p}/\mathfrak{p}_0} = \dim(R/\mathfrak{p}_0)_{\mathfrak{p}/\mathfrak{p}_0} = \operatorname{ht} \mathfrak{p}/\mathfrak{p}_0 = n, \tag{6.9.1}$$

where the first equality follows from [7, Theorem 2.1.3(b)]. By (6.9.1) there exists an M-regular sequence $\overline{\boldsymbol{x}} = \overline{x_1}, \dots, \overline{x_n}$ in $\mathfrak{p}/\mathfrak{p}_0$. Then $\boldsymbol{x} = x_1, \dots, x_n$ is an M-regular sequence in $\mathfrak{p}$. Set $C = M/\boldsymbol{x}M$. Since $M_{\mathfrak{p}/\mathfrak{p}_0} = M_{\mathfrak{p}}$, we see from (6.9.1) that $\operatorname{depth} M_{\mathfrak{p}} = n$. Using [7, Corollary 1.1.3(a)], we get

$$\operatorname{depth} C_{\mathfrak{p}} = \operatorname{depth} M_{\mathfrak{p}}/\boldsymbol{x}M_{\mathfrak{p}} = \operatorname{depth} M_{\mathfrak{p}} - n = 0. \tag{6.9.2}$$

Using [7, Corollary 1.1.3(a)] again, we see that for each $\mathfrak{q} \in \mathrm{V}(\mathfrak{p})$ there are equalities and inequalities

$$\begin{aligned} \operatorname{depth} C_{\mathfrak{q}} &= \operatorname{depth} M_{\mathfrak{q}}/\boldsymbol{x}M_{\mathfrak{q}} = \operatorname{depth} M_{\mathfrak{q}} - n = \operatorname{depth} M_{\mathfrak{q}/\mathfrak{p}_0} - n \\ &\geqslant \dim(R/\mathfrak{p}_0)_{\mathfrak{q}/\mathfrak{p}_0} - n = \operatorname{ht} \mathfrak{q}/\mathfrak{p}_0 - n \geqslant \operatorname{ht} \mathfrak{q}/\mathfrak{p} + \operatorname{ht} \mathfrak{p}/\mathfrak{p}_0 - n = \operatorname{ht} \mathfrak{q}/\mathfrak{p}. \end{aligned} \tag{6.9.3}$$

Lemma 6.6(2) says that $\mathfrak{p}$ is in $\mathrm{NF}(C)$. Applying [11, Proposition 3.1], we find an R-module D such that

$$\mathrm{NF}(D) = \mathrm{V}(\mathfrak{p}), \qquad \operatorname{depth} D_{\mathfrak{q}} = \inf\{\operatorname{depth} C_{\mathfrak{q}}, \operatorname{depth} R_{\mathfrak{q}}\} \text{ for all } \mathfrak{q} \in \mathrm{V}(\mathfrak{p}). \tag{6.9.4}$$

Since f is a moderate function, we have $0 \leqslant \operatorname{depth} R_{\mathfrak{p}} - f(\mathfrak{p}) \leqslant \operatorname{depth} R_{\mathfrak{p}}$. We set $E = \Omega_R^{\operatorname{depth} R_{\mathfrak{p}} - f(\mathfrak{p})} D$. It is seen from (6.9.2) and (6.9.4) that $\operatorname{depth} D_{\mathfrak{p}} = 0$. Applying Lemma 6.6(3), we obtain $\operatorname{depth} E_{\mathfrak{p}} = \operatorname{depth} R_{\mathfrak{p}} - f(\mathfrak{p})$, that is to say, $\operatorname{depth} R_{\mathfrak{p}} - \operatorname{depth} E_{\mathfrak{p}} = f(\mathfrak{p})$.

Fix a prime ideal $\mathfrak{q}$ of R. We want to deduce $\operatorname{depth} R_\mathfrak{q} - \operatorname{depth} E_\mathfrak{q} \leqslant f(\mathfrak{q})$. For this, first, suppose $\mathfrak{q} \notin \mathrm{V}(\mathfrak{p})$. Then $\mathfrak{q} \notin \mathrm{NF}(D)$ by (6.9.4). Hence $D_\mathfrak{q}$ is $R_\mathfrak{q}$-free, and so is $E_\mathfrak{q}$. We have $\operatorname{depth} E_\mathfrak{q} \geqslant \operatorname{depth} R_\mathfrak{q}$, and $\operatorname{depth} R_\mathfrak{q} - \operatorname{depth} E_\mathfrak{q} \leqslant 0 \leqslant f(\mathfrak{q})$ as desired. Next, suppose $\mathfrak{q} \in \mathrm{V}(\mathfrak{p})$. The depth lemma shows

$$\operatorname{depth} E_\mathfrak{q} \geqslant \inf\{\operatorname{depth} R_\mathfrak{q}, (\operatorname{depth} R_\mathfrak{p} - f(\mathfrak{p})) + \operatorname{depth} D_\mathfrak{q}\}.$$

Consider the case $\operatorname{depth} C_\mathfrak{q} \leqslant \operatorname{depth} R_\mathfrak{q}$. Then $\operatorname{depth} D_\mathfrak{q} = \operatorname{depth} C_\mathfrak{q} \geqslant \operatorname{ht} \mathfrak{q}/\mathfrak{p}$ by (6.9.3) and (6.9.4). As f is moderate, $\operatorname{depth} R_\mathfrak{p} - f(\mathfrak{p}) \geqslant \operatorname{depth} R_\mathfrak{q} - f(\mathfrak{q}) - \operatorname{ht} \mathfrak{q}/\mathfrak{p}$. We observe $\operatorname{depth} E_\mathfrak{q} \geqslant \operatorname{depth} R_\mathfrak{q} - f(\mathfrak{q})$.

Consider the case $\operatorname{depth} C_\mathfrak{q} > \operatorname{depth} R_\mathfrak{q}$. Then $\operatorname{depth} D_\mathfrak{q} = \operatorname{depth} R_\mathfrak{q}$ by (6.9.4), while $\operatorname{depth} R_\mathfrak{p} - f(\mathfrak{p}) \geqslant 0$ since f is a moderate function. We see that $\operatorname{depth} E_\mathfrak{q} \geqslant \operatorname{depth} R_\mathfrak{q}$, and hence $\operatorname{depth} E_\mathfrak{q} \geqslant \operatorname{depth} R_\mathfrak{q} - f(\mathfrak{q})$.

Thus, in either case, we have $\operatorname{depth} R_\mathfrak{q} - \operatorname{depth} E_\mathfrak{q} \leqslant f(\mathfrak{q})$. Consequently, E is a module which we desire to construct, and the proof of the theorem is completed. ■

As an application of the above theorem, we present an example. For this, we make a remark.

Remark 6.10. Assume that $\operatorname{Max} R \subseteq \operatorname{Ass} R$ (e.g., R is local and has depth zero). Then the only grade-consistent function on $\operatorname{Spec} R$ is 0. In fact, let f be a grade-consistent function on $\operatorname{Spec} R$. Let $\mathfrak{p}$ be a prime ideal of R. Then there exists a maximal ideal $\mathfrak{m}$ of R with $\mathfrak{p} \subseteq \mathfrak{m}$. It holds that $0 \leqslant f(\mathfrak{p}) \leqslant f(\mathfrak{m}) \leqslant \operatorname{grade} \mathfrak{m}$. Since $\mathfrak{m}$ is an associated prime of R, we have $\operatorname{grade} \mathfrak{m} = 0$. Therefore $f(\mathfrak{p}) = 0$, and thus $f = 0$.

We demonstrate how to apply our theorem in the following example.

Example 6.11. Let $R = k[\![x, y, z]\!]/(x^2, xy, xz)$ be a homomorphic image of a formal power series ring over a field k. Then R is a local ring, and we denote by $\mathfrak{m}$ the maximal ideal of R. Set $P = xR$.

(1) As the local ring R has depth zero, Remark 6.10 implies that

$$\{\text{grade-consistent function on } \operatorname{Spec} R\} = \{0\}.$$

Since the free R-modules are the only R-modules of finite projective dimension, $\operatorname{add}\{R\}$ is the only resolving subcategory of $\operatorname{mod} R$ contained in $\operatorname{fpd} R$; see also [11, Theorem 1.2].

(2) The local ring R is an isolated singularity. Indeed, let $\mathfrak{p} \neq \mathfrak{m}$ be a prime ideal of R. Then one of the elements $x, y, z \in R$ is outside $\mathfrak{p}$, from which we easily deduce that the localization $R_\mathfrak{p}$ is regular.

(3) The ring R is catenary as it is a complete local ring. One has $\operatorname{Min} R = \{P\}$. In particular, R is equidimensional. We see from (2) that $R_\mathfrak{p}$ is catenary and equidimensional for every $\mathfrak{p} \in \operatorname{Spec} R$.

(4) We claim that the moderate functions can be described as follows.

$$\{\text{moderate function on } \operatorname{Spec} R\} = \left\{ f \in \operatorname{Map}(\operatorname{Spec} R, \mathbb{N}) \;\middle|\; \begin{array}{l} \text{for each } \mathfrak{p} \in \operatorname{Spec} R \text{ one has} \\ f(\mathfrak{p}) = \begin{cases} 0 & (\text{if } \operatorname{ht} \mathfrak{p} \neq 1) \\ 0, 1 & (\text{if } \operatorname{ht} \mathfrak{p} = 1) \end{cases} \end{array} \right\}.$$

Indeed, (3) and Proposition 6.3(3) say that $f \in \operatorname{Map}(\operatorname{Spec} R, \mathbb{N})$ is moderate if and only if $f(\mathfrak{p}) \leqslant \operatorname{depth} R_\mathfrak{p}$ for all $\mathfrak{p} \in \operatorname{Spec} R$ and $f(\mathfrak{p}) + \operatorname{cmd} R_\mathfrak{p} \leqslant f(\mathfrak{q}) + \operatorname{cmd} R_\mathfrak{q}$ for all $\mathfrak{p}, \mathfrak{q} \in \operatorname{Spec} R$ with $\mathfrak{p} \subsetneq \mathfrak{q}$.

Let f be a moderate function on $\operatorname{Spec} R$. Then $0 \leqslant f(P) \leqslant \operatorname{depth} R_P = 0$ and $0 \leqslant f(\mathfrak{m}) \leqslant \operatorname{depth} R_\mathfrak{m} = 0$, which imply $f(P) = f(\mathfrak{m}) = 0$. For $\mathfrak{p} \in \operatorname{Spec} R$ with $\operatorname{ht} \mathfrak{p} = 1$ we have $0 \leqslant f(\mathfrak{p}) \leqslant \operatorname{depth} R_\mathfrak{p} \leqslant \dim R_\mathfrak{p} = 1$, whence $f(\mathfrak{p}) = 0, 1$. Thus f is in the right-hand side of the above equality.

Conversely, take any element f of the right-hand side of the above equality. Then for each $\mathfrak{p} \in \operatorname{Spec} R$, if $\operatorname{ht} \mathfrak{p} \neq 1$, then $f(\mathfrak{p}) = 0 \leqslant \operatorname{depth} R_\mathfrak{p}$, while if $\operatorname{ht} \mathfrak{p} = 1$, then $0 \leqslant f(\mathfrak{p}) \leqslant 1 = \dim R_\mathfrak{p} = \operatorname{depth} R_\mathfrak{p}$ by (2). If $\mathfrak{p}$ is a prime ideal of R with $P \subsetneq \mathfrak{p} \subsetneq \mathfrak{m}$, then we have $\operatorname{ht} \mathfrak{p} = 1$, and

$$f(P) = 0, \quad f(\mathfrak{p}) = 0, 1, \quad f(\mathfrak{m}) = 0, \quad \operatorname{cmd} R_P = 0, \quad \operatorname{cmd} R_\mathfrak{p} = 0, \quad \operatorname{cmd} R_\mathfrak{m} = 2,$$

which give rise to $f(P) + \operatorname{cmd} R_P \leqslant f(\mathfrak{p}) + \operatorname{cmd} R_\mathfrak{p} \leqslant f(\mathfrak{m}) + \operatorname{cmd} R_\mathfrak{m}$. Hence f is a moderate function.

(5) It is seen from (1) and (4) that

$$\{\text{grade-consistent function on } \operatorname{Spec} R\} \subsetneq \{\text{moderate function on } \operatorname{Spec} R\}.$$

(6) The ring R possesses uncountably many height one prime ideals. In fact, each element of $\mathfrak{m}$ is contained in some height one prime ideal of R, that is, $\mathfrak{m} \subseteq \bigcup_{\mathfrak{p} \in \operatorname{Spec} R,\, \operatorname{ht} \mathfrak{p} = 1} \mathfrak{p}$. Since the local ring R is complete and $\dim R = 2 > 1$, it is observed from [23, Corollary (2.2)] that the statement holds.

(7) We see from (4) and (6) that there exist uncountably many moderate functions on $\operatorname{Spec} R$. The ideal $P = xR$ is the only minimal prime ideal of R, and $R/P \cong k[\![y, z]\!]$ is a regular local ring. Theorem 6.9 implies that there exist uncountably many dominant resolving subcategories of $\operatorname{mod} R$.

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References

[1] T. Aihara; T. Araya; O. Iyama; R. Takahashi; M. Yoshiwaki, Dimensions of triangulated categories with respect to subcategories, *J. Algebra* **399** (2014), 205–219.

[2] L. Angeleri Hügel; D. Pospíšil; J. Šťovíček; J. Trlifaj, Tilting, cotilting, and spectra of commutative Noetherian rings, *Trans. Amer. Math. Soc.* **366** (2014), no. 7, 3487–3517.

[3] M. Auslander; M. Bridger, Stable module theory, *Mem. Amer. Math. Soc.* **94** (1969).

[4] M. Auslander; I. Reiten, Applications of contravariantly finite subcategories, *Adv. Math.* **86** (1991), no. 1, 111–152.

[5] L. L. Avramov; S. B. Iyengar; J. Lipman, Reflexivity and rigidity for complexes, I, Commutative rings, *Algebra Number Theory* **4** (2010), no. 1, 47–86.

[6] H. Bass; M. P. Murthy, Grothendieck groups and Picard groups of abelian group rings, *Ann. of Math. (2)* **86** (1967), 16–73.

[7] W. Bruns; J. Herzog, Cohen–Macaulay rings, revised edition, Cambridge Studies in Advanced Mathematics, **39**, *Cambridge University Press, Cambridge*, 1998.

[8] L. W. Christensen; H.-B. Foxby; A. Frankild, Restricted homological dimensions and Cohen–Macaulayness, *J. Algebra* **251** (2002), no. 1, 479–502.

[9] H. Dao; T. Kobayashi; R. Takahashi, Burch ideals and Burch rings, *Algebra Number Theory* **14** (2020), no. 8, 2121–2150.

[10] H. Dao; R. Takahashi, The radius of a subcategory of modules, *Algebra Number Theory* **8** (2014), no. 1, 141–172.

[11] H. Dao; R. Takahashi, Classification of resolving subcategories and grade consistent functions, *Int. Math. Res. Not. IMRN* **2015**, no. 1, 119–149.

[12] H. Dao; R. Takahashi, Upper bounds for dimensions of singularity categories, *C. R. Math. Acad. Sci. Paris* **353** (2015), no. 4, 297–301.

[13] H. Dao; R. Takahashi, The dimension of a subcategory of modules, *Forum Math. Sigma* **3** (2015), e19, 31 pp.

[14] A. A. Gerko, On homological dimensions (Russian); translated from *Mat. Sb.* **192** (2001), no. 8, 79–94, *Sb. Math.* **192** (2001), no. 7-8, 1165–1179.

[15] M. Hochster, Topics in the homological theory of modules over commutative rings, Expository lectures from the CBMS Regional Conference held at the University of Nebraska, Lincoln, Neb., June 24-28, 1974, Conference Board of the Mathematical Sciences Regional Conference Series in Mathematics **24**, *Published for the Conference Board of the Mathematical Sciences by the American Mathematical Society, Providence, R.I.*, 1975.

[16] M. Hochster, Homological conjectures and lim Cohen–Macaulay sequences, *Homological and computational methods in commutative algebra*, 173–197, Springer INdAM Ser. **20**, *Springer, Cham*, 2017.

[17] S. B. Iyengar; R. Takahashi, Annihilation of cohomology and strong generation of module categories, *Int. Math. Res. Not. IMRN* **2016**, no. 2, 499–535.

[18] T. Kawasaki, On Macaulayfication of Noetherian schemes, *Trans. Amer. Math. Soc.* **352** (2000), no. 6, 2517–2552.

[19] H. Matsumura, Commutative ring theory, Translated from the Japanese by M. Reid, Second edition, Cambridge Studies in Advanced Mathematics, **8**, *Cambridge University Press, Cambridge*, 1989.

[20] S. Nasseh; R. Takahashi. Local rings with quasi-decomposable maximal ideal, *Math. Proc. Cambridge Philos. Soc.* **168** (2020), no. 2, 305–322.

[21] A. Sadeghi; R. Takahashi, Resolving subcategories closed under certain operations and a conjecture of Dao and Takahashi, *Michigan Math. J.* (to appear), `arXiv:1710.10018`.

[22] W. Sanders, Classifying resolving subcategories, *Pacific J. Math.* **286** (2017), no. 2, 401–438.

[23] R. Y. Sharp; P. Vámos, Baire's category theorem and prime avoidance in complete local rings, *Arch. Math. (Basel)* **44** (1985), no. 3, 243–248.

[24] G. Stevenson, Subcategories of singularity categories via tensor actions, *Compos. Math.* **150** (2014), no. 2, 229–272.

[25] R. Takahashi, Modules in resolving subcategories which are free on the punctured spectrum, *Pacific J. Math.* **241** (2009), no. 2, 347–367.

[26] R. Takahashi, Classifying thick subcategories of the stable category of Cohen–Macaulay modules, *Adv. Math.* **225** (2010), no. 4, 2076–2116.

[27] R. Takahashi, Contravariantly finite resolving subcategories over commutative rings, *Amer. J. Math.* **133** (2011), no. 2, 417–436.

[28] R. Takahashi, Classifying resolving subcategories over a Cohen–Macaulay local ring, *Math. Z.* **273** (2013), no. 1-2, 569–587.

[29] R. Takahashi, Thick subcategories over Gorenstein local rings that are locally hypersurfaces on the punctured spectra, *J. Math. Soc. Japan* **65** (2013), no. 2, 357–374.

[30] R. Takahashi, Reconstruction from Koszul homology and applications to module and derived categories, *Pacific J. Math.* **268** (2014), no. 1, 231–248.

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