Cohen-Macaulay rings and related homological dimensions

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Contents

Preface

Commutative ring theory is a branch of algebra which was born in the late 19th century. In the 1870s, J. W. R. Dedekind introduced the notion of an ideal in his research of the rings of integers of algebraic number fields. After that, D. Hilbert proved several fundamental results on polynomial rings, containing the famous theorem which asserts that ideals of polynomial rings are finitely generated. (This theorem is now called Hilbert's basis theorem.) With that as a start, the theory of ideals of polynomial rings was further developed by algebraists such as E. Lasker and F. S. Macaulay around the turn of the century. Through a lot of great achievements of A. E. Noether, W. Krull, E. Artin, and others, commutative ring theory gradually occupied a respectable position in algebra in the early 20th century.

In the 1950s, commutative ring theory took a new turn. M. Auslander, D. A. Buchsbaum, D. Rees, D. G. Northcott, J.-P. Serre, and others introduced homological algebra into commutative ring theory. By virtue of this, commutative ring theory had a chance to make rapid progress. As a result worthy of special mention, it was at last established by Serre that any localization of any regular local ring is again regular, which had been conjectured but had never proved for a long time.

A Cohen-Macaulay ring was originally defined as a ring possessing a certain good property on the heights of ideals called the unmixedness theorem. By the innovation of homological algebra into commutative ring theory, the notion of a Cohen-Macaulay ring has grown up to play a quite important role in commutative ring theory. It is closely related not only to algebraic geometry but also to algebraic combinatorics, and has been greatly studied by a large number of mathematicians.

A Gorenstein ring is a significant example of Cohen-Macaulay rings. This ring satisfies a certain duality on the derived category of the category of finitely generated modules. The class of regular rings are contained in that of Gorenstein rings, and complete intersections constitute an intermediate class between those classes of rings. These four principal classes of commutative rings have the associated homological invariants of modules called projective dimension, complete intersection dimension (abbr. CI-dimension), Gorenstein dimension (abbr. G-dimension), and Cohen-Macaulay dimension (abbr. CM-dimension). Specifically, projective dimension (resp. CI-dimenison, G-dimension, CM-dimension) is a homological invariant characterizing the regular (resp. complete intersection, Gorenstein, Cohen-Macaulay) property of the base ring. Such a homological invariant is called a homological dimension.

Throughout this doctoral thesis, we fasten our eyes on the Cohen-Macaulay property of a given noetherian ring. In other words, we consider to what extent properties of Cohen-Macaulay rings are reflected in general noetherian rings. We work on this problem mainly by using homological methods, especially by using homological dimensions mentioned above.

The organization of this thesis is as follows; it consists of six chapters.

In Chapter 1, we state on Nagata criterion. This is a criterion for the openness of the locus of a property of local rings. On the other hand, there are ideal-theoretic and homological conditions called Serre's (R_n) and (S_n)-conditions. It is known that Nagata criterion holds for both the regular property and the Cohen-Macaulay property, and that Serre's (R_n) and (S_n)-conditions hold for the regular property and the Cohen-Macaulay property respectively. We prove in this chapter that Nagata criterion holds for Serre's conditions.

In Chapter 2, we define new homological dimensions called upper CM-dimension and upper CI-dimension. These dimensions are invariants which are very similar to CM-dimension and CI-dimension respectively: these also characterize the Cohen-Macaulay property and the complete intersection property of the base ring respectively. After that, we introduce relative versions of these dimensions, and observe the relationship between upper CM-dimension and G-dimension and the relationship between upper CI-dimension and projective dimension.

In Chapter 3, we generalize CM-dimension. CM-dimension was defined by A. A. Gerko as an invariant of finitely generated modules over a commutative noetherian local ring. In this chapter, we extend CM-dimension to invariants of bounded complexes over a noetherian ring which is neither necessarily local nor necessarily commutative. Considering CM-dimension under circumstances as general as possible, we grope for the quintessence of CM-dimension. In the final section of this chapter, using the generalized CM-dimensions, we give characterizations of Cohen-Macaulay rings and Gorenstein rings.

In Chapter 4, we observe how CM-dimension behaves over a commutative noetherian local ring of prime characteristic. Considering the algebra whose algebra structure is defined by the Frobenius map, we give characterizations of Cohen-Macaulay rings and Gorenstein rings, which are analogues of the characterization of regular rings by E. Kunz and A. G. Rodicio, and that of complete intersections by A. Blanco and J. Majadas. Also, we easily prove a result which generalizes the main theorem of the famous paper of J. Herzog published in 1974. The results in this chapter may be deeply related to the notion of tight closure which has been greatly studied in commutative ring theory and algebraic geometry recently.

In Chapter 5, we state a result which is related to the Peskine-Szpiro intersection theorem. This theorem was proved in the prime characteristic case by C. Peskine and L. Szpiro in the early 1970s, was proved in the equicharacteristic case by M. Hochster, and was proved in the general case by P. Roberts in the late 1980s. It is certainly one of the main results in commutative ring theory in the late 20th century. There is a conjecture on G-dimesion which generalizes an important corollary of this theorem. In this chapter, we give several criteria for the Gorenstein property in terms of G-dimension, one of which not only implies all the other criteria but also proves that the conjecture holds in a special case.

In Chapter 6, we investigate the category of modules of G-dimension zero. This category coincides with the category of maximal Cohen-Macaulay modules if the base ring is a Gorenstein local ring. The category of maximal Cohen-Macaulay modules (over a Cohen-Macaulay local ring) is also studied in representation theory (of algebras), and many are known concerning this category. We observe in this chapter whether a certain property of the category of maximal Cohen-Macaulay modules over a Gorenstein local ring (i.e. the category of modules of G-dimension zero over a Gorenstein local ring) is satisfied by the category of modules of G-dimension zero over a non-Gorenstein local ring, and prove as the main theorem that in several cases it is not. It has been expected that modules of G-dimension zero (over a general local ring) may behave similarly to maximal Cohen-Macaulay modules over a Gorenstein local ring, but the main theorem is against the expectation.

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1 Nagata criterion for Serre's conditions

The contents of this chapter are entirely contained in the author's own paper [47].

It is known that Nagata criterion holds for the regular property and the Cohen-Macaulay property. On the other hand, we easily see from definition that these properties can be described by using Serre's (R_n) and (S_n)-conditions, that is, a commutative noetherian ring A is regular (resp. Cohen-Macaulay) if and only if the ring A satisfies the condition (R_n) (resp. (S_n)) for every $n \geq 0$. Hence it is natural to expect that Nagata criterion may also hold for both (R_n) and (S_n) for each $n \geq 0$. We shall prove in this chapter that it is indeed true.

1.1 Introduction

Throughout this chapter, we assume that all rings are noetherian commutative rings.

First of all, we recall Serre's (R_n) and (S_n)-conditions for a ring A. These are defined as follows. Let n be an integer.

(R_n) : If $\mathfrak{p} \in \operatorname{Spec} A$ and $\operatorname{ht} \mathfrak{p} \leq n$, then $A_\mathfrak{p}$ is regular.

(S_n) : $\operatorname{depth} A_\mathfrak{p} \geq \inf\{n, \operatorname{ht} \mathfrak{p}\}$ for all $\mathfrak{p} \in \operatorname{Spec} A$.

Let $\mathcal{P}$ be a property of local rings. For a ring A we put

$$\mathcal{P}(A) = \{ \ \mathfrak{p} \in \operatorname{Spec} A \mid \mathcal{P} \text{ holds for } A_\mathfrak{p} \ \}$$

and call it the $\mathcal{P}$-locus of A. The following statement is called the (ring-theoretic) Nagata criterion for the property $\mathcal{P}$, and we abbreviate it to (NC).

(NC) : If A is a ring and if $\mathcal{P}(A/\mathfrak{p})$ contains a non-empty open subset of $\operatorname{Spec} A/\mathfrak{p}$ for every $\mathfrak{p} \in \operatorname{Spec} A$, then $\mathcal{P}(A)$ is open in $\operatorname{Spec} A$.

This statement was invented by Nagata in 1959. In algebraic geometry, there is a problem asking when the regular locus (that is, the non-singular locus) of a ring is open. He proposed the above criterion to consider this problem, and he proved that (NC) holds for $\mathcal{P}$ = regular [41]. There are some other properties $\mathcal{P}$ for which (NC) holds, for example, $\mathcal{P}$ = Cohen-Macaulay [37][38], Gorenstein [28][38], and complete intersection [28]. On the other hand, it is easy to see that (NC) holds for $\mathcal{P}$ = (integral) domain, coprimary (a ring A is called coprimary if $\sharp \operatorname{Ass} A = 1$), (R_0), (S_1), reduced, and normal. Moreover, as corollaries of these results, we easily see that the following proposition is true for $\mathcal{P}$ = Cohen-Macaulay [37][38], Gorenstein [38], domain, coprimary, (R_0), (S_1), and reduced.

Let $\mathcal{P}$ be a property for which (NC) holds. Then, for a ring A satisfying $\mathcal{P}$, the $\mathcal{P}$-locus of a homomorphic image of A is open.

It is known that the properties "regular", "Cohen-Macaulay", "reduced", and "normal" are described by using (R_n) and (S_n). Since (NC) holds for each of these properties, we naturally expect that (NC) may hold for (R_n) and (S_n) for every $n \geq 0$. This is in fact true, and the main purpose of this chapter is to give its complete proof.

1.2 For (S_n)-condition

Let A be a ring and f an element of A. For an A-module M, we denote by M_f the localization of M with respect to the multiplicatively closed set $\{1, f, f^2, \cdots\}$. The following lemma on 'generic flatness' should be referred to [37, §22].

Lemma 1.2.1 *Let A be an integral domain, B an finitely generated A-algebra, and N a finitely generated B-module. Then there exists $f(\neq 0) \in A$ such that N_f is a free (in particular flat) A_f-module.*

Now we can prove the main result of this section.

Theorem 1.2.2 (NC) *holds for $\mathcal{P} = (S_n)$ for any $n \geq 0$.*

PROOF We prove the theorem by induction on n. It is easy to see that (NC) holds for $\mathcal{P} = (S_0)$ and (S_1) respectively. Hence we assume $n \geq 2$ in the rest. Suppose that a ring A satisfies the assumption in (NC). We want to prove that the locus $S_n(A)$ is open in $\operatorname{Spec} A$. Since (S_n) implies (S_{n-1}), the locus $S_{n-1}(A)$ is open in $\operatorname{Spec} A$ by induction hypothesis. Therefore we can write $S_{n-1}(A) = \bigcup_{i=1}^{s} D(f_i)$ with $f_i \in A$, and hence we have $S_n(A) = \bigcup_{i=1}^{s}(S_n(A) \cap D(f_i)) = \bigcup_{i=1}^{s} S_n(A_{f_i})$. Since $S_{n-1}(A_{f_i}) = S_{n-1}(A) \cap D(f_i) = D(f_i) = \operatorname{Spec} A_{f_i}$, the condition (S_{n-1}) holds for A_{f_i}. Thus, replacing A by A_{f_i}, to prove the openness of $S_n(A)$ we may assume that the condition (S_{n-1}) holds for A.

Let $\mathcal{I}$ be the set of ideals $\mathfrak{a}$ of A such that $S_n(A)^c := \operatorname{Spec} A - S_n(A) \subseteq V(\mathfrak{a})$. This set $\mathcal{I}$ is non-empty because the zero ideal of A belongs to $\mathcal{I}$. Since A is noetherian, the set $\mathcal{I}$ has a maximal element I. If $I = A$, then we have $S_n(A) = \mathrm{Spec} A$, which is of course open in $\mathrm{Spec}\, A$. Therefore we assume that $I \subsetneq A$. It is easy to see from the maximality that $\sqrt{I} = I$ and that the closure $\overline{S_n(A)^c}$ of $S_n(A)^c$ in $\mathrm{Spec}\, A$ is the set $V(I)$. It follows from this that I has a primary decomposition of the form $I = \mathfrak{p}_1 \cap \mathfrak{p}_2 \cap \cdots \cap \mathfrak{p}_t$, where each $\mathfrak{p}_i$ is a prime ideal of A, and we may assume that there are no inclusion relations between the $\mathfrak{p}_i$'s and that $\mathrm{ht}\, \mathfrak{p}_1 \leq \mathrm{ht}\, \mathfrak{p}_i$ for all i.

Now we claim that

(1) $\mathrm{ht}\, I \geq n$,

(2) $\mathfrak{p}_i \in S_n(A)^c$ for all i,

(3) $S_n(A)^c = V(I)$.

It follows from (3) that $S_n(A) = D(I)$, which shows that $S_n(A)$ is open in $\mathrm{Spec}\, A$, proving the theorem. We prove these in turn.

(1) It suffices to prove that $\mathrm{ht}\, \mathfrak{p}_1 \geq n$. To prove this by contradiction, suppose that $l := \mathrm{ht}\, \mathfrak{p}_1 \leq n-1$. Since the condition (S_{n-1}) holds for A, we get $\mathrm{depth}\, A_{\mathfrak{p}_1} \geq \inf\{n-1, \mathrm{ht}\, \mathfrak{p}_1\} = \mathrm{ht}\, \mathfrak{p}_1 = l$, and hence there exist elements $c_1, c_2, \cdots, c_l \in \mathfrak{p}_1$ and $f \in A - \mathfrak{p}_1$ such that

$$\begin{cases} c_1, c_2, \cdots, c_l \text{ is an } A_f\text{-sequence in } \mathfrak{p}_1 A_f, \\ (c_1, c_2, \cdots, c_l)A_f \text{ is a } \mathfrak{p}_1 A_f\text{-primary ideal of } A_f. \end{cases}$$

Now we can take $g \in \bigcap_{i=2}^t \mathfrak{p}_i - \mathfrak{p}_1$ such that $IA_g = \mathfrak{p}_1 A_g$ because $\mathfrak{p}_i \not\subseteq \mathfrak{p}_1$ for all $i \geq 2$. Moreover, by the assumption in (NC), there exists $h \in A - \mathfrak{p}_1$ such that $D(h) \cap V(\mathfrak{p}_1) \subseteq S_n(A/\mathfrak{p}_1)$, and hence the condition (S_n) holds for $A_h/\mathfrak{p}_1 A_h$. Put $x = fgh$ $(\in A - \mathfrak{p}_1)$. Replacing A by A_x, we may assume that

$$\begin{cases} c_1, c_2, \cdots, c_l \text{ is an } A\text{-sequence in } \mathfrak{p}_1, \\ (c_1, c_2, \cdots, c_l) \text{ is } \mathfrak{p}_1\text{-primary} \\ \quad (\text{hence } \mathfrak{p}_1^r \subseteq (c_1, c_2, \cdots, c_l) \text{ for some } r \in \mathbb{N}), \\ I = \mathfrak{p}_1 \\ \quad (\text{hence } \overline{S_n(A)^c} = V(\mathfrak{p}_1)), \\ (S_n) \text{ holds for } A/\mathfrak{p}_1. \end{cases}$$

Moreover, by Lemma 1.2.1, replacing A by A_y with some $y \in A - \mathfrak{p}_1$, we may assume that

$$\mathfrak{p}_1^i / (\mathfrak{p}_1^{i+1} + (c_1, c_2, \cdots, c_l) \cap \mathfrak{p}_1^i)$$

is a free $A/\mathfrak{p}_1$-module for $i = 1, 2, \cdots, r-1$.

Now note that the set $S_n(A)^c$ is non-empty. In fact, if $S_n(A)^c = \emptyset$, then $V(\mathfrak{p}_1) = \overline{S_n(A)^c} = \emptyset$, and hence $\mathfrak{p}_1 = A$, which is contradiction. Therefore we have $S_n(A)^c \neq \emptyset$. We would like to prove that the ring $A_\mathfrak{p}$ satisfies the condition (S_n) for any $\mathfrak{p} \in S_n(A)^c$. If this is true, then we have a contradiction since $\mathfrak{p} \notin S_n(A)$. Therefore, we will have $\operatorname{ht}\mathfrak{p}_1 \geq n$ as desired. To prove that (S_n) holds for $A_\mathfrak{p}$, take $\mathfrak{p}' \in \operatorname{Spec} A$ with $\mathfrak{p}' \subseteq \mathfrak{p}$, and $\mathfrak{p}'' \in V(\mathfrak{p}' + \mathfrak{p}_1)$ such that $\operatorname{ht}(\mathfrak{p}' + \mathfrak{p}_1/\mathfrak{p}_1) = \operatorname{ht}(\mathfrak{p}''/\mathfrak{p}_1)$. (Since $\mathfrak{p}', \mathfrak{p}_1 \subseteq \mathfrak{p}$, we have $V(\mathfrak{p}' + \mathfrak{p}_1) \neq \emptyset$.) We should divide the proof into two cases.

$i)$ The case when $\operatorname{ht}(\mathfrak{p}' + \mathfrak{p}_1/\mathfrak{p}_1) \leq n$:
Since $\operatorname{ht}(\mathfrak{p}''/\mathfrak{p}_1) \leq n$, the ring $A_{\mathfrak{p}''}/\mathfrak{p}_1 A_{\mathfrak{p}''} = (A/\mathfrak{p}_1)_{\mathfrak{p}''/\mathfrak{p}_1}$ is Cohen-Macaulay. Replacing A by $A/(c_1, c_2, \cdots, c_l)$, we may assume that $\mathfrak{p}_1^r = (0)$ and that $\mathfrak{p}_1^i/\mathfrak{p}_1^{i+1}$ is a free $A/\mathfrak{p}_1$-module. Therefore we have

$$\begin{aligned} \operatorname{depth} A_{\mathfrak{p}''} &= \operatorname{depth}(A_{\mathfrak{p}''}/\mathfrak{p}_1^r A_{\mathfrak{p}''}) \\ &= \operatorname{depth}(A_{\mathfrak{p}''}/\mathfrak{p}_1 A_{\mathfrak{p}''}) \\ &= \operatorname{ht}(\mathfrak{p}''/\mathfrak{p}_1) \\ &= \operatorname{ht}\mathfrak{p}'', \end{aligned}$$

and hence $A_{\mathfrak{p}''}$ is a Cohen-Macaulay ring. It follows that $A_{\mathfrak{p}'} = (A_{\mathfrak{p}''})_{\mathfrak{p}' A_{\mathfrak{p}''}}$ is also a Cohen-Macaulay ring.

$ii)$ The case when $\operatorname{ht}(\mathfrak{p}' + \mathfrak{p}_1/\mathfrak{p}_1) \geq n$:
Let $\mathfrak{q}/\mathfrak{p}_1 \in V(\mathfrak{p}' + \mathfrak{p}_1/\mathfrak{p}_1)$. Then $\operatorname{ht}(\mathfrak{q}/\mathfrak{p}_1) \geq n$, and hence $\operatorname{depth}(A/\mathfrak{p}_1)_{\mathfrak{q}/\mathfrak{p}_1} \geq n$. Thus, we have $\operatorname{grade}(\mathfrak{p}' + \mathfrak{p}_1/\mathfrak{p}_1) \geq n$. Therefore there exist a sequence $c_1', c_2', \cdots, c_n'$ in $\mathfrak{p}'$ which forms an $A/\mathfrak{p}_1$-sequence in $\mathfrak{p}' + \mathfrak{p}_1/\mathfrak{p}_1$. Since $\mathfrak{p}_1^i/(\mathfrak{p}_1^{i+1} + (c_1, c_2, \cdots, c_l) \cap \mathfrak{p}_1^i)$ is a free $A/\mathfrak{p}_1$-module for $i = 1, 2, \cdots, r-1$, we can show that

$$c_1', c_2', \cdots, c_n' \text{ is an } A/(c_1, c_2, \cdots, c_l)\text{-sequence in } \mathfrak{p}' + \mathfrak{p}_1/\mathfrak{p}_1.$$

Hence $c_1, c_2, \cdots, c_l, c_1', c_2', \cdots, c_n'$ is an A-sequence in $\mathfrak{p}''$, and is an $A_{\mathfrak{p}''}$-sequence in $\mathfrak{p}'' A_{\mathfrak{p}''}$. Therefore we have

$$c_1', c_2', \cdots, c_n', c_1, c_2, \cdots, c_l \text{ is an } A_{\mathfrak{p}''}\text{-sequence in } \mathfrak{p}'' A_{\mathfrak{p}''}.$$

Hence $c_1', c_2', \cdots, c_n'$ is an $A_{\mathfrak{p}''}$-sequence in $\mathfrak{p}' A_{\mathfrak{p}''}$, and is an $A_{\mathfrak{p}'} = (A_{\mathfrak{p}''})_{\mathfrak{p}' A_{\mathfrak{p}''}}$-sequence in $\mathfrak{p}' A_{\mathfrak{p}'} = \mathfrak{p}'(A_{\mathfrak{p}''})_{\mathfrak{p}' A_{\mathfrak{p}''}}$. It follows that $\operatorname{depth} A_{\mathfrak{p}'} \geq n$.

As we have remarked above, it follows from $i)$, $ii)$ that $\operatorname{ht}\mathfrak{p}_1 \geq n$.

(2) To prove it by contradiction, suppose that $\mathfrak{p}_k \in S_n(A)$ for some k. Since $I \subseteq \mathfrak{p}_k$, we have $\operatorname{ht}\mathfrak{p}_k \geq n$, and hence $\operatorname{depth} A_{\mathfrak{p}_k} \geq \inf\{n, \operatorname{ht}\mathfrak{p}_k\} = n$. Therefore, there exist elements $c_1, c_2, \cdots, c_n \in \mathfrak{p}_k$ and $f \in A - \mathfrak{p}_k$ such that

$$\begin{cases} c_1, c_2, \cdots, c_n \text{ is an } A_f\text{-sequence in } \mathfrak{p}_k A_f, \\ I A_f = \mathfrak{p}_k A_f. \end{cases}$$

Since $\mathfrak{p}_k \in V(I) = \overline{S_n(A)^c}$, we have $D(f) \cap S_n(A)^c \neq \emptyset$. Let $\mathfrak{p}$ be a minimal element of this set. Since $\mathfrak{p} \in S_n(A)^c \subseteq V(I)$, we have $I \subseteq \mathfrak{p}$, and hence $\mathfrak{p}A_f \supseteq IA_f = \mathfrak{p}_k A_f$. Therefore $c_1, c_2, \cdots, c_n$ is an A_f-sequence in $\mathfrak{p}A_f$, and hence is an $A_\mathfrak{p} = (A_f)_{\mathfrak{p}A_f}$-sequence in $\mathfrak{p}A_\mathfrak{p}$. It follows that $\operatorname{depth} A_\mathfrak{p} \geq n = \inf\{n, \operatorname{ht}\mathfrak{p}\}$. On the other hand, if $\mathfrak{p}' \in \operatorname{Spec} A$ such that $\mathfrak{p}' \subsetneq \mathfrak{p}$, then we have $\mathfrak{p}' \notin D(f) \cap S_n(A)^c$ by the minimality of $\mathfrak{p}$. Since $\mathfrak{p} \in D(f)$, we have $\mathfrak{p}' \in D(f)$. Therefore we have $\mathfrak{p}' \notin S_n(A)^c$, and hence (S_n) holds for $A_{\mathfrak{p}'}$. Thus, we see that (S_n) holds for $A_\mathfrak{p}$, but this is contrary to the choice of $\mathfrak{p}$.

(3) We have $S_n(A)^c \subseteq \overline{S_n(A)^c} = V(I)$. Suppose that $S_n(A)^c \subsetneq V(I)$. Then there exists $\mathfrak{p} \in V(I)$ such that $\mathfrak{p} \notin S_n(A)^c$. Hence we have $\mathfrak{p}_k \subseteq \mathfrak{p}$ for some k and $\mathfrak{p} \in S_n(A)$. Therefore (S_n) holds for $(A_\mathfrak{p})_{\mathfrak{p}_k A_\mathfrak{p}} = A_{\mathfrak{p}_k}$. It follows that $\mathfrak{p}_k \in S_n(A)$, which is contrary to the claim (2). □

1.3 For (R_n)-condition

Consider the following condition. Let n be an integer and let A be a local ring.

(R'_n) : If $\mathfrak{p} \in \operatorname{Spec} A$ and $\operatorname{codim}\mathfrak{p} \leq n$, then $A_\mathfrak{p}$ is regular.

Here the codimension of an ideal I of A is defined as follows.

$$\operatorname{codim} I = \dim A - \dim A/I.$$

Lemma 1.3.1 *Let A be a local ring. Then (R_n) holds for A if and only if (R'_n) holds for $A_\mathfrak{p}$ for every $\mathfrak{p} \in \operatorname{Spec} A$.*

PROOF Suppose that (R_n) holds for A. Let $\mathfrak{p} \in \operatorname{Spec} A$, and let $\mathfrak{p}' \in \operatorname{Spec} A$ such that $\mathfrak{p}' \subseteq \mathfrak{p}$ and that $\operatorname{codim}\mathfrak{p}'A_\mathfrak{p} \leq n$. Then we have $\operatorname{ht}\mathfrak{p}'A_\mathfrak{p} \leq \operatorname{codim}\mathfrak{p}'A_\mathfrak{p} \leq n$, and hence the local ring $(A_\mathfrak{p})_{\mathfrak{p}'A_\mathfrak{p}}$ is regular since (R_n) holds for $A_\mathfrak{p}$. It follows that (R'_n) holds for $A_\mathfrak{p}$. Conversely, suppose that (R'_n) holds for $A_\mathfrak{p}$ for every $\mathfrak{p} \in \operatorname{Spec} A$. Let $\mathfrak{p} \in \operatorname{Spec} A$, and let $\mathfrak{p}' \in \operatorname{Spec} A$ such that $\mathfrak{p}' \subseteq \mathfrak{p}$ and that $\operatorname{ht}\mathfrak{p}'A_\mathfrak{p} \leq n$. Then we have $\operatorname{codim}\mathfrak{p}'A_{\mathfrak{p}'} = \operatorname{ht}\mathfrak{p}' = \operatorname{ht}\mathfrak{p}'A_\mathfrak{p} \leq n$, and hence the local ring $(A_\mathfrak{p})_{\mathfrak{p}'A_\mathfrak{p}} = A_{\mathfrak{p}'} = (A_{\mathfrak{p}'})_{\mathfrak{p}'A_{\mathfrak{p}'}}$ is regular. It follows that (R_n) holds for $A_\mathfrak{p}$. Therefore, (R_n) holds for A. □

The following theorem is the main result of this section.

Theorem 1.3.2 (NC) *holds for $\mathcal{P} = (R_n)$ for any $n \geq 0$.*

PROOF We prove this theorem by induction on n. It is easy to see that (NC) holds for $\mathcal{P} = (R_0)$, and hence we assume $n \geq 1$ in the rest. We discuss in the same way as the proof of Theorem 1.2.2. Suppose that a ring A satisfies the assumption in (NC). Let I be a maximal element of the set of ideals $\mathfrak{a}$ of the ring A such that $R_n(A)^c \subseteq V(\mathfrak{a})$. We may assume that

$$\begin{cases} (R_{n-1}) \text{ holds for } A, \\ I \subsetneq A, \\ \sqrt{I} = I, \\ \overline{R_n(A)^c} = V(I), \\ I = \mathfrak{p}_1 \cap \mathfrak{p}_2 \cap \cdots \cap \mathfrak{p}_t \text{ (for some } \mathfrak{p}_i \in \operatorname{Spec} A), \\ \text{there are no inclusion relations between the } \mathfrak{p}_i\text{'s}, \\ \operatorname{ht} \mathfrak{p}_1 \le \operatorname{ht} \mathfrak{p}_i \text{ for all } i. \end{cases}$$

Now we prove that $\operatorname{ht} \mathfrak{p}_1 \ge n$. To prove this by contradiction, suppose that $l := \operatorname{ht} \mathfrak{p}_1 \le n-1$. Since (R_{n-1}) holds for A, we see that $A_{\mathfrak{p}_1}$ is a regular local ring. Hence replacing A by A_x for some $x \in A - \mathfrak{p}_1$, we may assume that

$$\begin{cases} c_1, c_2, \cdots, c_l \text{ is an } A\text{-sequence in } \mathfrak{p}_1 \text{ (for some } c_i \in \mathfrak{p}_1), \\ (c_1, c_2, \cdots, c_l) = \mathfrak{p}_1, \\ I = \mathfrak{p}_1 \text{ (hence } \overline{R_n(A)^c} = V(\mathfrak{p}_1)), \\ (R_n) \text{ holds for } A/\mathfrak{p}_1. \end{cases}$$

Since $R_n(A)^c \neq \emptyset$, we can take a prime ideal $\mathfrak{p} \in R_n(A)^c$. Then we have $\mathfrak{p}_1 \subseteq \mathfrak{p}$. To show that $A_{\mathfrak{p}}$ satisfies (R'_n), we take a prime ideal $\mathfrak{p}' \in \operatorname{Spec} A$ such that $\mathfrak{p}' \subseteq \mathfrak{p}$ and that $\operatorname{codim} \mathfrak{p}' A_{\mathfrak{p}} \le n$. There exists $\mathfrak{p}'' \in V(\mathfrak{p}' + \mathfrak{p}_1)$ such that $\operatorname{codim}(\mathfrak{p}' + \mathfrak{p}_1/\mathfrak{p}_1)A_{\mathfrak{p}} = \operatorname{codim}(\mathfrak{p}''/\mathfrak{p}_1)A_{\mathfrak{p}}$ $(= \operatorname{codim} \mathfrak{p}'' A_{\mathfrak{p}}/\mathfrak{p}_1 A_{\mathfrak{p}})$. We have

$$\begin{cases} \begin{array}{rcl} \operatorname{codim}(\mathfrak{p}' + \mathfrak{p}_1/\mathfrak{p}_1)A_{\mathfrak{p}} & = & \operatorname{ht}(\mathfrak{p}/\mathfrak{p}_1) - \operatorname{ht}(\mathfrak{p}/\mathfrak{p}' + \mathfrak{p}_1) \\ & = & \operatorname{ht} \mathfrak{p} - l - \operatorname{ht}(\mathfrak{p}/\mathfrak{p}' + \mathfrak{p}_1), \\ \operatorname{codim}(\mathfrak{p}' + \mathfrak{p}_1/\mathfrak{p}')A_{\mathfrak{p}} & = & \operatorname{codim}(c_1, \cdots, c_l)A/\mathfrak{p}'_{\mathfrak{p}/\mathfrak{p}'} \\ & \le & l, \\ \operatorname{codim}(\mathfrak{p}' + \mathfrak{p}_1/\mathfrak{p}')A_{\mathfrak{p}} & = & \operatorname{ht}(\mathfrak{p}/\mathfrak{p}') - \operatorname{ht}(\mathfrak{p}/\mathfrak{p}' + \mathfrak{p}_1). \end{array} \end{cases}$$

It follows that

$$\begin{array}{rcl} \operatorname{codim}(\mathfrak{p}'' A_{\mathfrak{p}}/\mathfrak{p}_1 A_{\mathfrak{p}}) & = & \operatorname{codim}(\mathfrak{p}' + \mathfrak{p}_1/\mathfrak{p}_1)A_{\mathfrak{p}} \\ & \le & \operatorname{ht} \mathfrak{p} - \operatorname{ht}(\mathfrak{p}/\mathfrak{p}') \\ & = & \operatorname{codim} \mathfrak{p}' A_{\mathfrak{p}} \\ & \le & n. \end{array}$$

Since (R_n) holds for $A/\mathfrak{p}_1$, we see that (R_n) also holds for $(A/\mathfrak{p}_1)_{\mathfrak{p}/\mathfrak{p}_1} = A_{\mathfrak{p}}/\mathfrak{p}_1 A_{\mathfrak{p}}$. By Lemma 1.3.1, we see that the local ring $A_{\mathfrak{p}''}/\mathfrak{p}_1 A_{\mathfrak{p}''} =$

$(A_\mathfrak{p}/\mathfrak{p}_1A_\mathfrak{p})_{\mathfrak{p}''A_\mathfrak{p}/\mathfrak{p}_1A_\mathfrak{p}}$ is regular, which shows that the local ring $A_{\mathfrak{p}''}$ is also regular. It follows that $(A_\mathfrak{p})_{\mathfrak{p}'A_\mathfrak{p}} = (A_{\mathfrak{p}''})_{\mathfrak{p}'A_{\mathfrak{p}''}}$ is a regular local ring. Therefore we see that $A_\mathfrak{p}$ satisfies (R'_n). Let $\mathfrak{q} \in \operatorname{Spec} A$ such that $\mathfrak{q} \subseteq \mathfrak{p}$. If $\mathfrak{q} \in R_n(A)$, then (R_n) holds for $A_\mathfrak{q}$, and hence (R'_n) holds for $A_\mathfrak{q}$. If $\mathfrak{q} \in R_n(A)^c$, then we see that (R'_n) holds for $A_\mathfrak{q}$ by discussing in the same way as above. Thus, it follows from Lemma 1.3.1 that (R_n) holds for $A_\mathfrak{p}$. However, since $\mathfrak{p} \in R_n(A)^c$, we have contradiction. Thus we have shown that $\operatorname{ht}\mathfrak{p}_1 \geq n$, and hence $\operatorname{ht} I \geq n$.

Therefore we can arrange the order of $\mathfrak{p}_1, \mathfrak{p}_2, \cdots, \mathfrak{p}_t$ to satisfy the following conditions.

$$\operatorname{ht}\mathfrak{p}_i \begin{cases} = n \ (1 \leq i \leq s), \\ > n \ (s < i \leq t), \end{cases}$$

$$A_{\mathfrak{p}_i} \text{ is } \begin{cases} \text{non-regular } (1 \leq i \leq r), \\ \quad \text{regular } (r < i \leq s). \end{cases}$$

Put $J = \mathfrak{p}_1 \cap \mathfrak{p}_2 \cap \cdots \cap \mathfrak{p}_r$. Let $\mathfrak{p} \in R_n(A)^c$. Then there exists a prime ideal $\mathfrak{p}'$ of A such that $\mathfrak{p}' \subseteq \mathfrak{p}$, $\operatorname{ht}\mathfrak{p}' \leq n$, and that $A_{\mathfrak{p}'}$ is non-regular. Since (R_{n-1}) holds for A, we get $\operatorname{ht}\mathfrak{p}' = n$. Replacing $\mathfrak{p}$ by $\mathfrak{p}'$, we may assume that $\operatorname{ht}\mathfrak{p} = n$. Since $R_n(A)^c \subseteq V(I)$, we have $I \subseteq \mathfrak{p}$, and hence $\mathfrak{p}_k \subseteq \mathfrak{p}$ for some k. Since $\operatorname{ht}\mathfrak{p} = n$, we have $\mathfrak{p}_k = \mathfrak{p}$ and k is not bigger than s, and since $A_\mathfrak{p}$ is non-regular, k is not bigger than r. It follows that $J \subseteq \mathfrak{p}_k = \mathfrak{p}$, i.e., $\mathfrak{p} \in V(J)$. Therefore, we have $R_n(A)^c \subseteq V(J)$. Since the opposite inclusion is obvious by the choice of $\mathfrak{p}_i$, we have $R_n(A)^c = V(J)$. Thus, we get $R_n(A) = D(J)$, which shows that $R_n(A)$ is open in $\operatorname{Spec} A$. $\square$

2 Upper Cohen-Macaulay dimension and upper complete intersection dimension

The contents of this chapter are entirely contained in the author's paper [1] with T. Araya and Y. Yoshino, and in the author's own paper [48].

In this chapter, we define two homological invariants for finitely generated modules over a commutative noetherian local ring, which we call upper Cohen-Macaulay dimension and upper complete intersection dimension. These dimensions are quite similar to Cohen-Macaulay dimension introduced by Gerko [26] and complete intersection dimension introduced by Avramov, Gasharov, and Peeva [13], respectively. Also we define relative versions of upper Cohen-Macaulay dimension and upper complete intersection dimension. Relative upper Cohen-Macaulay dimension links upper Cohen-Macaulay dimension with Gorenstein dimension, and relative upper complete intersection dimension links upper complete intersection dimension with projective dimension.

2.1 Introduction

Throughout this chapter, all rings are assumed to be commutative noetherian rings, and all modules are assumed to be finitely generated modules.

Let R be a local ring with residue class field k. Projective dimension pd_R is one of the most classical homological dimensions. Gorenstein dimension (abbr. G-dimension) $\mathrm{G}\text{-}\mathrm{dim}_R$ was defined by Auslander [3], and was developed by him and Bridger [4]. These two dimensions have played important roles in the classification of modules and rings. Recently, complete intersection dimension (abbr. CI-dimension) $\mathrm{CI}\text{-}\mathrm{dim}_R$ has been introduced by Avramov, Gasharov, and Peeva [13], and Cohen-Macaulay dimension (abbr. CM-dimension) $\mathrm{CM}\text{-}\mathrm{dim}_R$ has been introduced by Gerko [26]. The former is defined by using projective dimension and the idea of a quasi-deformation, and the latter is defined by using G-dimension and the idea of a G-quasideformation.

Each of these dimensions is a homological invariant for R-modules which characterizes a certain property of local rings and satisfies a certain equality. Let i_R be a numerical invariant for R-modules, i.e. $i_R(M) \in \mathbb{N} \cup \{\infty\}$ for an

R-module M, and let $\mathcal{P}$ be a property of local rings. The following conditions hold for the pairs $(\mathcal{P}, i_R) =$ (regular, pd_R), (complete intersection, CI-$\dim_R$), (Gorenstein, G-$\dim_R$), and (Cohen-Macaulay, CM-$\dim_R$).

(a) The following conditions are equivalent.

i) R satisfies $\mathcal{P}$.

ii) $i_R(M) < \infty$ for any R-module M.

iii) $i_R(k) < \infty$.

(b) Let M be a non-zero R-module with $i_R(M) < \infty$. Then

$$i_R(M) = \operatorname{depth} R - \operatorname{depth}_R M.$$

Moreover, among these dimensions, there are inequalities as follows:

(c) For an R-module M, the following inequalities hold.

$$\text{CM-}\dim_R M \leq \text{G-}\dim_R M \leq \text{CI-}\dim_R M \leq \mathrm{pd}_R M.$$

The above inequalities yield the well-known implications for the local ring R:

$$\begin{aligned} \text{R is a regular ring} &\Rightarrow \text{R is a complete intersection ring} \\ &\Rightarrow \text{R is a Gorenstein ring} \\ &\Rightarrow \text{R is a Cohen-Macaulay ring} \end{aligned}$$

In this chapter, modifying the definition of CM-dimension and CI-dimension, we will define new homological invariants for R-modules which we will call upper Cohen-Macaulay dimension (abbr. CM*-dimension) and upper complete intersection dimension (abbr. CI*-dimension). We will denote these by CM*-$\dim_R$ and CI*-$\dim_R$, respectively. CM*-dimension (resp. CI*-dimension) interpolates between CM-dimension and G-dimension (resp. between CI-dimension and projective dimension). Specifically, for an R-module M, we have inequalities

$$\begin{cases} \text{CM-}\dim_R M \leq \text{CM}^*\text{-}\dim_R M \leq \text{G-}\dim_R M, \\ \text{CI-}\dim_R M \leq \text{CI}^*\text{-}\dim_R M \leq \mathrm{pd}_R M. \end{cases}$$

The equalities hold to the left of any finite dimension.

CM*-dimension (resp. CI*-dimension) is quite similar to CM-dimension (resp. CI-dimension): it has many properties analogous to those of CM-dimension (resp. CI-dimension). For example, the above two conditions (a), (b) also hold for the pairs ($\mathcal{P}$, i_R)=(Cohen-Macaulay, CM*-dim$_R$) and (complete intersection, CI*-dim$_R$).

Let $\phi : S \to R$ be a local homomorphism of local rings. The main purpose of this chapter is to provide new homological invariants for R-modules with respect to the homomorphism ϕ, which we will call *upper Cohen-Macaulay dimension relative to* ϕ and *upper complete intersection dimension relative to* ϕ, and will denote by CM*-dim$_\phi$ and CI*-dim$_\phi$, respectively. We will define them by using the idea of a G-factorization and a P-factorization.

In Section 2, we will make a list of properties of CM*-dimension. In our sense, it will be *absolute* CM*-dimension.

In Section 3, we will make the precise definition of *relative* CM*-dimension CM*-dim$_\phi$, and will study the properties of this dimension. Specifically, we shall prove the following:

(A) The following conditions are equivalent.

  i) R is a Cohen-Macaulay ring and S is a Gorenstein ring.

  ii) CM*-dim$_\phi M < \infty$ for any R-module M.

  iii) CM*-dim$_\phi k < \infty$.

(B) Let M be a non-zero R-module with CM*-dim$_\phi M < \infty$. Then

$$\text{CM}^*\text{-dim}_\phi M = \operatorname{depth} R - \operatorname{depth}_R M.$$

(C) Let M be an R-module.

  i) Suppose that ϕ is faithfully flat. Then

$$\text{CM}^*\text{-dim}_R M \leq \text{CM}^*\text{-dim}_\phi M \leq \text{G-dim}_R M.$$

  The equalities hold to the left of any finite dimension.

  ii) If S is the prime field of R and ϕ is the natural embedding, then

$$\text{CM}^*\text{-dim}_\phi M = \text{CM}^*\text{-dim}_R M.$$

  iii) If S is equal to R and ϕ is the identity map, then

$$\text{CM}^*\text{-dim}_\phi M = \text{G-dim}_R M.$$

The results (A), (B) are analogues of the conditions (a), (b). The result (C) says that relative CM*-dimension connects absolute CM*-dimension with G-dimension; relative CM*-dimension coincides with absolute CM*-dimension (resp. G-dimension) as a numerical invariant for R-modules if S is the "smallest" (resp. "largest") subring of R.

In Section 4 and 5, we will make the precise definitions of absolute CI*-dimension and relative CI*-dimension, and observe that they have a lot of properties analogous to absolute CM*-dimension and relative CM*-dimension. We shall prove the following:

(A') The following conditions are equivalent.

i) R is a complete intersection ring and S is a regular ring.

ii) $\text{CI}^*\text{-dim}_\phi M < \infty$ for any R-module M.

iii) $\text{CI}^*\text{-dim}_\phi k < \infty$.

(B') Let M be a non-zero R-module with $\text{CI}^*\text{-dim}_\phi M < \infty$. Then

$$\text{CI}^*\text{-dim}_\phi M = \operatorname{depth} R - \operatorname{depth}_R M.$$

(C') Let M be an R-module.

i) Suppose that ϕ is faithfully flat. Then

$$\text{CI}^*\text{-dim}_R M \leq \text{CI}^*\text{-dim}_\phi M \leq \operatorname{pd}_R M.$$

The equalities hold to the left of any finite dimension.

ii) Suppose that k is of characteristic zero. If S is the prime field of R and ϕ is the natural embedding, then

$$\text{CI}^*\text{-dim}_\phi M = \text{CI}^*\text{-dim}_R M.$$

iii) Suppose that $S = R$ and ϕ is the identity map. Then

$$\text{CI}^*\text{-dim}_\phi M = \operatorname{pd}_R M.$$

2.2 Upper Cohen-Macaulay dimension

Throughout this section, $(R, \mathfrak{m}, k)$ is always a local ring. We begin with recalling the definition of Gorenstein dimension (abbr. G-dimension). Denote by $\Omega_R^n M$ the nth syzygy module of an R-module M.

Definition 2.2.1 Let M be an R-module.

(1) If the following conditions hold, then we say that M has *G-dimension zero*, and write $\text{G-dim}_R M = 0$.

i) The natural homomorphism $M \to \mathrm{Hom}_R(\mathrm{Hom}_R(M,R),R)$ is an isomorphism.

ii) $\mathrm{Ext}^i_R(M,R) = 0$ for every $i > 0$.

iii) $\mathrm{Ext}^i_R(\mathrm{Hom}_R(M,R),R) = 0$ for every $i > 0$.

(2) If $\Omega^n_R M$ has G-dimension zero for a non-negative integer n, then we say that M has *G-dimension at most* n, and write $\text{G-dim}_R M \leq n$. If such an integer n does not exist, then we say that M has *infinite G-dimension*, and write $\text{G-dim}_R M = \infty$.

(3) If M has G-dimension at most n but does not have G-dimension at most $n-1$, then we say that M has *G-dimension* n, and write $\text{G-dim}_R M = n$.

For the properties of G-dimension, we refer to [4], [19], [36], and [57].

Now we recall the definition of Cohen-Macaulay dimension (abbr. CM-dimension) which has been introduced by Gerko.

Definition 2.2.2 [26, Definition 3.1, 3.2]

(1) An R-module M is called *G-perfect* if $\text{G-dim}_R M = \mathrm{grade}_R M$.

(2) A local homomorphism $\phi : S \to R$ of local rings is called a *G-deformation* if ϕ is surjective and R is G-perfect as an S-module.

(3) A diagram $S \overset{\phi}{\to} R' \overset{\alpha}{\leftarrow} R$ of local homomorphisms of local rings is called a *G-quasideformation* of R if α is faithfully flat and ϕ is a G-deformation.

(4) For an R-module M, the *Cohen-Macaulay dimension* of M is defined as follows:

$$\text{CM-dim}_R M = \inf \left\{ \begin{array}{r|l} \text{G-dim}_S(M \otimes_R R') & S \to R' \leftarrow R \text{ is a} \\ -\text{G-dim}_S R' & \text{G-quasideformation of } R \end{array} \right\}.$$

Modifying the above definition, we make the following definition.

Definition 2.2.3 (1) We call a diagram $S \overset{\phi}{\to} R' \overset{\alpha}{\leftarrow} R$ of local homomorphisms of local rings an *upper G-quasideformation* of R if it is a G-quasideformation and the closed fiber of α is regular.

(2) For an R-module M, we define the *upper Cohen-Macaulay dimension* (abbr. CM*-dimension) of M as follows:

$$\text{CM}^*\text{-dim}_R M = \inf \left\{ \begin{array}{r|l} \text{G-dim}_S(M \otimes_R R') & S \to R' \leftarrow R \text{ is an upper} \\ -\text{G-dim}_S R' & \text{G-quasideformation of } R \end{array} \right\}.$$

Comparing the definition of CM*-dimension with that of CM-dimension, one easily sees that

$$\text{CM-dim}_R M \leq \text{CM}^*\text{-dim}_R M$$

for any R-module M; the equality holds if CM*-dim$_R M < \infty$. CM*-dimension shares a lot of properties with CM-dimension. We shall exhibit a list of them in the rest of this section. We will omit the proofs of them because they can be proved quite similarly to the corresponding results of CM-dimension.

Theorem 2.2.4 *[26, Theorem 3.9] The following conditions are equivalent.*

i) *R is Cohen-Macaulay.*

ii) CM*-dim$_R M < \infty$ *for any R-module M.*

iii) CM*-dim$_R k < \infty$.

The CM*-dimension satisfies an equality analogous to the Auslander-Buchsbaum formula:

Theorem 2.2.5 *[26, Theorem 3.8] Let M be a non-zero R-module. If* CM*-dim$_R M < \infty$*, then*

$$\text{CM}^*\text{-dim}_R M = \operatorname{depth} R - \operatorname{depth}_R M.$$

Christensen defines a *semi-dualizing module* in his paper [20], which Gerko and Golod call a *suitable module* in [26] and [27]. Developing this concept a little, we make the following definition as a matter of convenience.

Definition 2.2.6 Let M and C be R-modules. We call C a *semi-dualizing module for M* if it satisfies the following conditions.

i) The natural homomorphism $R \to \text{Hom}_R(C, C)$ is an isomorphism.

ii) $\text{Ext}_R^i(C, C) = 0$ for any $i > 0$.

iii) The natural homomorphism $M \to \text{Hom}_R(\text{Hom}_R(M, C), C)$ is an isomorphism.

iv) $\mathrm{Ext}^i_R(M, C) = \mathrm{Ext}^i_R(\mathrm{Hom}_R(M, C), C) = 0$ for any $i > 0$.

It is worth noting that an R-module M has G-dimension zero if and only if R is a semi-dualizing module for M.

Refering to [22, Proposition 1.1], one can easily show that semi-dualizing modules enjoy the following properties.

Proposition 2.2.7 *Let C be a semi-dualizing R-module for some R-module. Then,*

(1) *C is faithful. In particular,* $\dim_R C = \dim R$.

(2) *A sequence $\boldsymbol{x} = x_1, x_2, \cdots, x_n$ in R is R-regular if and only if it is C-regular. In particular,* $\mathrm{depth}_R C = \mathrm{depth}\, R$.

It is possible to describe CM*-dimension in terms of a semi-dualizing module:

Theorem 2.2.8 *[26, Theorem 3.7] The following conditions are equivalent for an R-module M and a non-negative integer n.*

i) $\mathrm{CM}^*\text{-}\mathrm{dim}_R M \leq n$.

ii) *There exist a faithfully flat homomorphism $R \to R'$ of local rings whose closed fiber is regular, and an R'-module C such that C is a semi-dualizing module for $\Omega^n_R M \otimes_R R'$ as an R'-module.*

In particular, $\mathrm{CM}^*\text{-}\mathrm{dim}_R M \geq 0$ *for any R-module M.*

Corollary 2.2.9 *For an R-module M, we have*

$$\mathrm{CM}^*\text{-}\mathrm{dim}_R M \leq \mathrm{G}\text{-}\mathrm{dim}_R M.$$

The equality holds if $\mathrm{G}\text{-}\mathrm{dim}_R M < \infty$.

We end off this section by making a remark on G-dimension for later use:

Theorem 2.2.10 *[57, Theorem 2.7] An R-module M has finite G-dimension if and only if the natural morphism $M \to \mathbf{R}\mathrm{Hom}_R(\mathbf{R}\mathrm{Hom}_R(M, R), R)$ is an isomorphism in the derived category of the category of R-modules.*

2.3 Relative upper Cohen-Macaulay dimension

In this section, we observe CM*-dimension from a relative point of view. Throughout the section, ϕ always denotes a local homomorphism from a local ring $(S, \mathfrak{n}, \ell)$ to a local ring $(R, \mathfrak{m}, k)$.

We consider a commutative diagram

$$\begin{array}{ccc} S' & \xrightarrow{\phi'} & R' \\ {\scriptstyle\beta}\uparrow & & \uparrow{\scriptstyle\alpha} \\ S & \xrightarrow[\phi]{} & R \end{array}$$

of local homomorphisms of local rings, which we call a *G-factorization* of ϕ if β is a faithfully flat homomorphism and $S' \overset{\phi'}{\to} R' \overset{\alpha}{\leftarrow} R$ is an upper G-quasideformation of R. Using the idea of a G-factorization, we make the following definition.

Definition 2.3.1 Let M be an R-module. We define the *upper Cohen-Macaulay dimension* of M *relative* to ϕ, which is denoted by CM*-dim$_\phi M$, as follows:

$$\text{CM}^*\text{-dim}_\phi M = \inf \left\{ \begin{array}{r|l} \text{G-dim}_{S'}(M \otimes_R R') & S \to S' \to R' \leftarrow R \\ -\text{G-dim}_{S'} R' & \text{is a G-factorization of } \phi \end{array} \right\}.$$

In the rest of this section, the dimensions CM*-dim$_R$ and CM*-dim$_\phi$ will be often called *absolute* CM*-dimension and *relative* CM*-dimension, respectively.

We use the convention that the infimum of the empty set is ∞. It is natural to ask whether ϕ always has a G-factorization. The following example says that this is not true in general.

Example 2.3.2 Suppose that $R = \ell$ is the residue class field of S, and ϕ is the natural surjection from S to ℓ. Furthermore, suppose that S is not Gorenstein. Then ϕ does not have a G-factorization. (Hence we have CM*-dim$_\phi M = \infty$ for any R-module M.)

Indeed, assume that ϕ has a G-factorization $S \overset{\beta}{\to} S' \overset{\phi'}{\to} R' \overset{\alpha}{\leftarrow} R$. Then, since the closed fiber of α is regular, R' is a regular local ring. Let $\boldsymbol{x} = x_1, x_2, \cdots, x_n$ be a regular system of parameters of R'. Since G-dim$_{S'} R' =$ grade$_{S'} R' < \infty$ and $\boldsymbol{x}$ is an R'-regular sequence, we see that G-dim$_{S'} R'/(\boldsymbol{x}) < \infty$. Note that $R'/(\boldsymbol{x})$ is isomorphic to the residue class field of S'. Therefore S' is a Gorenstein local ring, and hence so is S because β is faithfully flat. This contradicts our assumption.

From the above example, we see that ϕ does not necessarily have a G-factorization in a general setting. However it seems that ϕ has a G-factorization at least when S is Gorenstein. We can prove it if we furthermore assume that S contains a field. To do this, we prepare a couple of lemmas.

Lemma 2.3.3 *Let $\phi : S \to R$ be a local homomorphism of complete local rings which have the same coefficient field k. Put $S' = S\widehat{\otimes}_k R$, and define $\lambda : S \to S'$ by $\lambda(b) = b\widehat{\otimes}1$, $\varepsilon : S' \to R$ by $\varepsilon(b\widehat{\otimes}a) = \phi(b)a$. Suppose that S is Gorenstein. Then $S \xrightarrow{\lambda} S' \xrightarrow{\varepsilon} R \xleftarrow{\mathrm{id}} R$ is a G-factorization of ϕ.*

Proof Take a minimal system of generators $y_1, y_2, \cdots, y_s$ of the maximal ideal of S. Put $J = \mathrm{Ker}\ \varepsilon$ and $dy_i = y_i\widehat{\otimes}1 - 1\widehat{\otimes}\phi(y_i) \in S'$ for each $1 \le i \le s$.

Claim 1 $J = (dy_1, dy_2, \cdots, dy_s)S'$.

Proof Put $J_0 = (dy_1, dy_2, \cdots, dy_s)$. Take an element $z = b\widehat{\otimes}a$ in J, and let $b = \sum b_{i_1 i_2 \cdots i_s} y_1^{i_1} y_2^{i_2} \cdots y_s^{i_s}$ be a power series expansion in $y_1, y_2, \cdots, y_s$ with coefficients $b_{i_1 i_2 \cdots i_s} \in k$. Then we have

$$\begin{array}{rcl} b\widehat{\otimes}1 & = & \sum b_{i_1 i_2 \cdots i_s} (y_1\widehat{\otimes}1)^{i_1} (y_2\widehat{\otimes}1)^{i_2} \cdots (y_s\widehat{\otimes}1)^{i_s} \\ & \equiv & \sum b_{i_1 i_2 \cdots i_s} (1\widehat{\otimes}\phi(y_1))^{i_1} (1\widehat{\otimes}\phi(y_2))^{i_2} \cdots (1\widehat{\otimes}\phi(y_s))^{i_s} \\ & = & 1\widehat{\otimes}\phi(b) \end{array}$$

modulo J_0. It follows that $z \equiv 1\widehat{\otimes}\phi(b)a$ modulo J_0. Since $\phi(b)a = \varepsilon(b\widehat{\otimes}a) = 0$, we have $z \equiv 0$ modulo J_0. Hence $z \in J_0$, and we see that $J = J_0$. $\square$

Claim 2 *If S is regular, then the sequence $dy_1, dy_2, \cdots, dy_s$ is an S'-regular sequence.*

Proof Since S is regular, we may assume that $S = k[[Y_1, Y_2, \cdots, Y_s]]$ and $S' = R[[Y_1, Y_2, \cdots, Y_s]]$ are formal power series rings, and $dy_i = Y_i - \phi(Y_i)$ for $1 \le i \le s$. Note that there is an automorphism on S' which sends Y_i to dy_i. Since the sequence $Y_1, Y_2, \cdots, Y_s$ is S'-regular, we see that $dy_1, dy_2, \cdots, dy_s$ also form a regular sequence on S'. $\square$

Now, let $T = k[[Y_1, Y_2, \cdots, Y_s]]$ be a formal power series ring and consider S to be a T-algebra in the natural way. Put $T' = T\widehat{\otimes}_k R$. Since the rings S, T are Gorenstein, we have $\mathbf{R}\mathrm{Hom}_T(S, T) \cong S[-e]$, where $e = \dim T - \dim S$. Note that T' is faithfully flat over T. Hence $\mathbf{R}\mathrm{Hom}_{T'}(S', T') \cong S'[-e]$. On the other hand, since T is regular, it follows from the claims that the sequence

$Y_1 - \phi(y_1), Y_2 - \phi(y_2), \cdots, Y_s - \phi(y_s)$ in T' is a T'-regular sequence. Hence we see that $\mathbf{R}\mathrm{Hom}_{T'}(R, T') \cong R[-s]$. Therefore we have

$$\begin{aligned} \mathbf{R}\mathrm{Hom}_{S'}(R, S') &\cong \mathbf{R}\mathrm{Hom}_{S'}(R, \mathbf{R}\mathrm{Hom}_{T'}(S', T')[e]) \\ &\cong \mathbf{R}\mathrm{Hom}_{T'}(R, T')[e] \\ &\cong R[e-s]. \end{aligned}$$

Thus it follows that $\text{G-dim}_{S'} R = \mathrm{grade}_{S'} R = s - e < \infty$. $\square$

To show the existence of G-factorizations, we need the following type of factorizations, which are called *Cohen factorizations*.

Lemma 2.3.4 *[12, Theorem 1.1] Let $\phi : (S, \mathfrak{n}) \to (R, \mathfrak{m})$ be a local homomorphism of local rings, and $\alpha : R \to \widehat{R}$ be the natural embedding into the $\mathfrak{m}$-adic completion. Then there exists a commutative diagram*

$$\begin{tikzcd} S' \arrow[r, "\phi'"] & \widehat{R} \\ S \arrow[u, "\beta"] \arrow[r, "\phi"'] & R \arrow[u, "\alpha"'] \end{tikzcd}$$

such that S' is a local ring, β is a faithfully flat homomorphism with regular closed fiber, and ϕ' is a surjective homomorphism.

Now we can prove the following theorem.

Theorem 2.3.5 *Let S be a Gorenstein local ring containing a field. Then any local homomorphism $\phi : S \to R$ of local rings has a G-factorization.*

Proof Replacing R and S with their completions respectively, we may assume that R and S are complete. By Lemma 2.3.4, ϕ has a Cohen factorization

$$\begin{tikzcd} & S' \arrow[dr, "\phi'"] & \\ S \arrow[ur, "\beta"] \arrow[rr, "\phi"'] & & R, \end{tikzcd}$$

where β is a faithfully flat homomorphism with regular closed fiber, and ϕ' is surjective. Hence S' is also Gorenstein. Thus, replacing S with S', we may assume that ϕ is surjective. In particular, the local rings R and S have the same coefficient field. Then it follows from Lemma 2.3.3 that ϕ has a G-factorization. $\square$

Conjecture 2.3.6 If S is an arbitrary Gorenstein local ring which may not contain a field, then every local homomorphism $\phi : S \to R$ has a G-factorization.

In the following theorem, we compare relative CM*-dimension with absolute CM*-dimension.

Theorem 2.3.7 *Let $\phi : (S, \mathfrak{n}) \to (R, \mathfrak{m})$ be a local homomorphism as before.*

(1) *For any R-module M, we have*

$$\text{CM}^*\text{-dim}_{\phi} M \geq \text{CM}^*\text{-dim}_R M.$$

In particular, $\text{CM}^*\text{-dim}_{\phi} M \geq 0$.

(2) *If S is regular and ϕ is faithfully flat, then*

$$\text{CM}^*\text{-dim}_{\phi} M = \text{CM}^*\text{-dim}_R M$$

for any R-module M.

Proof (1) If $S \xrightarrow{\beta} S' \xrightarrow{\phi'} R' \xleftarrow{\alpha} R$ is a G-factorization of ϕ, then $S' \xrightarrow{\phi'} R' \xleftarrow{\alpha} R$ is an upper G-quasideformation of R. Hence, comparing Definition 2.3.1 with Definition 2.2.3(2), we have the required inequality.

(2) It is enough to show that if $\text{CM}^*\text{-dim}_R M = n < \infty$ then $\text{CM}^*\text{-dim}_{\phi} M \leq n$. Theorem 2.2.8 says that there exist a faithfully flat homomorphism $\alpha : R \to R'$ of local rings with regular closed fiber, and a semi-dualizing R'-module C for $N := \Omega^n_{R'}(M \otimes_R R')$. Let $S' = R' \ltimes C$ be the trivial extension of R' by C. Let $\beta : S \to S'$ be the composite map of ϕ, α, and the natural inclusion $R' \to S'$, and let $\phi' : S' \to R'$ be the natural surjection.

Claim 1 *β is faithfully flat.*

Proof Let $\boldsymbol{y} = y_1, y_2, \cdots, y_n$ be a regular system of parameters of S. Since ϕ and α are faithfully flat, $\boldsymbol{y}$ is an R'-regular sequence, and hence is a C-regular sequence by Proposition 2.2.7(2). Note that the Koszul complex $K_\bullet(\boldsymbol{y}, S)$ is an S-free resolution of $S/(\boldsymbol{y}) = S/\mathfrak{n}$. Since $K_\bullet(\boldsymbol{y}, C) \cong K_\bullet(\boldsymbol{y}, S) \otimes_S C$ and $\boldsymbol{y}$ is a C-regular sequence, we have $\text{Tor}_1^S(S/\mathfrak{n}, C) \cong H_1(\boldsymbol{y}, C) = 0$. It follows from the local criteria of flatness that C is flat over S. Since R' is also flat over S, so is S'. Therefore β is a flat local homomorphism, and hence is faithfully flat. □

Claim 2 $\text{G-dim}_{S'} R' = 0$ *and* $\text{G-dim}_{S'}(M \otimes_R R') = n$.

Proof Note that $\mathbf{R}\mathrm{Hom}_{R'}(S', C) \cong S'$. Hence we have $\mathbf{R}\mathrm{Hom}_{S'}(R', S') \cong C$. Therefore we see that

$$\begin{aligned}\mathbf{R}\mathrm{Hom}_{S'}(\mathbf{R}\mathrm{Hom}_{S'}(R', S'), S') &\cong \mathbf{R}\mathrm{Hom}_{S'}(C, \mathbf{R}\mathrm{Hom}_{R'}(S', C)) \\ &\cong \mathbf{R}\mathrm{Hom}_{R'}(C, C) \\ &\cong R'\end{aligned}$$

because C is a semi-dualizing R'-module. It follows from Theorem 2.2.10 that $\text{G-dim}_{S'} R' < \infty$. Thus, we have $\text{G-dim}_{S'} R' = \operatorname{depth} S' - \operatorname{depth} R' = 0$. On the other hand, since C is a semi-dualizing module for N as an R'-module, it is easy to see that $\mathbf{R}\mathrm{Hom}_{R'}(N, C) \cong \mathrm{Hom}_{R'}(N, C)$ and

$$\begin{aligned}\mathbf{R}\mathrm{Hom}_{S'}(\mathbf{R}\mathrm{Hom}_{S'}(N, S'), S') &\cong \mathbf{R}\mathrm{Hom}_{R'}(\mathbf{R}\mathrm{Hom}_{R'}(N, C), C) \\ &\cong \mathbf{R}\mathrm{Hom}_{R'}(\mathrm{Hom}_{R'}(N, C), C) \\ &\cong \mathrm{Hom}_{R'}(\mathrm{Hom}_{R'}(N, C), C) \\ &\cong N.\end{aligned}$$

Applying Theorem 2.2.10 again, we see that $\text{G-dim}_{S'} N < \infty$. In the above we have shown that $\text{G-dim}_{S'} R' < \infty$. Hence $\text{G-dim}_{S'} F < \infty$ for any free R'-module F. Therefore we have $\text{G-dim}_{S'}(M \otimes_R R') < \infty$. Thus, we see that

$$\begin{aligned}\text{G-dim}_{S'}(M \otimes_R R') &= \operatorname{depth} S' - \operatorname{depth}(M \otimes_R R') \\ &= \operatorname{depth} R - \operatorname{depth} M \\ &= \text{CM}^*\text{-dim}_R M \\ &= n,\end{aligned}$$

as desired. $\square$

The above claims imply that $S \xrightarrow{\beta} S' \xrightarrow{\phi'} R' \xleftarrow{\alpha} R$ is a G-factorization of ϕ, and we have $\text{CM}^*\text{-dim}_\phi M \leq \text{G-dim}_{S'}(M \otimes_R R') - \text{G-dim}_{S'} R' = n$, which completes the proof of the theorem. $\square$

Let us consider the case that R contains a field K (e.g. K is the prime field of R). The second assertion of the above proposition especially says that if $S = K$ and $\phi : K \to R$ is the natural inclusion then $\text{CM}^*\text{-dim}_\phi M = \text{CM}^*\text{-dim}_R M$ for all R-module M. In other words, CM*-dimension relative to the map giving R the structure of a K-algebra, is absolute CM*-dimension. This leads us to the following conjecture.

Conjecture 2.3.8 If S is the prime local ring of R and ϕ is the natural inclusion, then relative CM*-dimension $\text{CM}^*\text{-dim}_\phi$ coincides with absolute CM*-dimension $\text{CM}^*\text{-dim}_R$.

Our next goal is to give some properties of relative CM*-dimension, which are similar to those of absolute CM*-dimension. First of all, relative CM*-dimension also satisfies an Auslander-Buchsbaum-type equality.

Theorem 2.3.9 *Let M be a non-zero R-module. If* $\mathrm{CM}^*\text{-}\dim_\phi M < \infty$*, then*

$$\mathrm{CM}^*\text{-}\dim_\phi M = \operatorname{depth} R - \operatorname{depth}_R M.$$

Hence we especially have $\mathrm{CM}^*\text{-}\dim_\phi M = \mathrm{CM}^*\text{-}\dim_R M$.

Proof Since $\mathrm{CM}^*\text{-}\dim_\phi M < \infty$, there exists a G-factorization $S \xrightarrow{\beta} S' \xrightarrow{\phi'} R' \xleftarrow{\alpha} R$ of ϕ such that $\mathrm{CM}^*\text{-}\dim_\phi M = \text{G-}\dim_{S'}(M \otimes_R R') - \text{G-}\dim_{S'} R' < \infty$. Hence we have

$$\begin{array}{rcl} \mathrm{CM}^*\text{-}\dim_\phi M & = & \text{G-}\dim_{S'}(M \otimes_R R') - \text{G-}\dim_{S'} R' \\ & = & (\operatorname{depth} S' - \operatorname{depth}_{S'}(M \otimes_R R')) \\ & - & (\operatorname{depth} S' - \operatorname{depth}_{S'} R') \\ & = & \operatorname{depth}_{S'} R' - \operatorname{depth}_{S'}(M \otimes_R R'). \end{array}$$

Since ϕ' is surjective and α, β are faithfully flat, we obtain two equalities

$$\begin{cases} \operatorname{depth}_{S'} R' = \operatorname{depth} R + \operatorname{depth} R'/\mathfrak{m}R', \\ \operatorname{depth}_{S'}(M \otimes_R R') = \operatorname{depth}_R M + \operatorname{depth} R'/\mathfrak{m}R'. \end{cases}$$

Therefore we see that $\mathrm{CM}^*\text{-}\dim_\phi M = \operatorname{depth} R - \operatorname{depth}_R M$ as desired. □

Corollary 2.3.10 *Suppose that S is a Gorenstein local ring containing a field. Then*

$$\mathrm{CM}^*\text{-}\dim_\phi F = 0$$

for any free R-module F.

Proof Theorem 2.3.5 says that ϕ has a G-factorization $S \xrightarrow{\beta} S' \xrightarrow{\phi'} R' \xleftarrow{\alpha} R$. Note that $\text{G-}\dim_{S'}(F \otimes_R R') = \text{G-}\dim_{S'} R' < \infty$. Hence we have $\mathrm{CM}^*\text{-}\dim_\phi F < \infty$. The assertion follows from the above theorem. □

Theorem 2.2.4 says that absolute CM^*-dimension $\mathrm{CM}^*\text{-}\dim_R$ characterizes the Cohen-Macaulayness of R. As an analogous result for relative CM^*-dimension, we have the following.

Theorem 2.3.11 *The following conditions are equivalent for a local homomorphism $\phi : (S, \mathfrak{n}, l) \to (R, \mathfrak{m}, k)$.*

i) *R is Cohen-Macaulay and S is Gorenstein.*

ii) $\mathrm{CM}^*\text{-}\dim_\phi M < \infty$ *for any R-module M.*

iii) $\mathrm{CM}^*\text{-}\dim_\phi k < \infty$.

Proof i) $\Rightarrow$ ii): By Lemma 2.3.4, there is a Cohen factorization $S \xrightarrow{\beta} S' \xrightarrow{\phi'} \widehat{R} \xleftarrow{\alpha} R$ of ϕ. Since the closed fiber of β is regular, the local ring S' is also Gorenstein. Hence we have $\mathbf{R}\mathrm{Hom}_{S'}(\widehat{R}, S') \cong K_{\widehat{R}}[-e]$, where $K_{\widehat{R}}$ is the canonical module of $\widehat{R}$ and $e = \dim S' - \dim \widehat{R}$. Note that $\text{G-dim}_{S'}\widehat{R} < \infty$ because S' is Gorenstein. Therefore we easily see that $\text{G-dim}_{S'}\widehat{R} = \mathrm{grade}_{S'}\widehat{R} = e$. Thus the Cohen factorization $S \xrightarrow{\beta} S' \xrightarrow{\phi'} \widehat{R} \xleftarrow{\alpha} R$ of ϕ is also a G-factorization of ϕ. The Gorensteinness of S' implies that $\text{G-dim}_{S'}(M \otimes_R \widehat{R}) < \infty$ for any R-module M. The assertion follows from this.

ii) $\Rightarrow$ iii): This is trivial.

iii) $\Rightarrow$ i): Theorem 2.3.7(1) implies that $\text{CM}^*\text{-dim}_R k < \infty$. Hence R is Cohen-Macaulay by virtue of Theorem 2.2.4. On the other hand, since $\text{CM}^*\text{-dim}_\phi k < \infty$, ϕ has a G-factorization $S \xrightarrow{\beta} S' \xrightarrow{\phi'} R' \xleftarrow{\alpha} R$ such that $\text{G-dim}_{S'}(k \otimes_R R') < \infty$. Note that the closed fiber $A := k \otimes_R R' \cong R'/\mathfrak{m}R'$ of α is regular. Let $\boldsymbol{x} = x_1, x_2, \cdots x_n$ be a regular system of parameters of A. Since $\text{G-dim}_{S'}A < \infty$ and $\boldsymbol{x}$ is an A-regular sequence, we have $\text{G-dim}_{S'}A/(\boldsymbol{x}) < \infty$. Hence S' is Gorenstein because $A/(\boldsymbol{x})$ is isomorphic to the residue class field of S'. It follows from the flatness of β that S is also Gorenstein. □

In the rest of this section, we consider the relationship between relative CM*-dimension and G-dimension. Let us consider the case that ϕ is faithfully flat. Then $S \xrightarrow{\phi} R \xrightarrow{\mathrm{id}} R \xleftarrow{\mathrm{id}} R$ is a G-factorization of ϕ. Hence, if the G-dimension of an R-module M is finite, then the CM*-dimension of M relative to ϕ is also finite. Since both relative CM*-dimension and G-dimension satisfy Auslander-Buchsbaum-type equalities, we have the following result that slightly generalizes Corollary 2.2.9.

Proposition 2.3.12 *Suppose that ϕ is faithfully flat. Then we have*

$$\text{CM}^*\text{-dim}_\phi M \leq \text{G-dim}_R M$$

for any R-module M. The equality holds if $\text{G-dim}_R M < \infty$.

Remark 2.3.13 Generally speaking, there is no inequality relation between relative CM*-dimension $\text{CM}^*\text{-dim}_\phi$ and G-dimension G-dim_R:

(1) If R is Gorenstein and S is not Gorenstein, then we have $\text{CM}^*\text{-dim}_\phi k = \infty$ and $\text{G-dim}_R k < \infty$. Hence $\text{CM}^*\text{-dim}_\phi k > \text{G-dim}_R k$.

(2) If R is not Gorenstein but Cohen-Macaulay and S is Gorenstein, then we have $\text{CM}^*\text{-dim}_\phi k < \infty$ and $\text{G-dim}_R k = \infty$. Hence $\text{CM}^*\text{-dim}_\phi k < \text{G-dim}_R k$.

(Both follow immediately from Theorem 2.3.11.)

As we have remarked after Theorem 2.3.7, relative CM*-dimension CM*-dim$_\phi$ coincides with absolute CM*-dimension CM*-dim$_R$ if S is the prime field of R (or maybe the prime local ring of R), in other words, S is the "smallest" local subring of R. In contrast with this, if S is the "largest" local subring of R, i.e. $S = R$, then relative CM*-dimension CM*-dim$_\phi$ coincides with G-dimension G-dim$_R$.

Theorem 2.3.14 *If $S = R$ and ϕ is the identity map of R, then*

$$\text{CM}^*\text{-dim}_\phi M = \text{G-dim}_R M$$

for any R-module M.

Proof By Proposition 2.3.12, we have only to prove that if CM*-dim$_\phi M = m < \infty$ then G-dim$_R M = m$. There exists a G-factorization $R \xrightarrow{\beta} S' \xrightarrow{\phi'} R' \xleftarrow{\alpha} R$ of $\phi = \mathrm{id}_R$ such that G-dim$_{S'}(M \otimes_R R') -$ G-dim$_{S'} R' = m$.

Claim 1 $\mathbf{R}\mathrm{Hom}_{S' \otimes_R k}(R' \otimes_R k, S' \otimes_R k) \cong \mathbf{R}\mathrm{Hom}_{S'}(R', S') \otimes^{\mathbf{L}}_R k$

Proof Let $F_\bullet$ be an S'-free resolution of R'. Since R' and S' are faithfully flat over R, it is easy to see that $F_\bullet \otimes_R k$ is an $(S' \otimes_R k)$-free resolution of $R' \otimes_R k$. Note that $\mathrm{Hom}_{S'}(F_\bullet, S')$ is a complex of free S'-modules, and hence is a complex of flat R-modules. Therefore we have

$$\begin{aligned} \mathbf{R}\mathrm{Hom}_{S'}(R', S') \otimes^{\mathbf{L}}_R k &\cong \mathrm{Hom}_{S'}(F_\bullet, S') \otimes_R k \\ &\cong \mathrm{Hom}_{S' \otimes_R k}(F_\bullet \otimes_R k, S' \otimes_R k) \\ &\cong \mathbf{R}\mathrm{Hom}_{S' \otimes_R k}(R' \otimes_R k, S' \otimes_R k), \end{aligned}$$

as desired. $\square$

Claim 2 *$S' \otimes_R k$ is a Gorenstein local ring.*

Proof Putting $g =$ G-dim$_{S'} R' = \mathrm{grade}_{S'} R'$ and $N = \mathrm{Ext}^g_{S'}(R', S')$, we have $N \cong \mathbf{R}\mathrm{Hom}_{S'}(R', S')[g]$. Then it follows from Claim 1 that

$$\mathbf{R}\mathrm{Hom}_{S' \otimes_R k}(R' \otimes_R k, S' \otimes_R k) \cong (N \otimes^{\mathbf{L}}_R k)[-g]. \tag{2.1}$$

In particular, we have $\mathrm{Ext}^n_{S' \otimes_R k}(R' \otimes_R k, S' \otimes_R k) \cong \mathrm{Tor}^R_{g-n}(N, k) = 0$ for all $n > g$. Now taking a regular system of parameters $\boldsymbol{x} = x_1, x_2, \cdots, x_r$ of $A := R' \otimes_R k$, we have $\mathrm{Ext}^n_{S' \otimes_R k}(A/(\boldsymbol{x}), S' \otimes_R k) = 0$ for all $n > g + r$. Since $A/(\boldsymbol{x})$ is isomorphic to the residue class field of $S' \otimes_R k$, the self injective dimension of $S' \otimes_R k$ is not bigger than $g + r$. Therefore $S' \otimes_R k$ is a Gorenstein local ring. $\square$

Claim 3 $R' \cong \mathbf{R}\mathrm{Hom}_{S'}(R', S')[g]$.

PROOF Note that, since the ring $R' \otimes_R k$ is regular, the canonical module of $R' \otimes_R k$ is isomorphic to $R' \otimes_R k$. Thus, it follows from (2.1) and Claim 2 that $N \otimes^{\mathbf{L}}_R k \cong \mathbf{R}\mathrm{Hom}_{S' \otimes_R k}(R' \otimes_R k, S' \otimes_R k)[g] \cong R' \otimes_R k$, and hence $N \otimes_R k \cong R' \otimes_R k$. Therefore we have $N \otimes_{R'} k' \cong k'$, where k' is the residue class field of R'. In other words, $N \cong R'/I$ for some ideal I of R'. On the other hand, since $\text{G-dim}_{S'} R' < \infty$, we have

$$\begin{aligned} \mathbf{R}\mathrm{Hom}_{R'}(N, N) &\cong \mathbf{R}\mathrm{Hom}_{R'}(\mathbf{R}\mathrm{Hom}_{S'}(R', S')[g], \mathbf{R}\mathrm{Hom}_{S'}(R', S')[g]) \\ &\cong \mathbf{R}\mathrm{Hom}_{S'}(\mathbf{R}\mathrm{Hom}_{S'}(R', S'), S') \\ &\cong R'. \end{aligned}$$

In particular, N is a semi-dualizing R'-module for R'. Hence by Proposition 2.2.7(1), we see that $I = 0$, i.e. $R' \cong N \cong \mathbf{R}\mathrm{Hom}_{S'}(R', S')[g]$. $\square$

Now we can prove that $\text{G-dim}_R M = m$. Since R' is flat over R and $\text{G-dim}_{S'}(M \otimes_R R') < \infty$, we see that

$$\begin{aligned} &\mathbf{R}\mathrm{Hom}_R(\mathbf{R}\mathrm{Hom}_R(M, R), R) \otimes_R R' \\ \cong\ &\mathbf{R}\mathrm{Hom}_{R'}(\mathbf{R}\mathrm{Hom}_{R'}(M \otimes_R R', R'), R') \\ \cong\ &\mathbf{R}\mathrm{Hom}_{S'}(\mathbf{R}\mathrm{Hom}_{S'}(M \otimes_R R', S'), S') \\ \cong\ &M \otimes_R R' \end{aligned}$$

by Claim 3. It follows from the faithful flatness of $\alpha : R \to R'$ that $\mathbf{R}\mathrm{Hom}_R(\mathbf{R}\mathrm{Hom}_R(M, R), R) \cong M$, and hence $\text{G-dim}_R M < \infty$. Note that Claim 3 implies $\mathbf{R}\mathrm{Hom}_{R'}(M \otimes_R R', R') \cong \mathbf{R}\mathrm{Hom}_{S'}(M \otimes_R R', S')[g]$. Therefore we have

$$\begin{aligned} \text{G-dim}_R M &= \text{G-dim}_{R'}(M \otimes_R R') \\ &= \text{G-dim}_{S'}(M \otimes_R R') - g \\ &= m, \end{aligned}$$

as desired. $\square$

2.4 Upper complete intersection dimension

Throughout this section, we assume that all rings are commutative noetherian rings, and all modules are finitely generated.

First of all, we recall the definition of the CI-dimension of a module over a local ring R. It is similar to that of virtual projective dimension introduced by Avramov [7]:

(1) A local homomorphism $\phi : S \to R$ of local rings is called a *deformation* if ϕ is surjective and the kernel of ϕ is generated by an S-regular sequence.

(2) A diagram $S \xrightarrow{\phi} R' \xleftarrow{\alpha} R$ of local homomorphisms of local rings is called a *quasi-deformation* of R if α is faithfully flat and ϕ is a deformation.

(3) For an R-module M, the *complete intersection dimension* of M is defined as follows:

$$\text{CI-dim}_R M = \inf \left\{ \begin{array}{r|l} \mathrm{pd}_S(M \otimes_R R') & S \to R' \leftarrow R \text{ is a} \\ -\mathrm{pd}_S R' & \text{quasi-deformation of } R \end{array} \right\}.$$

Now, slightly modifying the definition of CI-dimension, we define a homological invariant for modules over a local ring as follows.

Definition 2.4.1 (1) We call a diagram $S \xrightarrow{\phi} R' \xleftarrow{\alpha} R$ of local homomorphisms of local rings an *upper quasi-deformation* of R if α is faithfully flat, the closed fiber of α is regular, and ϕ is a deformation.

(2) For an R-module M, we define the *upper complete intersection dimension* (abbr. CI*-dimension) of M as follows:

$$\text{CI}^*\text{-dim}_R M = \inf \left\{ \begin{array}{r|l} \mathrm{pd}_S(M \otimes_R R') & S \to R' \leftarrow R \text{ is an} \\ -\mathrm{pd}_S R' & \text{upper quasi-deformation of } R \end{array} \right\}.$$

Here we itemize several properties of CI*-dimension, which are analogous to those of CI-dimension. We omit their proofs because we can prove them in the same way as the proofs of the corresponding results of CI-dimension given in [13]. Let R be a local ring with residue field k. We denote by $\Omega_R^r M$ the rth syzygy module of an R-module M.

(1) The following conditions are equivalent.

  i) R is a complete intersection.

  ii) $\text{CI}^*\text{-dim}_R M < \infty$ for any R-module M.

  iii) $\text{CI}^*\text{-dim}_R k < \infty$.

(2) Let M be a non-zero R-module with $\text{CI}^*\text{-dim}_R M < \infty$. Then

$$\text{CI}^*\text{-dim}_R M = \operatorname{depth} R - \operatorname{depth}_R M.$$

(3) Let M be a non-zero R-module and r an integer. Then

$$\text{CI}^*\text{-dim}_R(\Omega_R^r M) = \sup\{\text{CI}^*\text{-dim}_R M - r, 0\}.$$

(4) Let $\boldsymbol{x} = x_1, x_2, \cdots, x_n$ be a sequence in R. Then the following hold for an R-module M.

i) $\text{CI}^*\text{-dim}_R(M/\boldsymbol{x}M) = \text{CI}^*\text{-dim}_R M + n$ if $\boldsymbol{x}$ is M-regular.

ii) $\text{CI}^*\text{-dim}_{R/(\boldsymbol{x})}(M/\boldsymbol{x}M) \leq \text{CI}^*\text{-dim}_R M$ if $\boldsymbol{x}$ is R-regular and M-regular.
The equality holds if $\text{CI}^*\text{-dim}_R M < \infty$.

iii) $\text{CI}^*\text{-dim}_{R/(\boldsymbol{x})} M \leq \text{CI}^*\text{-dim}_R M - n$ if $\boldsymbol{x}$ is R-regular and $\boldsymbol{x}M = 0$.
The equality holds if $\text{CI}^*\text{-dim}_R M < \infty$.

(5) Let M be an R-module. Then

$$\text{CI-dim}_R M \leq \text{CI}^*\text{-dim}_R M \leq \text{pd}_R M.$$

If any one of these dimensions is finite, then it is equal to those to its left.

2.5 Relative upper complete intersection dimension

Throughout the section, $\phi : (S, \mathfrak{n}, l) \to (R, \mathfrak{m}, k)$ always denotes a local homomorphism of local rings.

In this section, we shall make the precise definition of the upper complete intersection dimension of an R-module relative to ϕ to observe CI*-dimension from a relative point of view. To do this, we need the notion of a P-factorization, instead of that of an upper quasi-deformation used in the definition of (absolute) CI*-dimension.

Definition 2.5.1 Let

$$\begin{tikzcd} S' \arrow[r, "\phi'"] & R' \\ S \arrow[u, "\beta"] \arrow[r, "\phi"'] & R, \arrow[u, "\alpha"'] \end{tikzcd}$$

be a commutative diagram of local homomorphisms of local rings. We call this diagram a *P-factorization* of ϕ if α and β are faithfully flat homomorphisms, the closed fiber of α is a regular local rings, and ϕ' is a deformation.

Note that this is an imitation of a G-factorization defined in Section 3. The existence of a P-factorization of ϕ transmits several properties of R to S:

Proposition 2.5.2 *Suppose that there exists a P-factorization of ϕ. Then, if R is a regular (resp. complete intersection, Gorenstein, Cohen-Macaulay) ring, so is S.*

Proof Let $S \xrightarrow{\beta} S' \xrightarrow{\phi'} R' \xleftarrow{\alpha} R$ be a P-factorization of ϕ. Suppose that R is a regular (resp. complete intersection, Gorenstein, Cohen-Macaulay) ring. Since α is a faithfully flat homomorphism with regular closed fiber, R' is also a regular (resp. ...) ring. Since ϕ' is a deformation, we easily see that S' is also a regular (resp. ...) ring, and so is S by the flatness of the homomorphism β. □

From now on, we consider the existence of a P-factorization of ϕ. First of all, the above proposition yields the following example which says that ϕ may not have a P-factorization.

Example 2.5.3 Suppose that $R = l$ is the residue class field of S and ϕ is the natural surjection from S to l. Then ϕ has no P-factorization unless S is regular by Proposition 2.5.2.

Although there does not necessarily exist a P-factorization of ϕ in general, a P-factorization of ϕ seems to exist whenever the local ring S is regular. We are able to show it if in addition we assume the condition that S contains a field:

Theorem 2.5.4 *Suppose that S is a regular local ring containing a field. Then every local homomorphism $\phi : S \to R$ of local rings has a P-factorization.*

In order to prove this theorem, we need the following two lemmas:

Lemma 2.5.5 *[12, Theorem 1.1] Let $\phi : (S, \mathfrak{n}) \to (R, \mathfrak{m})$ be a local homomorphism of local rings, and α be the natural embedding from R into its $\mathfrak{m}$-adic completion $\widehat{R}$. Then there exists a commutative diagram*

$$\begin{array}{ccc} S' & \xrightarrow{\phi'} & \widehat{R} \\ {\scriptstyle\beta}\uparrow & & \uparrow{\scriptstyle\alpha} \\ S & \xrightarrow[\phi]{} & R \end{array}$$

of local homomorphisms of local rings such that β is faithfully flat, the closed fiber of β is regular, and ϕ' is surjective. (Such a diagram is called a Cohen factorization *of ϕ.)*

Lemma 2.5.6 *Let $\phi : S \to R$ be a local homomorphism of complete local rings that admit the common coefficient field k. Put $S' = S\widehat{\otimes}_k R$. Let $\lambda : S \to S'$ be the injective homomorphism mapping $b \in S$ to $b\widehat{\otimes}1 \in S'$, and $\varepsilon : S' \to R$ be the surjective homomorphism mapping $b\widehat{\otimes}a \in S'$ to $\phi(b)a \in R$. Suppose that the local ring S is regular. Then $S \xrightarrow{\lambda} S' \xrightarrow{\varepsilon} R \xleftarrow{\mathrm{id}} R$ is a P-factorization of ϕ.*

Proof Let $y_1, y_2, \cdots, y_s$ be a minimal system of generators of the unique maximal ideal of S. Put $J = \operatorname{Ker}\varepsilon$ and $dy_i = y_i\widehat{\otimes}1 - 1\widehat{\otimes}\phi(y_i) \in S'$ for each $i = 1, 2, \cdots, s$.

Claim 1 *The ideal J of S' is generated by $dy_1, dy_2, \cdots, dy_s$.*

Proof Put $J_0 = (dy_1, dy_2, \cdots, dy_s)S'$. Let $z = b\widehat{\otimes}a$ be an element in J, and let $b = \sum b_{i_1 i_2 \cdots i_s} y_1^{i_1} y_2^{i_2} \cdots y_s^{i_s}$ be a power series expansion in $y_1, y_2, \cdots, y_s$ with coefficients $b_{i_1 i_2 \cdots i_s} \in k$. Then we have

$$\begin{array}{rcl} b\widehat{\otimes}1 & = & \sum b_{i_1 i_2 \cdots i_s} (y_1\widehat{\otimes}1)^{i_1} (y_2\widehat{\otimes}1)^{i_2} \cdots (y_s\widehat{\otimes}1)^{i_s} \\ & \equiv & \sum b_{i_1 i_2 \cdots i_s} (1\widehat{\otimes}\phi(y_1))^{i_1} (1\widehat{\otimes}\phi(y_2))^{i_2} \cdots (1\widehat{\otimes}\phi(y_s))^{i_s} \\ & = & 1\widehat{\otimes}\phi(b) \end{array}$$

modulo J_0. It follows that $z \equiv 1\widehat{\otimes}\phi(b)a$ modulo J_0. Since $\phi(b)a = \varepsilon(b\widehat{\otimes}a) = 0$, we have $z \equiv 0$ modulo J_0, that is, the element $z \in J$ belongs to J_0. Thus, we see that $J = J_0$. □

Claim 2 *The sequence $dy_1, dy_2, \cdots, dy_s$ is an S'-regular sequence.*

Proof Since S is regular, we may assume that $S = k[[Y_1, Y_2, \cdots, Y_s]]$ and $S' = R[[Y_1, Y_2, \cdots, Y_s]]$ are formal power series rings, and $dy_i = Y_i - \phi(Y_i) \in S'$ for each $1 \leq i \leq s$. Note that the endomorphism on S' which sends Y_i to dy_i is an automorphism. Since the sequence $Y_1, Y_2, \cdots, Y_s$ is S'-regular, we see that $dy_1, dy_2, \cdots, dy_s$ also form an S'-regular sequence. □

These claims prove that the homomorphism ε is a deformation. On the other hand, it is easy to see that λ is faithfully flat. Thus, the lemma is proved. □

Proof of Theorem 2.5.4 We may assume that R (resp. S) is complete in its $\mathfrak{m}$-adic (resp. $\mathfrak{n}$-adic) topology. Hence Lemma 2.5.5 implies that ϕ has a Cohen factorization

$$\begin{tikzcd} & S' \arrow[dr, "\phi'"] & \\ S \arrow[ur, "\beta"] \arrow[rr, "\phi"'] & & R, \end{tikzcd}$$

where β is a faithfully flat homomorphism with regular closed fiber, and ϕ' is a surjective homomorphism. Hence S' is also a regular local ring containing a field. Therefore, replacing S with S', we may assume that ϕ is a surjection. In particular R and S have the common coefficient field, and hence Lemma 2.5.6 implies that ϕ has a P-factorization, as desired. $\square$

Conjecture 2.5.7 Whenever S is regular, the local homomorphism $\phi : S \to R$ would have a P-factorization.

Now, by using the idea of a P-factorization, we define the CI*-dimension of a module in a relative sense.

Definition 2.5.8 For an R-module M, we put

$$\text{CI}^*\text{-dim}_\phi M = \inf \left\{ \begin{array}{r|l} \text{pd}_{S'}(M \otimes_R R') & S \to S' \to R' \leftarrow R \\ -\text{pd}_{S'} R' & \text{is a P-factorization of } \phi \end{array} \right\}$$

and call it the *upper complete intersection dimension* of M *relative* to ϕ.

By definition, $\text{CI}^*\text{-dim}_\phi M = \infty$ for an R-module M if ϕ has no P-factorization. Supppose that ϕ has at least one P-factorization $S \to S' \to R' \leftarrow R$. Then we have $\text{pd}_{S'}(F \otimes_R R') = \text{pd}_{S'} R'$ $(< \infty)$ for any free R-module F. Therefore the above theorem on the existence of a P-factorization yields the following result:

Proposition 2.5.9 *If S is a regular local ring that contains a field, then*

$$\text{CI}^*\text{-dim}_\phi F = 0 \ (< \infty)$$

for any free R-module F.

In the rest of this section, we observe the properties of relative CI*-dimension CI*-dim$_\phi$. We begin by proving that relative CI*-dimension also satisfies an Auslander-Buchsbaum-type equality:

Theorem 2.5.10 *Let M be a non-zero R-module. If $\text{CI}^*\text{-dim}_\phi M < \infty$, then*

$$\text{CI}^*\text{-dim}_\phi M = \operatorname{depth} R - \operatorname{depth}_R M.$$

Proof Since $\text{CI}^*\text{-dim}_\phi M < \infty$, there exists a P-factorization $S \xrightarrow{\beta} S' \xrightarrow{\phi'} R' \xleftarrow{\alpha} R$ of ϕ such that $\text{CI}^*\text{-dim}_\phi M = \text{pd}_{S'}(M \otimes_R R') - \text{pd}_{S'} R' < \infty$. Hence we see that

$$\begin{array}{rcl} \text{CI}^*\text{-dim}_\phi M & = & \text{pd}_{S'}(M \otimes_R R') - \text{pd}_{S'} R' \\ & = & (\operatorname{depth} S' - \operatorname{depth}_{S'}(M \otimes_R R')) \\ & - & (\operatorname{depth} S' - \operatorname{depth}_{S'} R') \\ & = & \operatorname{depth}_{S'} R' - \operatorname{depth}_{S'}(M \otimes_R R'). \end{array}$$

Note that ϕ' is surjective. Since α and β are faithfully flat, we obtain

$$\begin{cases} \operatorname{depth}_{S'} R' = \operatorname{depth} R + \operatorname{depth} R'/\mathfrak{m}R', \\ \operatorname{depth}_{S'}(M \otimes_R R') = \operatorname{depth}_R M + \operatorname{depth} R'/\mathfrak{m}R'. \end{cases}$$

It follows that $\text{CI}^*\text{-dim}_\phi M = \operatorname{depth} R - \operatorname{depth}_R M$. $\square$

In view of this theorem, we especially notice that the value of the relative CI*-dimension of an R-module is given independently of the ring S if it is finite.

Proposition 2.5.11 *Let M be an R-module. Then*

(1) $\text{CI}^*\text{-dim}_\phi M \geq \text{CI}^*\text{-dim}_R M$.
The equality holds if $\text{CI}^*\text{-dim}_\phi M < \infty$.

(2) $\text{CI}^*\text{-dim}_\phi M \leq \text{pd}_R M$ *if ϕ is faithfully flat.*
The equality holds if in addition $\text{pd}_R M < \infty$.

Proof (1) Since the inequality holds if $\text{CI}^*\text{-dim}_\phi M = \infty$, assume that $\text{CI}^*\text{-dim}_\phi M < \infty$. Let $S \xrightarrow{\beta} S' \xrightarrow{\phi'} R' \xleftarrow{\alpha} R$ be a P-factorization of ϕ such that $\text{pd}_{S'}(M \otimes_R R') - \text{pd}_{S'} R' < \infty$. Then by definition $S' \xrightarrow{\phi'} R' \xleftarrow{\alpha} R$ is a quasi-deformation of R, which shows that $\text{CI}^*\text{-dim}_R M < \infty$. Hence the assertion follows from Theorem 2.5.10 and the Auslander-Buchsbaum-type equality for CI*-dimension.

(2) Suppose that ϕ is faithfully flat. Since the inequality holds if $\text{pd}_R M = \infty$, assume that $\text{pd}_R M < \infty$. We easily see that the diagram $S \xrightarrow{\phi} R \xrightarrow{\text{id}} R \xleftarrow{\text{id}} R$ is a P-factorization of ϕ. Therefore we have $\text{CI}^*\text{-dim}_\phi M < \infty$. Hence the assertion follows from Theorem 2.5.10 and the Auslander-Buchsbaum formula for projective dimension. $\square$

The inequality in the second assertion of the above proposition may not hold without the faithful flatness of ϕ; see Remark 2.5.17 below.

Now, recall that

$$\text{CI}^*\text{-dim}_R M \leq \text{pd}_R M$$

for any R-module M. Hence the above proposition says that relative CI*-dimension is inserted between absolute CI*-dimension and projective dimension if ϕ is faithfully flat.

It is natural to ask when relative CI*-dimension $\text{CI}^*\text{-dim}_\phi$ coincides with absolute one $\text{CI}^*\text{-dim}_R$ as an invariant for R-modules. It seems to happen if S is the prime field of R.

Let us consider the case that the characteristic $\text{char}\, k$ of k is zero. Then we easily see that $\text{char}\, R = 0$. It follows that R has the prime field $\mathbb{Q}$. Let $S' \xrightarrow{\phi'} R' \xleftarrow{\alpha} R$ be a quasi-deformation of R. Since α is injective and ϕ' is surjective, the residue class field of R' is of characteristic zero, and so is that of S'. Hence we see that $\text{char} S' = 0$, and there exists a commutative diagram

$$\begin{tikzcd} S' \arrow[r, "\phi'"] & R' \\ \mathbb{Q} \arrow[u, "\beta"] \arrow[r, "\phi"'] & R, \arrow[u, "\alpha"'] \end{tikzcd}$$

where ϕ and β denote the natural embeddings. Note that β is faithfully flat because $\mathbb{Q}$ is a field. Therefore this diagram is a P-factorization of ϕ. Thus, Proposition 2.5.11(1) yields the following:

Proposition 2.5.12 *Suppose that k is of characteristic zero. If S is the prime field of R, then*

$$\text{CI}^*\text{-dim}_\phi M = \text{CI}^*\text{-dim}_R M$$

for any R-module M.

Conjecture 2.5.13 If S is the prime field of R, then it would always hold that $\text{CI}^*\text{-dim}_\phi M = \text{CI}^*\text{-dim}_R M$ for any R-module M.

As we have observed in Proposition 2.5.11, the relative CI*-dimension $\text{CI}^*\text{-dim}_\phi M$ of an R-module M is always smaller or equal to its projective dimension $\text{pd}_R M$, as long as ϕ is faithfully flat. The next theorem gives a sufficient condition for these dimensions to coincide with each other as invariants for R-modules.

Theorem 2.5.14 *Suppose that $S = R$ and ϕ is the identity map of R. Then*

$$\text{CI}^*\text{-dim}_\phi M = \text{pd}_R M$$

for every R-module M.

Proof The assumption in the theorem in particular implies that ϕ is faithfully flat. Hence Proposition 2.5.11(2) yields one inequality relation in the theorem. Thus we have only to prove the other inequality relation $\text{CI}^*\text{-dim}_\phi M \geq \text{pd}_R M$. There is nothing to show if $\text{CI}^*\text{-dim}_\phi M = \infty$. Hence assume that $\text{CI}^*\text{-dim}_\phi M < \infty$. Then the identity map ϕ of R has a P-factorization $R \xrightarrow{\beta} S' \xrightarrow{\phi'} R' \xleftarrow{\alpha} R$ such that $\text{pd}_{S'}(M \otimes_R R') < \infty$. Let l' denote the residue class field of S'. Taking an S'-sequence $\boldsymbol{x} = x_1, x_2, \cdots, x_r$ generating the kernel of ϕ', we have

$$\begin{aligned} \mathbf{R}\text{Hom}_{S'}(R', l') &\cong \text{Hom}_{S'}(K_\bullet(\boldsymbol{x}), l') \\ &\cong \textstyle\bigoplus_{i=0}^r l'^{\binom{r}{i}}[-i], \end{aligned}$$

where $K_\bullet(\boldsymbol{x})$ is the Koszul complex of $\boldsymbol{x}$ over S'. Noting that both α and β are faithfully flat, we see that

$$\begin{aligned} \mathbf{R}\text{Hom}_{S'}(M \otimes_R R', l') &\cong \mathbf{R}\text{Hom}_{S'}((M \otimes_R^{\mathbf{L}} S') \otimes_{S'}^{\mathbf{L}} R', l') \\ &\cong \mathbf{R}\text{Hom}_{S'}(M \otimes_R^{\mathbf{L}} S', \mathbf{R}\text{Hom}_{S'}(R', l')) \\ &\cong \mathbf{R}\text{Hom}_{S'}(M \otimes_R S', \textstyle\bigoplus_{i=0}^r l'^{\binom{r}{i}}[-i]) \\ &\cong \textstyle\bigoplus_{i=0}^r \mathbf{R}\text{Hom}_{S'}(M \otimes_R S', l')^{\binom{r}{i}}[-i]. \end{aligned}$$

It follows from this that

$$\begin{aligned} \text{Ext}_{S'}^j(M \otimes_R R', l') &\cong H^j(\mathbf{R}\text{Hom}_{S'}(M \otimes_R R', l')) \\ &\cong H^j(\textstyle\bigoplus_{i=0}^r \mathbf{R}\text{Hom}_{S'}(M \otimes_R S', l')^{\binom{r}{i}}[-i]) \\ &\cong \textstyle\bigoplus_{i=0}^r \text{Ext}_{S'}^{j-i}(M \otimes_R S', l')^{\binom{r}{i}}. \end{aligned}$$

Note that $\text{Ext}_{S'}^j(M \otimes_R R', l') = 0$ for any $j \gg 0$ because $\text{pd}_{S'}(M \otimes_R R') < \infty$. Hence we obtain $\text{Ext}_{S'}^j(M \otimes_R S', l') = 0$ for any $j \gg 0$, which implies that $\text{pd}_{S'}(M \otimes_R S') < \infty$. Thus we get $\text{pd}_R M < \infty$. Then the Auslander-Buchsbaum-type equalities for projective dimension and CI*-dimension yield that $\text{CI}^*\text{-dim}_\phi M = \text{pd}_R M = \text{depth}\, R - \text{depth}_R M$. $\square$

We know that $\text{CI}^*\text{-dim}_R M < \infty$ for any R-module M if R is a complete intersection and that R is a complete intersection if $\text{CI}^*\text{-dim}_R k < \infty$. We can prove the following result similar to this:

Theorem 2.5.15 *The following conditions are equivalent.*

i) *R is a complete intersection and S is a regular ring.*

ii) $\text{CI}^*\text{-dim}_\phi M < \infty$ *for any R-module M.*

iii) $\text{CI}^*\text{-dim}_\phi k < \infty$.

Proof i) $\Rightarrow$ ii): It follows from Lemma 2.5.5 that there is a Cohen factorization $S \xrightarrow{\beta} S' \xrightarrow{\phi'} \widehat{R} \xleftarrow{\alpha} R$ of ϕ. Since both the ring S and the closed fiber of β are regular, so is S' by the faithful flatness of β. On the other hand, since R is a complete intersection, so is its $\mathfrak{m}$-adic completion $\widehat{R}$. Hence the homomorphism ϕ' is a deformation. (A surjective homomorphism from a regular local ring to a local complete intersection must be a deformation; see [18, Theorem 2.3.3].) Thus, we see that the factorization $S \xrightarrow{\beta} S' \xrightarrow{\phi'} \widehat{R} \xleftarrow{\alpha} R$ is a P-factorization of ϕ. The regularity of the ring S' implies that every S'-module is of finite projective dimension over S', from which the condition ii) follows.

ii) $\Rightarrow$ iii): This is trivial.

iii) $\Rightarrow$ i): The condition iii) says that ϕ has a P-factorization $S \xrightarrow{\beta} S' \xrightarrow{\phi'} R' \xleftarrow{\alpha} R$ such that $\mathrm{pd}_{S'}(k \otimes_R R') < \infty$. Put $A = k \otimes_R R'$. Note that A is a regular local ring because it is the closed fiber of α. Let $\boldsymbol{a} = a_1, a_2, \cdots, a_t$ be a regular system of parameters of A. Since $\boldsymbol{a}$ is an A-regular sequence, we have $\mathrm{pd}_{S'} A/(\boldsymbol{a}) = \mathrm{pd}_{S'} A + t < \infty$. Since ϕ' is surjective, we see that the quotient ring $A/(\boldsymbol{a})$ is isomorphic to the residue class field l' of S'. Hence we obtain $\mathrm{pd}_{S'} l' < \infty$, which implies that S' is regular, and so is S. On the other hand, it follows from Theorem 2.5.11(1) that R is a complete intersection. $\square$

Suppose that R is regular. Then, by Proposition 2.5.2, S is also regular if ϕ has at least one P-factorization. Thus the above theorem implies the following corollary:

Corollary 2.5.16 *Suppose that R is regular. If $\mathrm{CI}^*\text{-}\dim_\phi N < \infty$ for some R-module N, then $\mathrm{CI}^*\text{-}\dim_\phi M < \infty$ for every R-module M.*

Remark 2.5.17 Relating to the second assertion of Proposition 2.5.11, there is no inequality relation between relative CI*-dimension and projective dimension in a general setting. In fact, the following results immediately follow from Theorem 2.5.15:

(1) $\mathrm{CI}^*\text{-}\dim_\phi k < \mathrm{pd}_R k$ if R is a complete intersection which is not regular and S is a regular ring.

(2) $\mathrm{CI}^*\text{-}\dim_\phi k > \mathrm{pd}_R k$ if R is regular and S is not regular.

We can calculate the relative CI*-dimension of each of the syzygy modules of an R-module M by using the relative CI*-dimension of M:

Proposition 2.5.18 *For an R-module M and an integer $n \geq 0$,*

$$\mathrm{CI}^*\text{-}\dim_\phi(\Omega_R^n M) = \sup\{\mathrm{CI}^*\text{-}\dim_\phi M - n, 0\}.$$

PROOF We claim that $\text{CI}^*\text{-dim}_\phi M < \infty$ if and only if $\text{CI}^*\text{-dim}_\phi(\Omega_R^1 M) < \infty$. Indeed, let $S \to S' \to R' \leftarrow R$ be a P-factorization of ϕ. There is a short exact sequence

$$0 \to \Omega_R^1 M \to R^m \to M \to 0$$

with some integer m. Since R' is flat over R, we obtain

$$0 \to \Omega_R^1 M \otimes_R R' \to R'^m \to M \otimes_R R' \to 0.$$

Note that $\text{pd}_{S'} R' < \infty$. Hence we see that $\text{pd}_{S'}(M \otimes_R R') < \infty$ if and only if $\text{pd}_{S'}(\Omega_R^1 M \otimes_R R') < \infty$. This implies the claim.

It follows from the claim that $\text{CI}^*\text{-dim}_\phi M < \infty$ if and only if $\text{CI}^*\text{-dim}_\phi(\Omega_R^n M) < \infty$. Thus, in order to prove the proposition, we may assume that $\text{CI}^*\text{-dim}_\phi M < \infty$ and $\text{CI}^*\text{-dim}_\phi(\Omega_R^n M) < \infty$. In particular, we have $\text{CI}^*\text{-dim}_R M < \infty$ by Proposition 2.5.11(1), and hence we also have $\text{CI-dim}_R M < \infty$. Therefore [13, (1.9)] gives us the equality

$$\text{depth}_R(\Omega_R^n M) = \min\{\text{depth}_R M + n, \text{depth}\, R\}.$$

Consequently we obtain

$$\begin{aligned} \text{CI}^*\text{-dim}_\phi(\Omega_R^n M) &= \text{depth}\, R - \text{depth}_R(\Omega_R^n M) \\ &= \max\{\text{depth}\, R - \text{depth}_R M - n, 0\} \\ &= \max\{\text{CI}^*\text{-dim}_\phi M - n, 0\}, \end{aligned}$$

as desired. □

As the last result of this section, we state the relationship between relative CI^*-dimension and regular sequences.

Proposition 2.5.19 *Let $\boldsymbol{x} = x_1, x_2, \cdots, x_m$ (resp. $\boldsymbol{y} = y_1, y_2, \cdots, y_n$) be a sequence in R (resp. S). Denote by $\overline{\phi}$ (resp. $\widetilde{\phi}$) the local homomorphism $S/(\boldsymbol{y}) \to R/\boldsymbol{y}R$ (resp. $S \to R/(\boldsymbol{x})$) induced by ϕ. Then*

(1) $\text{CI}^*\text{-dim}_\phi(M/\boldsymbol{x}M) = \text{CI}^*\text{-dim}_\phi M + m$ *if $\boldsymbol{x}$ is M-regular.*

(2) $\text{CI}^*\text{-dim}_{\overline{\phi}}(M/\boldsymbol{y}M) \leq \text{CI}^*\text{-dim}_\phi M$ *if $\boldsymbol{y}$ is S-regular, R-regular, and M-regular.*
The equality holds if $\text{CI}^*\text{-dim}_\phi M < \infty$.

(3) $\text{CI}^*\text{-dim}_{\widetilde{\phi}} M \leq \text{CI}^*\text{-dim}_\phi M - m$ *if $\boldsymbol{x}$ is R-regular and R-regular and $\boldsymbol{x}M = 0$.*
The equality holds if $\text{CI}^*\text{-dim}_\phi M < \infty$.

PROOF (1) According to Theorem 2.5.10, we have only to show that CI*-dim$_\phi(M/\boldsymbol{x}M) < \infty$ if and only if CI*-dim$_\phi M < \infty$. Let $S \to S' \to R' \leftarrow R$ be a P-factorization of ϕ. Since R' is flat over R, the sequence $\boldsymbol{x}$ is also $(M \otimes_R R')$-regular. Hence we obtain

$$\mathrm{pd}_{S'}(M \otimes_R R')/\boldsymbol{x}(M \otimes_R R') = \mathrm{pd}_{S'}(M \otimes_R R') + m.$$

Note that

$$(M \otimes_R R')/\boldsymbol{x}(M \otimes_R R') \cong (M/\boldsymbol{x}M) \otimes_R R'.$$

Therefore we see that $\mathrm{pd}_{S'}(M/\boldsymbol{x}M)\otimes_R R' < \infty$ if and only if $\mathrm{pd}_{S'}(M\otimes_R R') < \infty$. Thus the desired result is proved.

(2) We may assume that CI*-dim$_\phi M < \infty$ because the assertion immediately follows if CI*-dim$_\phi M = \infty$. It suffices to prove that the left side of the inequality is also finite, because the equality is implied by Theorem 2.5.10. There exists a P-factorization $S \to S' \to R' \leftarrow R$ of ϕ such that $\mathrm{pd}_{S'}(M \otimes_R R') < \infty$. Since $\boldsymbol{y}$ is both S-regular and R-regular, it is easy to see that the induced diagram

$$S/(\boldsymbol{y}) \to S'/\boldsymbol{y}S' \to R'/\boldsymbol{y}R' \leftarrow R/\boldsymbol{y}R$$

is a P-factorization of $\overline{\phi}$. As $\boldsymbol{y}$ is M-regular, it is also $(M \otimes_R R')$-regular, and we have

$$\begin{aligned} \mathrm{pd}_{S'/\boldsymbol{y}S'}((M/\boldsymbol{y}M) \otimes_R R') &= \mathrm{pd}_{S'/\boldsymbol{y}S'}((M \otimes_R R')/\boldsymbol{y}(M \otimes_R R')) \\ &= \mathrm{pd}_{S'}(M \otimes_R R') \\ &< \infty. \end{aligned}$$

Hence we have CI*-dim$_{\overline{\phi}}(M/\boldsymbol{y}M) < \infty$.

(3) Suppose that CI*-dim$_\phi M < \infty$. It is enough to prove that CI*-dim$_{\widetilde{\phi}} M < \infty$ by Theorem 2.5.10. Let $S \to S' \to R' \leftarrow R$ of ϕ be a P-factorization of ϕ with $\mathrm{pd}_{S'}(M \otimes_R R') < \infty$. Then we easily see that the induced diagram

$$S \to S' \to R'/\boldsymbol{x}R' \leftarrow R/(\boldsymbol{x})$$

is a P-factorization of $\widetilde{\phi}$. Since $M \otimes_{R/(\boldsymbol{x})} R'/\boldsymbol{x}R' \cong M \otimes_R R'$ has finite projective dimension over S', we have CI*-dim$_{\widetilde{\phi}} M < \infty$, as desired. □

3 Generalizations of Cohen-Macaulay dimension

The contents of this chapter are entirely contained in the author's paper [2] with T. Araya and Y. Yoshino.

Cohen-Macaulay dimension has been defined by Gerko as an homological invariant for finitely generated modules over a commutative noetherian local ring. This invariant shares many properties with projective dimension and Gorenstein dimension. The main purpose of this chapter is to extend the notion of Cohen-Macaulay dimension: in this chapter, as generalizations of Cohen-Macaulay dimension, we define homological invariants for bounded complexes over noetherian rings which are not necessarily commutative, by generalizing the notion of a semi-dualizing module which has been introduced by Christensen [20].

3.1 Introduction

The Cohen-Macaulay dimension for a module over a commutative noetherian local ring has been defined by A. A. Gerko [26]. That is to be a homological invariant of a module that shares a lot of properties with projective dimension and Gorenstein dimension. The aim of this chapter is to extend this invariant of modules to that of chain complexes, even over non-commutative rings. We try to pursue it in the most general context as possible as we can.

The key role will be played by semi-dualizing bimodules, which we introduce in this paper to generalize semi-dualizing modules in the sense of Christensen [20]. The advantage to consider an (R, S)-bimodule structure on a semi-dualizing module C is in the duality theorem. Actually we shall show that $\mathrm{Hom}_R(-, C)$ (resp. $\mathrm{Hom}_S(-, C)$) gives a duality between subcategories of R-mod and mod-S. We take such an idea from non-commutative ring theory, in particular, Morita duality and tilting theory.

In Section 2 we present a precise definition of semi-dualizing bimodules and show several properties of them. Associated to a semi-dualizing (R, S)-bimodule C, of most importance is the notion of the category ${}_R\mathcal{A}(C)$ and the ${}_R\mathcal{A}(C)$-dimension of an R-module. Under some special conditions the ${}_R\mathcal{A}(C)$-dimension will coincide with the CM dimension of a module.

In Section 3 we extend these notions to the derived category, and hence to chain complexes. We introduce the notion of trunk module of a complex, and as one of the main results of this chapter, we shall show that the ${}_R\mathcal{A}(C)$-dimension of a complex is essentially given by that of its trunk module (Theorem 3.3.12). By virtue of this theorem, we can show that many of the assertions concerning ${}_R\mathcal{A}(C)$-dimensions of modules will hold true for ${}_R\mathcal{A}(C)$-dimensions of complexes.

In Section 4 we shall show that a semi-dualizing bimodule, more generally a semi-dualizing complex of bimodules, yields a duality between subcategories of the derived categories. This second main result (Theorem 3.4.4) of the chapter gives an advantage to consider a bimodule structure of semi-dualizing modules.

In Section 5 we apply the theory to the case that base rings are commutative. Surprisingly, if the both rings R, S are commutative, then we shall see that a semi-dualizing (R, S)-bimodule is nothing but a semi-dualizing R-module, and actually $R = S$ (Lemma 3.5.1). In this case, we are able to argue $\mathcal{A}(C)$-dimension or G-dimension of dualizing complex and get in Corollary 3.5.7 a new characterization of Gorenstein rings.

3.2 $\mathcal{A}(C)$-dimensions for modules

Throughout this chapter, we assume that R (resp. S) is a left (resp. right) noetherian ring. Let R-mod (resp. mod-S) denote the category of finitely generated left R-modules (resp. right S-modules). When we say simply an R-module (resp. an S-module), we mean a finitely generated left R-module (resp. a finitely generated right S-module).

In this section, we shall define the notion of $\mathcal{A}(C)$-dimension of a module, and study its properties. For this purpose, we begin with defining semi-dualizing bimodules.

Definition 3.2.1 We call an (R, S)-bimodule C a *semi-dualizing bimodule* if the following conditions hold.

(1) The right homothety morphism $S \to \mathrm{Hom}_R(C, C)$ is a bijection.

(2) The left homothety morphism $R \to \mathrm{Hom}_S(C, C)$ is a bijection.

(3) $\mathrm{Ext}^i_R(C, C) = \mathrm{Ext}^i_S(C, C) = 0$ for all $i > 0$.

Example 3.2.2 (1) If R is a left and right noetherian ring, then the ring R itself is a semi-dualizing (R, R)-bimodule.

(2) Let R be a commutative noetherian ring, and let $S = M_n(R)$ be the n-th matrix ring over R. Then a free R-module of rank n has a structure of semi-dualizing (R, S)-bimodule.

(3) Let $R' \to R$ be a finite homomorphism of commutative noetherian rings. Suppose $\text{G-dim}_{R'} R = \text{grade}_{R'} R = g$. Then it is easy to see that $\text{Ext}^g_{R'}(R, R')$ is a semi-dualizing (R, R)-bimodule. In this case we say that R is a G-perfect R'-module in the sense of Golod.

In the rest of this section, C always denotes a semi-dualizing (R, S)-bimodule.

Definition 3.2.3 We say that an R-module M is *C-reflexive* if the following conditions are satisfied.

(1) $\text{Ext}^i_R(M, C) = 0$ for all $i > 0$.

(2) $\text{Ext}^i_S(\text{Hom}_R(M, C), C) = 0$ for all $i > 0$.

(3) The natural morphism $M \to \text{Hom}_S(\text{Hom}_R(M, C), C)$ is a bijection.

One can of course consider the same for right S-modules by symmetry.

Definition 3.2.4 If the following conditions hold for $N \in \text{mod-}S$, we say that N is C-reflexive.

(1) $\text{Ext}^i_S(N, C) = 0$ for all $i > 0$.

(2) $\text{Ext}^i_R(\text{Hom}_S(N, C), C) = 0$ for all $i > 0$.

(3) The natural morphism $N \to \text{Hom}_R(\text{Hom}_S(N, C), C)$ is a bijection.

Example 3.2.5 Both of the ring R and the semi-dualizing module C are C-reflexive R-modules. Similarly, S and C are C-reflexive S-modules.

Lemma 3.2.6 (1) *Let $0 \to L_1 \to L_2 \to L_3 \to 0$ be a short exact sequence either in R-mod or in mod-S. Assume that L_3 is C-reflexive. Then, L_1 is C-reflexive if and only if so is L_2.*

(2) *If L is a C-reflexive module, then so is any direct summand of L. In particular, any projective module is C-reflexive.*

(3) *The functors* $\text{Hom}_R(-, C)$ *and* $\text{Hom}_S(-, C)$ *yield a duality between the full subcategory of R-mod consisting of all C-reflexive R-modules and the full subcategory of mod-S consisting of all C-reflexive S-modules.*

Proof (1) Let $0 \to L_1 \to L_2 \to L_3 \to 0$ be a short exact sequence in R-mod. Suppose that L_3 is C-reflexive. Applying the functor $\mathrm{Hom}_R(-,C)$ to this sequence, we see that the sequence

$$0 \to \mathrm{Hom}_R(L_3,C) \to \mathrm{Hom}_R(L_2,C) \to \mathrm{Hom}_R(L_1,C) \to 0$$

is exact, and $\mathrm{Ext}^i_R(L_2,C) \cong \mathrm{Ext}^i_R(L_1,C)$ for $i>0$. Now applying the functor $\mathrm{Hom}_S(-,C)$, we will have an exact sequence

$$\begin{array}{rcccl} 0 & \longrightarrow & \mathrm{Hom}_S(\mathrm{Hom}_R(L_1,C),C) & \longrightarrow & \mathrm{Hom}_S(\mathrm{Hom}_R(L_2,C),C) \\ & \longrightarrow & \mathrm{Hom}_S(\mathrm{Hom}_R(L_3,C),C) & \longrightarrow & \mathrm{Ext}^1_S(\mathrm{Hom}_R(L_1,C),C) \\ & \longrightarrow & \mathrm{Ext}^1_S(\mathrm{Hom}_R(L_2,C),C) & \longrightarrow & 0 \end{array}$$

and the isomorphisms $\mathrm{Ext}^i_S(\mathrm{Hom}_R(L_1,C),C) \cong \mathrm{Ext}^i_S(\mathrm{Hom}_R(L_2,C),C)$ for $i>1$. It is now easy to see from the daigram chasing that L_1 is C-reflexive if and only if L_2 is C-reflexive.

(2) Trivial.

(3) Let $X \in R$-mod be a C-reflexive R-module. Putting $Y = \mathrm{Hom}_R(X,C) \in$ mod-S, we have that $\mathrm{Ext}^i_S(Y,C) = \mathrm{Ext}^i_S(\mathrm{Hom}_R(X,C),C) = 0$ for $i>0$, and that

$$\begin{aligned} \mathrm{Ext}^i_R(\mathrm{Hom}_S(Y,C),C) &= \mathrm{Ext}^i_R(\mathrm{Hom}_S(\mathrm{Hom}_R(X,C),C),C) \\ &\cong \mathrm{Ext}^i_R(X,C) \\ &= 0 \end{aligned}$$

for $i>0$. Since

$$\begin{aligned} Y &= \mathrm{Hom}_R(X,C) \\ &\cong \mathrm{Hom}_R(\mathrm{Hom}_S(\mathrm{Hom}_R(X,C),C),C) \\ &= \mathrm{Hom}_R(\mathrm{Hom}_S(Y,C),C), \end{aligned}$$

we see that Y is a C-reflexive S-module. On the other hand, note that $X \cong \mathrm{Hom}_S(\mathrm{Hom}_R(X,C),C) = \mathrm{Hom}_S(Y,C)$. Hence the functor

$$\mathrm{Hom}_R(-,C) : \{C\text{-reflexive } R\text{-modules}\} \to \{C\text{-reflexive } S\text{-modules}\}$$

is well-defined and dense. The full faithfulness of the functor follows from the next lemma. □

Lemma 3.2.7 *Let X be a C-reflexive R-module. Then the natural homomorphism*

$$\mathrm{Hom}_R(M,X) \to \mathrm{Hom}_S(\mathrm{Hom}_R(X,C),\mathrm{Hom}_R(M,C))$$

is isomorphic for any $M \in R$-mod.

PROOF Note first that there is a natural transformation of functors Φ : $\mathrm{Hom}_R(-, X) \to \mathrm{Hom}_S(\mathrm{Hom}_R(X, C), \mathrm{Hom}_R(-, C))$. If M is a free R-module, then it easy to see that $\Phi(M)$ is an isomorphism, since X is C-reflexive. Suppose that M is not R-free. Considering a finite R-free presentation $F_1 \to F_0 \to M \to 0$ of M, we can see that $\Phi(M)$ is an isomorphism, since the both functors $\mathrm{Hom}_R(-, X)$ and $\mathrm{Hom}_S(\mathrm{Hom}_R(X, C), \mathrm{Hom}_R(-, C))$ are contravariant left exact functors. $\square$

Lemma 3.2.8 *The following conditions are equivalent for $M \in R$-mod and $n \in \mathbb{Z}$.*

(1) *There exists an exact sequence*

$$0 \to X_n \to X_{n-1} \to \cdots \to X_0 \to M \to 0$$

such that each X_i is a C-reflexive R-module.

(2) *For any projective resolution*

$$P_\bullet : \cdots \to P_{m+1} \to P_m \to \cdots \to P_0 \to M \to 0$$

of M and for any $m \geq n$, we have that $\mathrm{Coker}(P_{m+1} \to P_m)$ is a C-reflexive R-module.

(3) *For any exact sequence*

$$\cdots \to X_{m+1} \to X_m \to \cdots \to X_0 \to M \to 0$$

with each X_i being C-reflexive, and for any $m \geq n$, we have that $\mathrm{Coker}(X_{m+1} \to X_m)$ is a C-reflexive R-module.

PROOF (1) $\Rightarrow$ (2) : Since $P_\bullet$ is a projective resolution of M, there is a chain map $\sigma_\bullet : P_\bullet \to X_\bullet$ of complexes over R :

$$\begin{tikzcd}
\cdots \arrow[r] & P_{n+1} \arrow[r] \arrow[d, "\sigma_{n+1}"'] & P_n \arrow[r] \arrow[d, "\sigma_n"'] & \cdots \arrow[r] & P_0 \arrow[r] \arrow[d, "\sigma_0"'] & M \arrow[r] \arrow[d, equal] & 0 \\
 & 0 \arrow[r] & X_n \arrow[r] & \cdots \arrow[r] & X_0 \arrow[r] & M \arrow[r] & 0
\end{tikzcd}$$

Taking the mapping cone of $\sigma_\bullet$, we see that there is an exact sequence

$$\begin{aligned}
\cdots \longrightarrow\ & P_m \longrightarrow \cdots \longrightarrow P_n \\
\longrightarrow\ & P_{n-1} \oplus X_n \longrightarrow P_{n-2} \oplus X_{n-1} \longrightarrow \cdots \longrightarrow P_0 \oplus X_1 \\
\longrightarrow\ & X_0 \longrightarrow 0.
\end{aligned}$$

It follows from a successive use of Lemma 3.2.6(1) that $\mathrm{Coker}(P_{m+1} \to P_m)$ is a C-reflexive R-module for $m \geq n$.

(2) $\Rightarrow$ (3) : Let $m \geq n$, and put $X = \mathrm{Coker}(X_{m+1} \to X_m)$ and $P = \mathrm{Coker}(P_{m+1} \to P_m)$. Since $P_\bullet$ is a projective resolution of M, there is a chain map $\sigma_\bullet : P_\bullet \to X_\bullet$ of complexes over R :

$$\begin{tikzcd}
0 \arrow[r] & P \arrow[r] \arrow[d, "\sigma_m"'] & P_{m-1} \arrow[r] \arrow[d, "\sigma_{m-1}"'] & \cdots \arrow[r] & P_0 \arrow[r] \arrow[d, "\sigma_0"'] & M \arrow[r] \arrow[d, equal] & 0 \\
0 \arrow[r] & X \arrow[r] & X_{m-1} \arrow[r] & \cdots \arrow[r] & X_0 \arrow[r] & M \arrow[r] & 0
\end{tikzcd}$$

Taking the mapping cone of $\sigma_\bullet$, we see that there is an exact sequence

$$0 \to P \to P_{m-1} \oplus X \to P_{m-2} \oplus X_{m-1} \to \cdots \to P_0 \oplus X_1 \to X_0 \to 0.$$

It then follows from Lemma 3.2.6(1) and 3.2.6(2) that X is a C-reflexive R-module.

(3) $\Rightarrow$ (1) : Trivial. $\square$

Imitating the way of defining the G-dimension in [4], we make the following definition.

Definition 3.2.9 For $M \in R$-mod, we define the ${}_R\mathcal{A}(C)$-dimension of M by

$$_R\mathcal{A}(C)\text{-dim}\, M = \inf \left\{ n \;\middle|\; \begin{array}{l} \text{there exists an exact sequence} \\ 0 \to X_n \to \cdots \to X_0 \to M \to 0, \\ \text{where } X_i \text{ is a } C\text{-reflexive } R\text{-module.} \end{array} \right\}.$$

Here we should note that we adopt the ordinary convention that $\inf \emptyset = +\infty$.

Remark 3.2.10 First of all we should notice that in the case $R = S = C$, the ${}_R\mathcal{A}(R)$-dimension is the same as the G-dimension.

Furthermore, comparing with Theorem 3.5.3 below, we are able to see that the ${}_R\mathcal{A}(C)$-dimension extends the Cohen-Macaulay dimension over a commutative ring R. More precisely, suppose that R and S are commutative local rings. If there is a semi-dualizing (R, S)-bimodule, then R must be isomorphic to S as we will show later in Lemma 3.5.1. Thus semi-dualizing bimodules are nothing but semi-dualizing R-modules in this case. One can define the Cohen-Macaulay dimension of an R-module M as

$$\text{CM-dim} M = \inf\{{}_R\mathcal{A}(C)\text{-dim}\, M \mid C \text{ is a semi-dualizing } R\text{-module}\}.$$

Let C_1 and C_2 be semi-dualizing R-modules. And suppose that an R-module M satisfies ${}_R\mathcal{A}(C_1)\text{-}\dim M < \infty$ and ${}_R\mathcal{A}(C_2)\text{-}\dim M < \infty$. Then we can show that ${}_R\mathcal{A}(C_1)\text{-}\dim M = {}_R\mathcal{A}(C_2)\text{-}\dim M$ ($=$ depth $R-$depth M) (c.f. Lemma 3.5.2). In other words, if the rings R and S are commutative, then the value of the ${}_R\mathcal{A}(C)$-dimension is constant for any choice of semi-dualizing modules C whenever it is finite. However, if R is non-commutgative, this is no longer true.

Example 3.2.11 Let Q be a quiver $e_1 \longrightarrow e_2$, and let $R = kQ$ be the path algebra over an algebraic closed field k. Put $P_1 = Re_1$, $P_2 = Re_2$, $I_1 = \mathrm{Hom}_k(e_1R, k)$, and $I_2 = \mathrm{Hom}_k(e_2R, k)$. Then, it is easy to see that the only indecomposable left R-modules are P_1, $P_2(\cong I_1)$, and I_2, up to isomorphisms. Putting $C_1 = P_1 \oplus P_2 = R$ and $C_2 = I_1 \oplus I_2$, we note that $\mathrm{End}_R(C_1) = \mathrm{End}_R(C_2) = R^{op}$, and that C_1 and C_2 are semi-dualizing (R, R)-bimodules. In this case we have that ${}_R\mathcal{A}(C_1)\text{-}\dim I_2 (= \text{G-dim } I_2) = 1$ and ${}_R\mathcal{A}(C_2)\text{-}\dim I_2 = 0$, which take different finite values.

Theorem 3.2.12 *If ${}_R\mathcal{A}(C)\text{-}\dim M < \infty$ for a module $M \in R$-mod, then*

$$ {}_R\mathcal{A}(C)\text{-}\dim M = \sup\{\ n \mid \mathrm{Ext}^n_R(M, C) \neq 0\ \}. $$

Proof We prove the theorem by induction on ${}_R\mathcal{A}(C)\text{-}\dim M$. Assume first that ${}_R\mathcal{A}(C)\text{-}\dim M = 0$. Then M is a C-reflexive module, and hence we have

$$ \sup\{\ n \mid \mathrm{Ext}^n_R(M, C) \neq 0\ \} = 0 $$

from the definition.

Assume next that ${}_R\mathcal{A}(C)\text{-}\dim M = 1$. Then there exists an exact sequence

$$ 0 \to X_1 \to X_0 \to M \to 0 $$

where X_0 and X_1 are C-reflexive R-modules. Then it is clear that $\mathrm{Ext}^i_R(M, C) = 0$ for $i > 1$. We must show that $\mathrm{Ext}^1_R(M, C) \neq 0$. To do this, supposed $\mathrm{Ext}^1_R(M, C) = 0$. Then we would have an exact sequence

$$ 0 \to \mathrm{Hom}_R(M, C) \to \mathrm{Hom}_R(X_0, C) \to \mathrm{Hom}_R(X_1, C) \to 0. \tag{3.1} $$

Then, writing the functor $\mathrm{Hom}_S(\mathrm{Hom}_R(-, C), C)$ as F, we get from this the commutative diagram with exact rows

$$ \begin{array}{ccccccccc} 0 & \longrightarrow & X_1 & \longrightarrow & X_0 & \longrightarrow & M & \longrightarrow & 0 \\ & & \cong\downarrow & & \cong\downarrow & & \downarrow & & \\ 0 & \longrightarrow & F(X_1) & \longrightarrow & F(X_0) & \longrightarrow & F(M) & \longrightarrow & 0, \end{array} $$

hence the natural map $M \to F(M)$ is also an isomorphism. Furthermore, it also follows from (3.1) that $\mathrm{Ext}^i_R(\mathrm{Hom}_R(M, C), C) = 0$ for $i > 0$. Therefore we would have ${}_R\mathcal{A}(C)$-dim $M = 0$, a contradiction. Hence $\mathrm{Ext}^1_R(M, C) \neq 0$ as desired.

Finally assume that ${}_R\mathcal{A}(C)$-dim $M = m > 1$. Then there exists an exact sequence

$$0 \to X_m \to X_{m-1} \to \cdots \to X_0 \to M \to 0$$

such that each X_i is a C-reflexive R-module. Putting $M' = \mathrm{Coker}(X_2 \to X_1)$, we note that the sequence $0 \to M' \to X_0 \to M \to 0$ is exact and ${}_R\mathcal{A}(C)$-dim $M' = m - 1 > 0$. Therefore, the induction hypothesis implies that

$$\sup\{\ n \mid \mathrm{Ext}^n_R(M', C) \neq 0\ \} = m - 1.$$

Since $\mathrm{Ext}^n_R(X_0, C) = 0$ for $n > 0$, it follows that

$$\sup\{\ n \mid \mathrm{Ext}^n_R(M, C) \neq 0\ \} = m$$

as desired. □

If R is a left and right noetherian ring and if $R = S = C$, then the equality ${}_R\mathcal{A}(R)$- dim M = G-dim M holds by definition. We should remark that if R is a Gorenstein commutative ring, then any R-module M has finite G-dimension and it can be embedded in a short exact sequence of the form $0 \to F \to X \to M \to 0$ with $\mathrm{pd}\, F < \infty$ and G-dim $X = 0$. Such a short exact sequence is called a Cohen-Macaulay approximation of M. For the details, see [5].

We can prove an analogue of this result. To state our theorem, we need several notations from [5]. Now let C be a semi-dualizing (R, S)-bimodule as before. We denote by $\mathcal{X}$ the full subcategory of R-mod consisting of all C-reflexive R-modules, and by $\widehat{\mathcal{X}}$ the full subcategory consisting of R-modules of finite ${}_R\mathcal{A}(C)$-dimension. And $\mathrm{add}(C)$ denotes the subcategory of all direst summands of direct sums of copies of C. It is obvious that $\mathrm{add}(C) \subseteq \mathcal{X}$ and that the objects of $\mathrm{add}(C)$ are injective objects in $\mathcal{X}$, because $\mathrm{Ext}^i_R(X, C) = 0$ for $X \in \mathcal{X}$ and $i > 0$. The following lemma says that C is an injective cogenerator of $\mathcal{X}$.

Lemma 3.2.13 *Suppose an R-module X is C-reflexive, and hence $X \in \mathcal{X}$. Then there exists an exact sequence $0 \to X \to C^0 \to X^1 \to 0$ where $C^0 \in \mathrm{add}(C)$ and $X^1 \in \mathcal{X}$. In particular, we can resolve X by objects in $\mathrm{add}(C)$ as*

$$0 \to X \to C^0 \to C^1 \to C^2 \to \cdots,\ (C_i \in \mathrm{add}(C)).$$

Proof It follows from Lemma 3.2.6(3) that $Y = \mathrm{Hom}_R(X, C)$ is a C-reflexive S-module. Take an exact sequence $0 \to Y' \to S^{\oplus n} \to Y \to 0$ to get the the first syzygy S-module Y' of Y. Applying the functor $\mathrm{Hom}_S(-, C)$, we obtain an exact sequence

$$0 \to X \to C^{\oplus n} \to \mathrm{Hom}_S(Y', C) \to 0.$$

Since Y' is a C-reflexive S-module, we see that $\mathrm{Hom}_S(Y', C)$ is a C-reflexive R-module again. $\square$

To state the theorem, let us denote

$$\widehat{\mathrm{add}(C)} = \left\{ F \in R\text{-mod} \;\middle|\; \begin{array}{l} \text{there exists an exact sequence} \\ 0 \to C_n \to C_{n-1} \to \cdots \to C_0 \to F \to 0 \\ \text{with } C_i \in \mathrm{add}(C) \end{array} \right\}.$$

Then it is easy to prove the following result in a completely similar way to the proof of [5, Theorem 1.1].

Theorem 3.2.14 *Let $M \in R$-mod, and suppose ${}_R\mathcal{A}(C)$-$\dim M < \infty$, and hence $M \in \widehat{\mathcal{X}}$. Then there exist short exact sequences*

$$0 \to F_M \to X_M \to M \to 0 \tag{3.2}$$

$$0 \to M \to F^M \to X^M \to 0 \tag{3.3}$$

where X_M and X^M are in $\mathcal{X}$, and F_M and F^M are in $\widehat{\mathrm{add}(C)}$.

Remark 3.2.15 Let X be a C-reflexive R-module. Since $\mathrm{Ext}^i(X, C) = 0$ for $i > 0$, it follows that $\mathrm{Ext}^i(X, F) = 0$ for $F \in \widehat{\mathrm{add}(C)}$ and $i > 0$. Hence, from (3.2), we have an exact sequence

$$0 \to \mathrm{Hom}_R(X, F_M) \to \mathrm{Hom}_R(X, X_M) \to \mathrm{Hom}_R(X, M) \to 0.$$

This means that any homomorphism from any C-reflexive R-module X to M factors through the map $X_M \to M$. In this sense, the exact sequence (3.2) gives an approximation of M by the subcategory $\mathcal{X}$.

Remark 3.2.16 We can of course define $\mathcal{A}_S(C)$-$\dim N$ for an S-module N as in the same manner as we define ${}_R\mathcal{A}(C)$-dimension. And it is clear by symmetry that it satisfies that

$$\mathcal{A}_S(C)\text{-}\dim N = \sup\{n |\ \mathrm{Ext}^n_S(N, C) \neq 0\}$$

etc.

3.3 $\mathcal{A}(C)$-dimensions for complexes

Again in this section, we assume that R (resp. S) is a left (resp. right) noetherian ring. We denote by $\mathfrak{D}^b(R\text{-mod})$ (resp. $\mathfrak{D}^b(\text{mod-}S)$) the derived category of R-mod (resp. mod-S) consisting of complexes with bounded finite homologies.

For a complex $M^\bullet$ we always write it as

$$\cdots \to M^{n-1} \xrightarrow{\partial_M^n} M^n \xrightarrow{\partial_M^{n+1}} M^{n+1} \xrightarrow{\partial_M^{n+2}} M^{n+2} \to \cdots ,$$

and the shifted complex $M^\bullet[m]$ is the complex with $M^\bullet[m]^n = M^{m+n}$.

According to Foxby [25], we define the *supremum*, the *infimum* and the *amplitude* of a complex $M^\bullet$ as follows;

$$\begin{cases} s(M^\bullet) = \sup\{\ n \mid H^n(M^\bullet) \neq 0\ \}, \\ i(M^\bullet) = \inf\{\ n \mid H^n(M^\bullet) \neq 0\ \}, \\ a(M^\bullet) = s(M^\bullet) - i(M^\bullet). \end{cases} \tag{3.4}$$

Note that $M^\bullet \cong 0$ iff $s(M^\bullet) = -\infty$ iff $i(M^\bullet) = +\infty$ iff $a(M^\bullet) = -\infty$.

Suppose in the following that $M^\bullet \ncong 0$. A complex $M^\bullet$ is called bounded if $s(M^\bullet) < \infty$ and $i(M^\bullet) > -\infty$ (hence $a(M^\bullet) < \infty$). And $\mathfrak{D}^b(R\text{-mod})$ is, by definition, consisting of bounded complexes with finitely generated homologies. Thus, whenever $M^\bullet \in \mathfrak{D}^b(R\text{-mod})$, we have

$$-\infty < i(M^\bullet) \leq s(M^\bullet) < +\infty.$$

and $a(M^\bullet)$ is a non-negative integer.

We remark that the category R-mod can be identified with the full subcategory of $\mathfrak{D}^b(R\text{-mod})$ consisting of all the complexes $M^\bullet \in \mathfrak{D}^b(R\text{-mod})$ with $s(M^\bullet) = i(M^\bullet) = a(M^\bullet) = 0$ or otherwise $M^\bullet \cong 0$. Through this identification we always think of R-mod as the full subcategory of $\mathfrak{D}^b(R\text{-mod})$.

For a complex $P^\bullet$, if each component P^i is a finitely generated projective module, then we say that $P^\bullet$ is a projective complex. For any complex $M^\bullet \in \mathfrak{D}^b(R\text{-mod})$, we can construct a projective complex $P^\bullet$ and a chain map $P^\bullet \to M^\bullet$ that yields an isomorphism in $\mathfrak{D}^b(R\text{-mod})$. We call such $P^\bullet \to M^\bullet$ a projective resolution of $M^\bullet$. If $M^\bullet \ncong 0$ and $s = s(M^\bullet)$, then we can take a projective resolution $P^\bullet$ of $M^\bullet$ in the form;

$$\cdots \to P^{s-2} \xrightarrow{\partial_P^{s-1}} P^{s-1} \xrightarrow{\partial_P^s} P^s \to 0 \to 0 \to \cdots ,\ \text{(i.e. } P^i = 0 \text{ for } i > s).$$

We call such a projective resolution with this additional property a *standard* projective resolution of $M^\bullet$.

For a projective complex $P^\bullet(\ncong 0)$ and an integer n, we can consider two kinds of truncated complexes:

$$\begin{cases} \tau^{\leq n}P^\bullet = (\cdots \to P^{n-2} \xrightarrow{\partial_P^{n-1}} P^{n-1} \xrightarrow{\partial_P^n} P^n \to 0 \to 0 \to \cdots) \\ \tau^{\geq n}P^\bullet = (\cdots \to 0 \to 0 \to P^n \xrightarrow{\partial_P^{n+1}} P^{n+1} \xrightarrow{\partial_P^{n+2}} P^{n+2} \to \cdots) \end{cases} \tag{3.5}$$

Definition 3.3.1 (ω-operation) Let $M^\bullet(\ncong 0) \in \mathfrak{D}^b(R\text{-mod})$ and $s = s(M^\bullet)$. Taking a standard projective resolution $P^\bullet$ of $M^\bullet$, we define the projective complex $\omega P^\bullet$ by

$$\omega P^\bullet = (\tau^{\leq s-1}P^\bullet)[-1]. \tag{3.6}$$

Note from this definition that $\omega P^\bullet$ and $P^\bullet[-1]$ share the same components in degree $\leq s$. We can also see from the definition that there is a triangle of the form

$$\omega P^\bullet \to P^s[-s] \to M^\bullet \to \omega P^\bullet[1]. \tag{3.7}$$

Therefore, if $M^\bullet$ is a module $M \in R$-mod, then $\omega P^\bullet$ is isomorphic to a first syzygy module of M. Note that $\omega P^\bullet$ is not uniquely determined by $M^\bullet$. Actually it depends on the choice of a standard projective resolution $P^\bullet$, but is unique up to a projective summand in degree s. It is easy to prove the following lemma.

Lemma 3.3.2 *Let $M^\bullet(\ncong 0) \in \mathfrak{D}^b(R\text{-mod})$ and let $P^\bullet$ be a standard projective resolution of $M^\bullet$. Now suppose that $a(M^\bullet) > 0$. Then,*

(1) $i(\omega P^\bullet) = i(M^\bullet) + 1$,

(2) $0 \leq a(\omega P^\bullet) < a(M^\bullet)$.

Proof Let $s = s(M^\bullet)$. Since the complexes $P^\bullet$ and $\omega P^\bullet[1]$ share the same components in degree $\leq s-1$, we have that $H^i(M^\bullet) = H^i(P^\bullet) = H^{i+1}(\omega P^\bullet)$ for $i \leq s-2$ and that $H^{s-1}(M^\bullet) = H^{s-1}(P^\bullet)$ is embedded into $H^s(\omega P^\bullet)$. The lemma follows from this observation. □

It follows from this lemma that taking ω-operation in several times to a given projective complex $P^\bullet$, we will have a complex with amplitude 0, i.e. a shifted module.

Definition 3.3.3 Let $M^\bullet$ and $P^\bullet$ be as in the lemma. Then there is the least integer b with $\omega^b P^\bullet$ having amplitude 0. Thus there is a module $T \in R$-mod such that $\omega^b P^\bullet \cong T[-c]$ for some $c \in \mathbb{Z}$. We call such a module T the *trunk module* of the complex $M^\bullet$.

Remark 3.3.4 Let $M^\bullet$ and $P^\bullet$ be as in the lemma. Set $i = i(M^\bullet)$, and we see that the trunk module T is isomorphic to $\tau^{\leq i} P^\bullet[i]$ in $\mathfrak{D}^b(R\text{-mod})$, and hence $T \cong \mathrm{Coker}(P^{i-1} \to P^i)$. Note that the trunk module T is unique only in the stable category $\underline{R\text{-mod}}$.

Note that the integer b in Definition 3.3.3 is not necessarily equal to $a(M^\bullet)$. For instance, consider the complex $M^\bullet = P^\bullet = R[2] \oplus R$. Then $a(M^\bullet) = 2$ and $T = \omega^1 P^\bullet[-1] = R$.

Now we fix a semi-dualizing (R, S)-bimodule C. Associated to it, we can consider the following subcategory of $\mathfrak{D}^b(R\text{-mod})$.

Definition 3.3.5 For a semi-dualizing (R, S)-bimodule C, we denote by ${}_R\mathcal{A}(C)$ the full subcategory of $\mathfrak{D}^b(R\text{-mod})$ consisting of all complexes $M^\bullet$ that satisfy the following two conditions.

(1) $\mathbf{R}\mathrm{Hom}_R(M^\bullet, C) \in \mathfrak{D}^b(\text{mod-}S)$.

(2) The natural morphism $M^\bullet \to \mathbf{R}\mathrm{Hom}_S(\mathbf{R}\mathrm{Hom}_R(M^\bullet, C), C)$ is an isomorphism in $\mathfrak{D}^b(R\text{-mod})$.

If R is a left and right noetherian ring and if $R = S = C$, then we should note from the papers of Avramov-Foxby [11, (4.1.7)] and Yassemi [57, (2.7)] that ${}_R\mathcal{A}(R) = \{\ M^\bullet \in \mathfrak{D}^b(R\text{-mod}) \mid \text{G-dim}\ M^\bullet < \infty\}$.

First of all we should notice the following fact.

Lemma 3.3.6 *Let C be a semi-dualizing (R, S)-bimodule as above.*

(1) *The subcategory ${}_R\mathcal{A}(C)$ of $\mathfrak{D}^b(R\text{-mod})$ is a triangulated subcategory which contains R, and is closed under direct summands. In particular, ${}_R\mathcal{A}(C)$ contains all projective R-modules.*

(2) *Let $P^\bullet$ be a projective complex in $\mathfrak{D}^b(R\text{-mod})$. Then, $P^\bullet \in {}_R\mathcal{A}(C)$ if and only if $\omega P^\bullet \in {}_R\mathcal{A}(C)$.*

(3) *Let $M^\bullet \in \mathfrak{D}^b(R\text{-mod})$ and let T be a trunk module of $M^\bullet$. Then $M^\bullet \in {}_R\mathcal{A}(C)$ if and only if $T \in {}_R\mathcal{A}(C)$.*

Proof One can prove (1) in standard way, and we omit it. For (2) and (3), in the triangel (3.7), noting that $P[-s] \in {}_R\mathcal{A}(C)$ and that ${}_R\mathcal{A}(C)$ is a triangulated category, we see that $P^\bullet \in {}_R\mathcal{A}(C)$ is equivalent to that $\omega P^\bullet \in {}_R\mathcal{A}(C)$. Since $T \cong \omega^b P^\bullet[c]$ as in Definition 3.3.3, this is also equivalent to that $T \in {}_R\mathcal{A}(C)$. $\square$

The following lemma says that R-modules in ${}_R\mathcal{A}(C)$ form the subcategory of modules of finite ${}_R\mathcal{A}(C)$-dimension.

Lemma 3.3.7 *Let M be an R-module. Then the following two conditions are equivalent.*

(1) ${}_R\mathcal{A}(C)\text{-dim}\, M < \infty$,

(2) $M \in {}_R\mathcal{A}(C)$.

Proof (1) $\Rightarrow$ (2): Note from the definition that every C-reflexive module belongs to ${}_R\mathcal{A}(C)$. Since ${}_R\mathcal{A}(C)\text{-dim}\, M < \infty$, there is a finite exact sequence

$$0 \to X_n \to X_{n-1} \to \cdots \to X_0 \to M \to 0$$

where each X_i is C-reflexive. Since each X_i belongs to ${}_R\mathcal{A}(C)$ and since ${}_R\mathcal{A}(C)$ is closed under making triangles, we see that $M \in {}_R\mathcal{A}(C)$.

(2) $\Rightarrow$ (1): Suppose that $M \in {}_R\mathcal{A}(C)$ and let $P^\bullet$ be a (standard) projective resolution of M. Since $\mathbf{R}\mathrm{Hom}_R(M, C)$ is a bounded complex, note that $s = s(\mathbf{R}\mathrm{Hom}_R(M, C))$ is a (finite) non-negative integer. Since the complexes $\mathrm{Hom}_R(\omega^s P^\bullet, C)$ and $\mathrm{Hom}_R(P^\bullet[-s], C)$ share the same component in non-negative degree, we see that

$$H^i(\mathbf{R}\mathrm{Hom}_R(\omega^s P^\bullet, C)) = H^{i+s}(\mathbf{R}\mathrm{Hom}_R(P^\bullet, C)) = 0$$

for $i \geq 1$. Noting that $\omega^s P^\bullet$ is isomorphic to the s-th syzygy module $\Omega^s M$ of M, we see from this that $\mathrm{Ext}^i(\Omega^s M, C) = 0$ for $i > 0$. Since $\omega^s P^\bullet \in {}_R\mathcal{A}(C)$, the natural map $\Omega^s M \to \mathbf{R}\mathrm{Hom}_S(\mathrm{Hom}_R(\Omega^s M, C), C)$ is an isomorphism, equivealently

$$\begin{cases} \Omega^s M \cong \mathrm{Hom}_S(\mathrm{Hom}_R(\Omega^s M, C), C), \\ \mathrm{Ext}^i(\mathrm{Hom}_R(\Omega^s M, C), C) = 0 \end{cases}$$

for $i > 0$. Consequently, we see that $\Omega^s M$ is a C-reflexive R-module, and hence ${}_R\mathcal{A}(C)\text{-dim}\, M \leq s < \infty$. $\square$

Recall from Theorem 3.2.12 that if an R-module M has finite ${}_R\mathcal{A}(C)$-dimension, then we have ${}_R\mathcal{A}(C)\text{-dim}\, M = s(\mathbf{R}\mathrm{Hom}_R(M, C))$. Therefore it will be reasonable to make the following definition.

Definition 3.3.8 Let C be a semi-dualizing (R, S)-bimodule and let $M^\bullet$ be a complex in $\mathfrak{D}^b(R\text{-mod})$. We define the ${}_R\mathcal{A}(C)$-dimension of $M^\bullet$ to be

$$\begin{cases} {}_R\mathcal{A}(C)\text{-dim}\, M^\bullet = s(\mathbf{R}\mathrm{Hom}_R(M^\bullet, C)) & \text{if} \quad M^\bullet \in {}_R\mathcal{A}(C), \\ {}_R\mathcal{A}(C)\text{-dim}\, M^\bullet = +\infty & \text{if} \quad M^\bullet \notin {}_R\mathcal{A}(C). \end{cases}$$

Note that this definition is comptatible with that of ${}_R\mathcal{A}(C)$-dimension for R-modules in Section 2. Just noting an obvious equality

$$s(\mathbf{R}\mathrm{Hom}_R(M^\bullet[m], C)) = s(\mathbf{R}\mathrm{Hom}_R(M^\bullet, C)) + m$$

for $M^\bullet \in \mathfrak{D}^b(R\text{-mod})$ and $m \in \mathbb{Z}$, we have the following lemma.

Lemma 3.3.9 *Let $M^\bullet$ be a complex in $\mathfrak{D}^b(R\text{-mod})$ and let m be an integer. Then we have*

$$ {}_R\mathcal{A}(C)\text{-}\dim M^\bullet[m] = {}_R\mathcal{A}(C)\text{-}\dim M^\bullet + m.$$

Lemma 3.3.10 *Let $M^\bullet$ be a complex in $\mathfrak{D}^b(R\text{-mod})$. Then the following inequality holds:*

$$ {}_R\mathcal{A}(C)\text{-}\dim M^\bullet + i(M^\bullet) \geq 0.$$

PROOF If $M^\bullet \cong 0$, then since $i(M^\bullet) = +\infty$, the inequality holds obviously. We may thus assume that $M^\bullet \not\cong 0$. If $M^\bullet \notin {}_R\mathcal{A}(C)$, then ${}_R\mathcal{A}(C)\text{-}\dim M^\bullet = +\infty$ by definition, and there is nothing to prove. Hence we assume $M^\bullet \in {}_R\mathcal{A}(C)$. In particular, we have $M^\bullet \cong \mathbf{R}\mathrm{Hom}_S(\mathbf{R}\mathrm{Hom}_R(M^\bullet, C), C)$. Therefore we have that

$$\begin{array}{rcl} i(M^\bullet) & = & i(\mathbf{R}\mathrm{Hom}_S(\mathbf{R}\mathrm{Hom}_R(M^\bullet, C), C)) \\ & \geq & i(C) - s(\mathbf{R}\mathrm{Hom}_R(M^\bullet, C)) \\ & = & -s(\mathbf{R}\mathrm{Hom}_R(M^\bullet, C)) \\ & = & -{}_R\mathcal{A}(C)\text{-}\dim M^\bullet. \end{array}$$

(For the inequality see Foxby [24, Lemma 2.1].) □

Proposition 3.3.11 *For a given complex $M^\bullet \in \mathfrak{D}^b(R\text{-mod})$, suppose that $a(M^\bullet) > 0$. Taking a standard projective resolution $P^\bullet$ of $M^\bullet$, we have an equality*

$$ {}_R\mathcal{A}(C)\text{-}\dim M^\bullet = {}_R\mathcal{A}(C)\text{-}\dim \omega P^\bullet + 1.$$

PROOF Note from Lemma 3.3.6(2) that ${}_R\mathcal{A}(C)\text{-}\dim M^\bullet < \infty$ if and only if ${}_R\mathcal{A}(C)\text{-}\dim \omega P^\bullet < \infty$. Assume that $n = {}_R\mathcal{A}(C)\text{-}\dim M^\bullet = s(\mathbf{R}\mathrm{Hom}_R(P^\bullet, C)) < \infty$ and let $s = s(M^\bullet)$. We should note from Lemma 3.3.10 that

$$\begin{array}{rcl} n + s & = & {}_R\mathcal{A}(C)\text{-}\dim M^\bullet + s(M^\bullet) \\ & > & {}_R\mathcal{A}(C)\text{-}\dim M^\bullet + i(M^\bullet) \\ & \geq & 0. \end{array}$$

Since the complex $\mathrm{Hom}_R(\omega P^\bullet, C)$ shares the components in degree $\geq -s$ with $\mathrm{Hom}_R(P^\bullet, C)[1]$, we see that $H^i(\mathrm{Hom}_R(\omega P^\bullet, C)) = H^{i+1}(\mathrm{Hom}_R(P^\bullet, C))$ for $i > -s$. Since $n > -s$ as above, it follows that $s(\mathrm{Hom}_R(\omega P^\bullet, C)) = s(\mathrm{Hom}_R(P^\bullet, C)) - 1$. □

As we show in the next theorem, the ${}_R\mathcal{A}(C)$-dimension of a complex is essentially the same as that of its trunk module. In that sense, every argument concerning ${}_R\mathcal{A}(C)$-dimension of complexes will be reduced to that of modules.

Theorem 3.3.12 *Let T be the trunk module of a complex $M^\bullet \in \mathfrak{D}^b(R\text{-}mod)$ as in Definition 3.3.3. Then there is an equality*

$$ {}_R\mathcal{A}(C)\text{-}\dim M^\bullet = {}_R\mathcal{A}(C)\text{-}\dim T - i(M^\bullet). $$

PROOF If $M^\bullet \notin {}_R\mathcal{A}(C)$, then the both sides take infinity and the equality holds. We assume that $M^\bullet \in {}_R\mathcal{A}(C)$ hence ${}_R\mathcal{A}(C)\text{-}\dim M^\bullet < \infty$.

We prove the equality by induction on $a(M^\bullet)$. If $a(M^\bullet) = 0$ then $M^\bullet \cong T[-i]$ for the trunk module T and for $i = i(M^\bullet)$. Therefore it follows from Lemma 3.3.9 ${}_R\mathcal{A}(C)\text{-}\dim M^\bullet = {}_R\mathcal{A}(C)\text{-}\dim T - i$.

Now assume that $a(M^\bullet) > 0$, and let $P^\bullet$ be a standard projective resolution of $M^\bullet$. Noting from Lemma 3.3.2 that we can apply the induction hypothesis on $\omega P^\bullet$, we get the following equalities from the previous proposition.

$$ \begin{aligned} {}_R\mathcal{A}(C)\text{-}\dim M^\bullet &= {}_R\mathcal{A}(C)\text{-}\dim \omega P^\bullet + 1 \\ &= {}_R\mathcal{A}(C)\text{-}\dim T - i(\omega P^\bullet) + 1 \\ &= {}_R\mathcal{A}(C)\text{-}\dim T - i(P^\bullet) \\ &= {}_R\mathcal{A}(C)\text{-}\dim T - i(M^\bullet). \end{aligned} $$

These equalities prove the theorem. □

As one of the applications of this theorem, we can show the following theorem that generalizes Lemma 3.2.8 to the category of complexes.

Theorem 3.3.13 *Let $M^\bullet$ be a complex in $\mathfrak{D}^b(R\text{-}mod)$. Then the following conditions are equivalent.*

(1) ${}_R\mathcal{A}(C)\text{-}\dim M^\bullet < \infty$,

(2) *There is a complex $X^\bullet$ of finite length consisting of C-reflexive modules and there is a chain map $X^\bullet \to M^\bullet$ that is an isomorphism in $\mathfrak{D}^b(R\text{-}mod)$.*

PROOF (2) $\Rightarrow$ (1): Note that every C-reflexive R-module belongs to ${}_R\mathcal{A}(C)$ and that ${}_R\mathcal{A}(C)$ is closed under making triangles. Therefore any complexes $X^\bullet$ of finite length consisting of C-reflexive modules are also in ${}_R\mathcal{A}(C)$, and hence ${}_R\mathcal{A}(C)\text{-}\dim X^\bullet < \infty$.

(1) $\Rightarrow$ (2): Assume that ${}_R\mathcal{A}(C)\text{-}\dim M^\bullet < \infty$ hence $M^\bullet \in {}_R\mathcal{A}(C)$. We shall prove by induction on $a(M^\bullet)$ that the second assertion holds. If

$a(M^\bullet) = 0$, then there is an R-module T such that $M^\bullet \cong T[-i]$ where $i = i(M^\bullet)$. Since ${}_R\mathcal{A}(C)\text{-dim}\, T < \infty$, there are an acyclic complex

$$X^\bullet = [\ 0 \to X_m \to \cdots \to X_2 \to X_1 \to X_0 \to 0\ ]$$

with each X_i being C-reflexive and a chain map $X^\bullet \to T$ that is an isomorphism in $\mathfrak{D}^b(R\text{-mod})$. Thus the complex $X^\bullet[-s]$ is the desired complex for $M^\bullet$.

Now suppose $a = a(M^\bullet) > 0$ and take a standard projective resolution $P^\bullet$ of $M^\bullet$. As in (3.7), we have chain maps $\varphi : P^s[-s] \to M^\bullet$ and $\psi : \omega P^\bullet \to P^s[-s]$ that make the triangle

$$\omega P^\bullet \xrightarrow{\psi} P^s[-s] \xrightarrow{\varphi} M^\bullet \to \omega P^\bullet[1].$$

Since $a(\omega P^\bullet) < a(M^\bullet)$, it follows from the induction hypothesis that there is a chain map $\rho : X^\bullet \to \omega P^\bullet$ that gives an isomorphism in $\mathfrak{D}^b(R\text{-mod})$, where $X^\bullet$ is a complex of finite length with each X^i being C-reflexive. Thus we also have a triangle

$$X^\bullet \xrightarrow{\psi\cdot\rho} P^s[-s] \xrightarrow{\varphi} M^\bullet \to X^\bullet[1].$$

Now take a mapping cone $Y^\bullet$ of $\psi \cdot \rho$. Then it is obvious that $Y^\bullet$ has finite length and each modules in $Y^\bullet$ is C-reflexive, since Y^i is a module X^i with at most directly summing P^s. Furthermore it follows from the above triangle that there is a chain map $Y^\bullet \to M^\bullet$ that yields an isomorphism in $\mathfrak{D}^b(R\text{-mod})$. $\square$

Also in the category $\mathfrak{D}^b(\text{mod-}S)$, we can construct the notion similar to that in $\mathfrak{D}^b(R\text{-mod})$.

Definition 3.3.14 Let C be a semi-dualizing (R,S)-bimodule. We denote by $\mathcal{A}_S(C)$ the full subcategory of $\mathfrak{D}^b(\text{mod-}S)$ consisting of all complexes $N^\bullet$ that satisfy the following two conditions.

(1) $\mathbf{R}\mathrm{Hom}_S(N^\bullet, C) \in \mathfrak{D}^b(R\text{-mod})$.

(2) The natural morphism $N^\bullet \to \mathbf{R}\mathrm{Hom}_R(\mathbf{R}\mathrm{Hom}_S(N^\bullet, C), C)$ is an isomorphism in $\mathfrak{D}^b(\text{mod-}S)$.

Definition 3.3.15 Let C be a semi-dualizing (R,S)-bimodule and let $N^\bullet$ be a complex in $\mathfrak{D}^b(\text{mod-}S)$. We define the $\mathcal{A}_S(C)$-dimension of $N^\bullet$ to be

$$\begin{cases} \mathcal{A}_S(C)\text{-dim}\, N^\bullet = s(\mathbf{R}\mathrm{Hom}_S(N^\bullet, C)) & \text{if} \quad N^\bullet \in \mathcal{A}_S(C), \\ \mathcal{A}_S(C)\text{-dim}\, N^\bullet = +\infty & \text{if} \quad N^\bullet \notin \mathcal{A}_S(C). \end{cases}$$

Note that all the properties concerning ${}_R\mathcal{A}(C)$ and ${}_R\mathcal{A}(C)$-dimension hold true for $\mathcal{A}_S(C)$ and $\mathcal{A}_S(C)$-dimension by symmetry.

Lemma 3.3.16 *Let C be a semi-dualizing (R, S)-bimodule as above. Then the functors $\mathbf{R}\mathrm{Hom}_R(-, C)$ and $\mathbf{R}\mathrm{Hom}_S(-, C)$ yield a duality between the categories ${}_R\mathcal{A}(C)$ and $\mathcal{A}_S(C)$.*

We postpone the proof of this lemma until Theorem 3.4.4 in the next section, where we prove the duality in more general setting. Using this lemma we are able to prove the following theorem, which generalizes Theorem 3.2.14. We recall that $\mathrm{add}(C)$ is the additive full subcategory of R-mod consisting of modules that are isomorphic to direct summands of direct sums of copies of C.

Theorem 3.3.17 *Let $M^\bullet \in \mathfrak{D}^b(R\text{-}mod)$ and suppose that ${}_R\mathcal{A}(C)\text{-}\dim M^\bullet < \infty$. Then there exists a triangle*

$$F_M^\bullet \to X_M^\bullet \to M^\bullet \to F_M^\bullet[1] \tag{3.8}$$

where $X_M^\bullet$ is a shifted C-reflexive R-module, and $F_M^\bullet$ is a complex that is isomorphic to a complex of finite length consisting of modules in $\mathrm{add}(C)$.

Proof Let $N^\bullet = \mathbf{R}\mathrm{Hom}_R(M^\bullet, C)$ and let T be a trunk module of $N^\bullet$ in the category $\mathfrak{D}^b(\text{mod-}S)$. We have a triangle of the following type:

$$T[-i] \to P^\bullet \to N^\bullet \to T[-i+1],$$

where $i = i(N^\bullet)$ and $P^\bullet$ is a projective S-complex of length $a(N^\bullet)$. Note that $n = \mathcal{A}_S(C)\text{-}\dim T$ is finite as well as $\mathcal{A}_S(C)\text{-}\dim N^\bullet < \infty$ by Lemma 3.3.16. Take the n-th syzygy module of T, and we have a C-reflexive S-module U with the triangle

$$U[-i-n] \to Q^\bullet \to N^\bullet \to U[-i-n+1],$$

where $Q^\bullet$ is again a projective S-complex of finite length. Applying the functor $\mathbf{R}\mathrm{Hom}_S(-, C)$, we have a triangle

$$\begin{aligned} &\mathbf{R}\mathrm{Hom}_S(U, C)[i+n-1] \ \to M^\bullet \to \mathbf{R}\mathrm{Hom}_S(Q^\bullet, C) \\ \to\ &\mathbf{R}\mathrm{Hom}_S(U, C)[i+n]. \end{aligned}$$

Note that $\mathbf{R}\mathrm{Hom}_S(U, C)$ is isomorphic to a C-reflexive R-module and that $\mathbf{R}\mathrm{Hom}_S(Q^\bullet, C)$ is a complex of finite length, each component of which is a module in $\mathrm{add}(C)$. $\square$

3.4 $\mathcal{A}(C^\bullet)$-dimensions for complexes

The notion of a semi-dualizing bimodule is naturally extended to that of a semi-dualizing complex of bimodules. For this purpose, let $C^\bullet$ be a complex consisting of (R,S)-bimodules and (R,S)-bimodule homomorphisms. Then for a complex $M^\bullet \in \mathfrak{D}^b(R\text{-mod})$, take an R-projective resolution $P^\bullet$ of $M^\bullet$, and we understand $\mathbf{R}\mathrm{Hom}_R(M^\bullet, C^\bullet)$ as the class of complexes of S-modules that are isomorphic in $\mathfrak{D}^b(\text{mod-}S)$ to the complex $\mathrm{Hom}_R(P^\bullet, C^\bullet)$. In this way, $\mathbf{R}\mathrm{Hom}_R(-, C^\bullet)$ yields a functor $\mathfrak{D}^b(R\text{-mod}) \to \mathfrak{D}^b(\text{mod-}S)$. Likewise, $\mathbf{R}\mathrm{Hom}_S(-, C^\bullet)$ yields a functor $\mathfrak{D}^b(\text{mod-}S) \to \mathfrak{D}^b(R\text{-mod})$.

Let $s \in S$. Then we see that the right multiplication $\rho(s) : C^\bullet \to C^\bullet$ is a chain map of R-complexes. Take a projective resolution $P^\bullet$ of $C^\bullet$ as a complex in $\mathfrak{D}^b(R\text{-mod})$ and a chain map $\psi : P^\bullet \to C^\bullet$ of R-complexes. Combining these two, we have a chain map $h(s) = \rho(s) \cdot \psi : P^\bullet \to C^\bullet$, which defines an element of degree 0 in the complex $\mathrm{Hom}_R(P^\bullet, C^\bullet)$. In such a way, we obtain the morphism $h : S \to \mathbf{R}\mathrm{Hom}_R(C^\bullet, C^\bullet)$ in $\mathfrak{D}^b(\text{mod-}S)$, which we call the right homothety morphism. Likewise, we have the left homothety morphism $R \to \mathbf{R}\mathrm{Hom}_S(C^\bullet, C^\bullet)$ in $\mathfrak{D}^b(R\text{-mod})$.

Definition 3.4.1 Let $C^\bullet$ be a complex consisting of (R,S)-bimodules and (R,S)-bimodule homomorphisms as above. We call $C^\bullet$ a *semi-dualizing complex of bimodules* if the following conditions hold.

(1) The complex $C^\bullet$ is bounded, that is, there are only a finite number of i with $H^i(C^\bullet) \neq 0$.

(2) The right homothety morphism $S \to \mathbf{R}\mathrm{Hom}_R(C^\bullet, C^\bullet)$ is an isomorphism in $\mathfrak{D}^b(\text{mod-}S)$.

(3) The left homothety morphism $R \to \mathbf{R}\mathrm{Hom}_S(C^\bullet, C^\bullet)$ is an isomorphism in $\mathfrak{D}^b(R\text{-mod})$.

Definition 3.4.2 We denote by ${}_R\mathcal{A}(C^\bullet)$ the full subcategory of $\mathfrak{D}^b(R\text{-mod})$ consisting of all complexes $M^\bullet \in \mathfrak{D}^b(R\text{-mod})$ that satisfy the following conditions.

(1) The complex $\mathbf{R}\mathrm{Hom}_R(M^\bullet, C^\bullet)$ of S-modules belongs to $\mathfrak{D}^b(\text{mod-}S)$.

(2) The natural morphism $M^\bullet \to \mathbf{R}\mathrm{Hom}_S(\mathbf{R}\mathrm{Hom}_R(M^\bullet, C^\bullet), C^\bullet)$ gives an isomorphism in $\mathfrak{D}^b(R\text{-mod})$.

Similarly we can define $\mathcal{A}_S(C^\bullet)$ as the full subcategory of $\mathfrak{D}^b(\text{mod-}S)$ consisting of all complexes $N^\bullet$ that satisfy the following conditions.

(1') The complex $\mathbf{R}\mathrm{Hom}_S(N^\bullet, C^\bullet)$ of R-modules belongs to $\mathfrak{D}^b(R\text{-mod})$.

(2') The natural morphism $N^\bullet \to \mathbf{R}\mathrm{Hom}_R(\mathbf{R}\mathrm{Hom}_S(N^\bullet, C^\bullet), C^\bullet)$ gives an isomorphism in $\mathfrak{D}^b(\text{mod-}S)$.

Definition 3.4.3 (1) For a complex $M^\bullet \in \mathfrak{D}^b(R\text{-mod})$, we define the ${}_R\mathcal{A}(C^\bullet)$-dimension of $M^\bullet$ as

$$ {}_R\mathcal{A}(C^\bullet)\text{-}\dim M^\bullet = \begin{cases} s(\mathbf{R}\mathrm{Hom}_R(M^\bullet, C^\bullet)) & \text{if } M^\bullet \in {}_R\mathcal{A}(C^\bullet), \\ +\infty & \text{otherwise.} \end{cases} $$

(2) Similarly we define the $\mathcal{A}_S(C^\bullet)$-dimension of a complex $N^\bullet \in \mathfrak{D}^b(\text{mod-}S)$ as

$$ \mathcal{A}_S(C^\bullet)\text{-}\dim N^\bullet = \begin{cases} s(\mathbf{R}\mathrm{Hom}_S(N^\bullet, C^\bullet)) & \text{if } N^\bullet \in {}_R\mathcal{A}(C^\bullet), \\ +\infty & \text{otherwise.} \end{cases} $$

Theorem 3.4.4 *Let $C^\bullet$ be a semi-dualizing complex of (R, S)-bimodules. Then the functors $\mathbf{R}\mathrm{Hom}_R(-, C^\bullet)$ and $\mathbf{R}\mathrm{Hom}_S(-, C^\bullet)$ give rise to a duality between ${}_R\mathcal{A}(C^\bullet)$ and $\mathcal{A}_S(C^\bullet)$.*

Proof Let $X^\bullet \in {}_R\mathcal{A}(C^\bullet)$. Putting $Y^\bullet = \mathbf{R}\mathrm{Hom}_R(X^\bullet, C^\bullet)$, we should note that $\mathbf{R}\mathrm{Hom}_S(Y^\bullet, C^\bullet) = \mathbf{R}\mathrm{Hom}_S(\mathbf{R}\mathrm{Hom}_R(X^\bullet, C^\bullet), C^\bullet) \cong X^\bullet$, which belongs to $\mathfrak{D}^b(R\text{-mod})$. We also have

$$ \begin{aligned} Y^\bullet &= \mathbf{R}\mathrm{Hom}_R(X^\bullet, C^\bullet) \\ &\cong \mathbf{R}\mathrm{Hom}_R(\mathbf{R}\mathrm{Hom}_S(\mathbf{R}\mathrm{Hom}_R(X^\bullet, C^\bullet), C^\bullet), C^\bullet) \\ &= \mathbf{R}\mathrm{Hom}_R(\mathbf{R}\mathrm{Hom}_S(Y^\bullet, C^\bullet), C^\bullet). \end{aligned} $$

It hence follows that $Y^\bullet \in \mathcal{A}_S(C^\bullet)$. Since

$$ \begin{aligned} X^\bullet &\cong \mathbf{R}\mathrm{Hom}_S(\mathbf{R}\mathrm{Hom}_R(X^\bullet, C^\bullet), C^\bullet) \\ &= \mathbf{R}\mathrm{Hom}_S(Y^\bullet, C^\bullet), \end{aligned} $$

we see that the functor $\mathbf{R}\mathrm{Hom}_R(-, C^\bullet)$ is well-defined and dense. It follows from the next lemma that this is fully faithful. $\Box$

Lemma 3.4.5 *Let $X^\bullet \in {}_R\mathcal{A}(C^\bullet)$ and $M^\bullet \in \mathfrak{D}^b(R\text{-mod})$. Then the natural morphism*

$$ \mathbf{R}\mathrm{Hom}_R(M^\bullet, X^\bullet) \to \mathbf{R}\mathrm{Hom}_S(\mathbf{R}\mathrm{Hom}_R(X^\bullet, C^\bullet), \mathbf{R}\mathrm{Hom}_R(M^\bullet, C^\bullet)) $$

is an isomorphism.

Proof First of all, note that there is a natural morphism of functors

$$\Phi : \mathbf{R}\mathrm{Hom}_R(-, X^\bullet) \to \mathbf{R}\mathrm{Hom}_S(\mathbf{R}\mathrm{Hom}_R(X^\bullet, C^\bullet), \mathbf{R}\mathrm{Hom}_R(-, C^\bullet)),$$

considering the both functors to be functors from $\mathfrak{D}^b(R\text{-mod})$ to the category of abelian groups.

We shall prove that $\Phi(M^\bullet)$ gives an isomorphism for any $M^\bullet \in \mathfrak{D}^b(R\text{-mod})$. If $M^\bullet \cong R$, then this is obvious because $X^\bullet \in {}_R\mathcal{A}(C^\bullet)$. Hence, the assertion also holds if $\mathrm{pd} M^\bullet < \infty$, i.e. $M^\bullet$ is isomorphic in $\mathfrak{D}^b(R\text{-mod})$ to a projective complex of finite length. Now to discuss the general case, let $P^\bullet$ be a standard projective resolution of $M^\bullet$. For any $n, m \in \mathbb{Z}$, since there is a natural morphism of complexes $\tau^{\geq m} P^\bullet \to P^\bullet$, and since $\mathrm{pd}\, \tau^{\geq m} P^\bullet < \infty$, we have the following commutative diagram.

$$\begin{array}{ccc}
H^n(\mathbf{R}\mathrm{Hom}_R(M^\bullet, X^\bullet)) & \xrightarrow{H^n\Phi(M^\bullet)} & H^n(\mathbf{R}\mathrm{Hom}_S((X^\bullet)^\dagger, (M^\bullet)^\dagger)) \\
\downarrow & & \downarrow \\
H^n(\mathbf{R}\mathrm{Hom}_R(\tau^{\geq m} P^\bullet, X^\bullet)) & \xrightarrow{\cong} & H^n(\mathbf{R}\mathrm{Hom}_S((X^\bullet)^\dagger, (\tau^{\geq m} P^\bullet)^\dagger))
\end{array}$$

Here, we set $(-)^\dagger = \mathbf{R}\mathrm{Hom}_R(-, C^\bullet)$.

For a given n, we shall prove the above vertical arrows are isomorphisms if we choose as m a small enough number to compare with n, i.e. $m \ll n$. If one can choose such m, then one sees that $H^n\Phi(M^\bullet)$ is an isomorphism for any $n \in \mathbb{Z}$, and consequently $\Phi(M^\bullet)$ is a quasi-isomorphism of complexes as desired.

Let $P^\bullet$ be a standard R-projective resolution of $M^\bullet$, and hence $P^i = 0$ for $i > s(M^\bullet)$. Similarly let

$$I^\bullet = \cdots \longrightarrow I^{\ell-1} \longrightarrow I^{\ell+1} \longrightarrow I^{\ell+1} \longrightarrow \cdots$$

be an R-injective resolution of $X^\bullet$ with $I^i = 0$ for $i < i(X^\bullet)$. For any integer $n \in \mathbb{Z}$, we take an integer $m \in \mathbb{Z}$ with $m \leq -n + i(X^\bullet) + 1$. Note that for integers a, b, k, if $b - a = k$, $k \leq n + 1$ and $b \geq i(X^\bullet)$ then $a \geq m$. Note also that the complexes $P^\bullet$ and $\tau^{\geq m} P^\bullet$ have the common components in the degrees not less than m. Therefore, for $k \leq n+1$, the k-th component of the complex $\mathrm{Hom}_R(\tau^{\geq m} P^\bullet, I^\bullet)$ that is of the form

$$\bigoplus_{\substack{a \geq m \\ b-a=k}} \mathrm{Hom}_R(P^a, I^b)$$

coincides with the k-th component of $\mathrm{Hom}_R(P^\bullet, I^\bullet)$ that is of the form

$$\bigoplus_{b-a=k} \mathrm{Hom}_R(P^a, I^b).$$

In particular we see that

$$H^n(\mathbf{R}\mathrm{Hom}_R(\tau^{\geq m}P^\bullet, I^\bullet)) \cong H^n(\mathbf{R}\mathrm{Hom}_R(P^\bullet, I^\bullet))$$

whenever $m \leq -n + i(X^\bullet) + 1$.

Similarly, we have an isomorphism

$$\begin{aligned} & H^n(\mathbf{R}\mathrm{Hom}_S(\mathbf{R}\mathrm{Hom}_R(X^\bullet, C^\bullet), \mathbf{R}\mathrm{Hom}_R(M^\bullet, C^\bullet))) \\ \cong \; & H^n(\mathbf{R}\mathrm{Hom}_S(\mathbf{R}\mathrm{Hom}_R(X^\bullet, C^\bullet), \mathbf{R}\mathrm{Hom}_R(\tau^{\geq m}P^\bullet_{M^\bullet}, C^\bullet))) \end{aligned}$$

if $m \leq -n + i(C^\bullet) - s(\mathbf{R}\mathrm{Hom}_R(X^\bullet, C^\bullet)) + 1$. Thus the proof is completed. □

3.5 $\mathcal{A}(C^\bullet)$-dimension in commutative case

In this final section of the chapter, we shall observe several properties of ${}_R\mathcal{A}(C)$-dimension in case when R and S are commutative local rings. We begin with the following lemma.

Lemma 3.5.1 *Let R and S be commutative noetherian rings. Suppose that there exists a semi-dualizing complex $C^\bullet$ of (R, S)-bimodules. Then R is isomorphic to S.*

Proof Let $\phi : R \to \mathbf{R}\mathrm{Hom}_R(C^\bullet, C^\bullet) = S$ and $\psi : S \to \mathbf{R}\mathrm{Hom}_S(C^\bullet, C^\bullet) = R$ be the homothety morphisms. Since R and S are commutative, we see that they are well-defined ring homomorphism and that $\psi\phi$ (resp. $\phi\psi$) is the identity map on R (resp. S). Hence $R \cong S$ as desired. □

In view of this lemma, we may assume that R coincides with S for our purpose of this section. Thus we may call a semi-dualizing complex of (R, S)-bimodules simply a semi-dualizing complex. For a semi-dualizing complex $C^\bullet$, we simply write $\mathcal{A}(C^\bullet)$ for ${}_R\mathcal{A}(C^\bullet)$. Note that $\mathcal{A}(C^\bullet)$-dim $M^\bullet$ is essentially the same as G-dim$_{C^\bullet} M^\bullet$ in [20] and $\mathrm{G}_{C^\bullet}$- dim $M^\bullet$ in [26].

From now on, we assume that R is a commutative noetherian local ring with unique maximal ideal $\mathfrak{m}$ and residue class field $k = R/\mathfrak{m}$. It is known that $\mathcal{A}(C^\bullet)$-dim $M^\bullet$ satisfies the Auslander-Buchsbaum-type equality as well as G-dim$_{C^\bullet} M^\bullet$.

Lemma 3.5.2 [20, Theorem 3.14] *For $M^\bullet \in \mathcal{A}(C^\bullet)$,*

$$\mathcal{A}(C^\bullet)\text{-dim}\, M^\bullet = \operatorname{depth} R - \operatorname{depth} M^\bullet + s(C^\bullet),$$

where the depth depth $M^\bullet$ *of a complex $M^\bullet$ is defined to be $i(\mathbf{R}\mathrm{Hom}(k, M^\bullet))$.*

We are now able to state the main result of this section.

Theorem 3.5.3 *The following conditions are equivalent for a local ring $(R, \mathfrak{m}, k)$.*

(1) *R is a Cohen-Macaulay local ring that is a homomorphic image of a Gorenstein local ring.*

(2) *For any $M \in R$-mod there exists a semi-dualizing module C such that $\mathcal{A}(C)$-$\dim M < \infty$.*

(3) *There exists a semi-dualizing module C such that $\mathcal{A}(C)$-$\dim k < \infty$.*

(4) *For any $M^\bullet \in \mathfrak{D}^b(R$-mod$)$ there exists a semi-dualizing module C such that $\mathcal{A}(C)$-$\dim M^\bullet < \infty$.*

(5) *There exists a semi-dualizing module C such that $\mathcal{A}(C) = \mathfrak{D}^b(R$-mod$)$.*

(6) *The dualizing complex $D^\bullet$ exists and there exists a semi-dualizing module C such that $\mathcal{A}(C)$-$\dim D^\bullet < \infty$.*

Proof The implications (5) $\Rightarrow$ (4) $\Rightarrow$ (2) $\Rightarrow$ (3) are trivial.

(3) $\Rightarrow$ (1): Since $\mathcal{A}(C)$-$\dim k < \infty$, we have $\mathrm{Ext}_R^n(k, C) = 0$ for $n \gg 0$. Hence we see from Lemma 3.5.2 that the injective dimension of C is finite. Therefore R is Cohen-Macaulay. (It is well-known that a commutative local ring which admits a finitely generated module of finite injective dimension is Cohen-Macaulay. For example, see [44].) Note that

$$\begin{aligned} \operatorname{depth} C &= -\mathcal{A}(C)\text{-}\dim C + \operatorname{depth} R + s(C) \\ &= \operatorname{depth} R \\ &= \dim R. \end{aligned}$$

That is to say, C is a maximal Cohen-Macaulay module. Since the isomorphism $\mathrm{Ext}_R^d(\mathrm{Ext}_R^d(k, C), C) \cong k$, where $d = \dim R$, holds, one can show that C is the canonical module of R. The existence of the canonical module of R implies that R is a homomorphic image of a Gorenstein local ring. (See Reiten [43, Theorem (3)] or Foxby [22, Theorem 4.1].)

(1) $\Rightarrow$ (6): It follows from the condition (1) that R admits the canonical module K_R. Note that K_R is a semi-dualizing module and isomorphic to the dualizing complex in $\mathfrak{D}^b(R$-mod$)$. Hence $\mathcal{A}(K_R)$-$\dim K_R = 0 < \infty$.

(6) $\Rightarrow$ (5): We may assume that $i(D^\bullet) = 0$. Then note that $\operatorname{depth} D^\bullet = \dim R$. It follows from Lemma 3.5.2 that

$$\begin{aligned} \mathcal{A}(C)\text{-}\dim D^\bullet &= \operatorname{depth} R - \operatorname{depth} D^\bullet + s(C) \\ &= \operatorname{depth} R - \dim R \\ &\leq 0. \end{aligned}$$

On the other hand, from Lemma 3.3.10 we have that $\mathcal{A}(C)\text{-}\dim D^\bullet = \mathcal{A}(C)\text{-}\dim D^\bullet + i(D^\bullet) \geq 0$. Consequently, we have $\dim R = \operatorname{depth} R$. Hence R is Cohen-Macaulay. And this implies that $D^\bullet$ is isomorphic to the canonical module K_R of R. It is obvious that K_R is a semi-dualizing module and every maximal Cohen-Macaulay module is K_R-reflexive. As a result, every R-module has finite $\mathcal{A}(K_R)$-dimension, hence $\mathcal{A}(K_R)$ contains all R-modules. Then it follows from Theorem 3.3.12 that $\mathcal{A}(K_R)$ contains all complexes in $\mathfrak{D}^b(R\text{-mod})$, and hence $\mathcal{A}(K_R) = \mathfrak{D}^b(R\text{-mod})$. □

Similarly to the above theorem, we can get a result for semi-dualizing complexes.

Theorem 3.5.4 *The following conditions are equivalent for a local ring* $(R, \mathfrak{m}, k)$.

(1) *R is a homomorphic image of a Gorenstein local ring.*

(2) *For any $M \in R$-mod, there exists a semi-dualizing complex $C^\bullet$ such that $\mathcal{A}(C^\bullet)\text{-}\dim M < \infty$.*

(3) *There exists a semi-dualizing complex $C^\bullet$ such that $\mathcal{A}(C^\bullet)\text{-}\dim k < \infty$.*

(4) *For any $M^\bullet \in \mathfrak{D}^b(R\text{-mod})$, there exists a semi-dualizing complex $C^\bullet$ such that $\mathcal{A}(C^\bullet)\text{-}\dim M < \infty$.*

(5) *There exists a semi-dualizing complex $C^\bullet$ such that $\mathcal{A}(C^\bullet) = \mathfrak{D}^b(R\text{-mod})$.*

(6) *The dualizing complex $D^\bullet$ exists.*

Proof It is easy to prove the implications (1) $\Rightarrow$ (6) $\Rightarrow$ (5) $\Rightarrow$ (4) $\Rightarrow$ (2) $\Rightarrow$ (3) $\Rightarrow$ (6). The remaining implication (6) $\Rightarrow$ (1) that is the most difficult to prove follows from [33, Theorem 1.2]. □

As final part of the chapter we are discussing a kind of uniqueness property of semi-dualizing complexes.

Theorem 3.5.5 *Let $C_1^\bullet$ and $C_2^\bullet$ be semi-dualizing complexes. Suppose that $C_1^\bullet \in \mathcal{A}(C_2^\bullet)$ and $C_2^\bullet \in \mathcal{A}(C_1^\bullet)$. Then $C_1^\bullet \cong C_2^\bullet[a]$ for some $a \in \mathbb{Z}$. In particular, we have $\mathcal{A}(C_1^\bullet) = \mathcal{A}(C_2^\bullet)$.*

For the proof this theorem we need the notion of Poincare and Bass series of a complex.

Remark 3.5.6 Let $(R, \mathfrak{m}, k)$ be a commutative noetherian local ring. For a complex $M^\bullet \in \mathfrak{D}^b(R\text{-mod})$, consider two kinds of formal power series in variable t;

$$\begin{aligned} \mathrm{P}_{M^\bullet}(t) &= \textstyle\sum_{n\in\mathbb{Z}} \dim_k H^{-n}(M^\bullet \otimes^{\mathbf{L}}_R k) \cdot t^n, \\ \mathrm{I}^{M^\bullet}(t) &= \textstyle\sum_{n\in\mathbb{Z}} \dim_k H^n(\mathbf{R}\mathrm{Hom}_R(k, M^\bullet)) \cdot t^n. \end{aligned}$$

These series are called respectively the Poincare series and the Bass series of $M^\bullet$. As it is shown in Foxby [24, Theorem 4.1(a)], the following equality holds for $M^\bullet, N^\bullet \in \mathfrak{D}^b(R\text{-mod})$.

$$\mathrm{I}^{\mathbf{R}\mathrm{Hom}(M^\bullet,N^\bullet)}(t) = \mathrm{P}_{M^\bullet}(t) \cdot \mathrm{I}^{N^\bullet}(t) \tag{3.9}$$

Proof Since $C_1^\bullet \in \mathcal{A}(C_2^\bullet)$, we have

$$C_1^\bullet \cong \mathbf{R}\mathrm{Hom}(\mathbf{R}\mathrm{Hom}(C_1^\bullet, C_2^\bullet), C_2^\bullet).$$

Hence, we have from (3.9) that

$$\mathrm{I}^{C_1^\bullet}(t) = \mathrm{P}_{\mathbf{R}\mathrm{Hom}(C_1^\bullet,C_2^\bullet)}(t) \cdot \mathrm{I}^{C_2^\bullet}(t).$$

Likewise, it follows from $C_2^\bullet \cong \mathbf{R}\mathrm{Hom}(\mathbf{R}\mathrm{Hom}(C_2^\bullet, C_1^\bullet), C_1^\bullet)$ that

$$\mathrm{I}^{C_2^\bullet}(t) = \mathrm{P}_{\mathbf{R}\mathrm{Hom}(C_2^\bullet,C_1^\bullet)}(t) \cdot \mathrm{I}^{C_1^\bullet}(t).$$

Therefore we have $\mathrm{P}_{\mathbf{R}\mathrm{Hom}(C_1^\bullet,C_2^\bullet)}(t) \cdot \mathrm{P}_{\mathbf{R}\mathrm{Hom}(C_2^\bullet,C_1^\bullet)}(t) = 1$. Now, write

$$\begin{cases} \mathrm{P}_{\mathbf{R}\mathrm{Hom}(C_1^\bullet,C_2^\bullet)}(t) = \alpha_{-a}t^{-a} + \alpha_{-a+1}t^{-a+1} + \cdots & (\alpha_{-a} \neq 0) \\ \mathrm{P}_{\mathbf{R}\mathrm{Hom}(C_2^\bullet,C_1^\bullet)}(t) = \beta_{-b}t^{-b} + \beta_{-b+1}t^{-b+1} + \cdots & (\beta_{-b} \neq 0), \end{cases}$$

and we have

$$(\alpha_{-a}t^{-a} + \alpha_{-a+1}t^{-a+1} + \cdots)(\beta_{-b}t^{-b} + \beta_{-b+1}t^{-b+1} + \cdots) = 1$$

Noting that each of α_i and β_j is a non-negative integer, we see that this equality implies that $\alpha_{-a} = 1$ and $\alpha_i = 0$ for any $i \neq -a$. Therefore we have that $\mathbf{R}\mathrm{Hom}(C_1^\bullet, C_2^\bullet) \cong R[-a]$. Thus it follows that

$$\begin{aligned} C_1^\bullet &\cong \mathbf{R}\mathrm{Hom}(\mathbf{R}\mathrm{Hom}(C_1^\bullet, C_2^\bullet), C_2^\bullet) \\ &\cong \mathbf{R}\mathrm{Hom}(R[-a], C_2^\bullet) \\ &\cong C_2^\bullet[a], \end{aligned}$$

as desired. □

Finally we have an interesting corollary of this theorem.

Corollary 3.5.7 *Suppose that R admits the dualizing complex $D^\bullet$. Then R is a Gorenstein ring if and only if* G-dim $D^\bullet < \infty$.

Proof If R is Gorenstein then $D^\bullet \cong R$ thus G-dim $D^\bullet =$ G-dim $R = 0$. Conversely, assume G-dim$D^\bullet < \infty$. Then we have $D^\bullet \in \mathcal{A}(R)$. On the other hand, we have $R \in \mathcal{A}(D^\bullet)$, more generally $\mathcal{A}(D^\bullet)$ contains all R-modules by the definition of dualizing complex. Hence it follows from the theorem that $D^\bullet \cong R[a]$ for some $a \in \mathbb{Z}$, which means R is a Gorenstein ring. □

4 Cohen-Macaulay dimension over a ring of characteristic p

The contents of this chapter are entirely contained in the author's paper [53] with Y. Yoshino.

Let R be a commutative noetherian local ring of prime characteristic. Denote by ${}^eR$ the ring R regarded as an R-algebra through e-times composition of the Frobenius map. Suppose that R is F-finite, i.e. the R-algebra ${}^1R$ is finitely generated as an R-module. We shall prove in this chapter that R is Cohen-Macaulay if and only if the R-modules ${}^eR$ have finite Cohen-Macaulay dimensions for infinitely many integers e. We will also give a similar criterion for the Gorenstein property of R.

4.1 Introduction

Throughout this chapter, we assume that all rings are commutative and noetherian.

Let R be a local ring of characteristic p, where p is a prime number. Let $f : R \to R$ be the Frobenius map, which is given by $a \mapsto a^p$. For an integer e, we denote by $f^e : R \to R$ the e-th power of f, that is, it is given by $a \mapsto a^{p^e}$. We denote by ${}^eR$ the R-algebra R whose R-algebra structure is given via f^e. The ring R is said to be *F-finite* if ${}^1R$, and hence every ${}^eR$, is a finitely generated R-module.

Kunz [35] proved that the local ring R is regular if and only if ${}^1R$ is flat over R. Rodicio [46] gave a generalization of this result as follows.

Theorem 4.1.1 (Rodicio) *A local ring R of prime characteristic is regular if and only if the R-module ${}^1R$ has finite flat dimension.*

A similar result concerning the complete intersection property and complete intersection dimension (abbr. CI-dimension) was proved by Blanco and Majadas [17]. They actually proved that a local ring R of prime characteristic is a complete intersection if and only if there exists a Cohen factorization $R \to S \to \widehat{{}^1R}$ such that the CI-dimension of $\widehat{{}^1R}$ over S is finite, where $\widehat{{}^1R}$ denotes the completion of the local ring ${}^1R$. (For the definition of a Cohen factorization, see [12].) Therefore, we have in particular the following.

Theorem 4.1.2 (Blanco-Majadas) *An F-finite local ring R is a complete intersection if and only if the R-module ${}^1R$ has finite CI-dimension.*

The main purpose of this chapter is to give a similar theorem for Cohen-Macaulay dimension to characterize Cohen-Macaulay local rings of prime characteristic. In Section 3, we shall consider F-finite local rings, and prove the following theorem. (In the following theorem, $\nu(R)$ is an invariant of the local ring R, whose definition we will give in Section 3.)

Theorem 4.3.6 *Let R be an F-finite local ring of characteristic p. Then the following conditions are equivalent.*

(1) *R is Cohen-Macaulay.*

(2) $\text{CM-dim}_R {}^eR < \infty$ *for all integers e.*

(3) $\text{CM-dim}_R {}^eR < \infty$ *for infinitely many integers e.*

(4) $\text{CM-dim}_R {}^eR < \infty$ *for some integer e with $p^e \geq \nu(R)$.*

To prove this theorem, the following lemma will be a key which plays an essential role in proving several other results of this chapter.

Lemma 4.3.3 *Let $(R, \mathfrak{m}, k) \to (S, \mathfrak{n}, l)$ be a homomorphism of local rings. Suppose that $\mathfrak{m}S \subseteq \mathfrak{n}^{\nu(S)}$. Then, there is an S-regular sequence $\boldsymbol{y}$ such that k is isomorphic to a direct summand of $S/(\boldsymbol{y})$ as an R-module.*

In Section 5, we shall also make a consideration for local rings which are not necessarily F-finite, and prove the following result that generalizes Theorem 4.3.6.

Theorem 4.5.3 *Let $(R, \mathfrak{m}, k)$ be a local ring of characteristic p. Then the following conditions are equivalent.*

(1) *R is Cohen-Macaulay.*

(2) *There exist a local ring homomorphism $R \to S$ and a finitely generated S-module N satisfying the following two conditions.*

  i) *There is a maximal R-regular sequence that is also N-regular.*

  ii) *There is an integer e such that $p^e \geq \nu(R)$ and that $\mathrm{Ext}_R^i({}^eR, N) = 0$ for any sufficiently large integer i.*

The organization of this chapter is as follows. In Section 2, we recall the definitions and elementary properties of Cohen-Macaulay dimension and Gorenstein dimension without proofs. In Section 3, we shall give proofs of Lemma 4.3.3 and Theorem 4.3.6. In Section 4, we make a generalization of a theorem of Herzog [30], which gives a sufficient condition for a module to have finite projective dimension or finite injective dimension. Using Lemma 4.3.3 and a result due to Avramov and Foxby [10], we shall actually refine the result of Herzog in Theorem 4.4.5. In Section 5, we shall give a proof of Theorem 4.5.3. In Section 6, we give a characterization of F-finite Gorenstein local rings. It is similar to Theorem 4.3.6, but is stated in a slightly stronger form.

4.2 Preliminaries

In this section, we shall recall the definitions of Cohen-Macaulay dimension defined by Gerko [26] and Gorenstein dimension defined by Auslander [3], and state several properties of those dimensions for later use.

Throughout this section, R always denotes a local ring with residue field k.

In order to define Cohen-Macaulay dimension and Gorenstein dimension, we begin with observing the notion of semi-dualizing module, which has been studied by Foxby [22], Golod [27], Christensen [20], and Gerko [26]. (A semi-dualizing module is a *PG-module* which appears in [22], and is called a *suitable module* in [27] and [26].) We denote by $\mathcal{D}^b(R)$ the derived category of the category of finitely generated R-modules.

Definition 4.2.1 (1) A finitely generated R-module C is said to be *semi-dualizing* if the natural homomorphism $R \to \mathrm{Hom}_R(C, C)$ is an isomorphism and $\mathrm{Ext}^i_R(C, C) = 0$ for $i > 0$, equivalently the natural morphism $R \to \mathbf{R}\mathrm{Hom}_R(C, C)$ is an isomorphism in $\mathcal{D}^b(R)$.

(2) For a semi-dualizing R-module C, we say that a finitely generated R-module M is *totally C-reflexive* if the natural homomorphism $M \to \mathrm{Hom}_R(\mathrm{Hom}_R(M, C), C)$ is an isomorphism and $\mathrm{Ext}^i_R(M, C) = \mathrm{Ext}^i_R(\mathrm{Hom}_R(M, C), C) = 0$ for $i > 0$.

One should note that R is always a semi-dualizing R-module.

Semi-dualizing modules satisfy several properties as in the following proposition. For their proofs, one refers to [20] and [22]. Here, for a finitely generated R-module M, $\mu_R(M)$ denotes the minimal number of generators of M, and $r_R(M)$ denotes the type of M, i.e. $\mu_R(M) = \dim_k(M \otimes_R k)$ and $r_R(M) = \dim_k \mathrm{Ext}^t_R(k, M)$ where $t = \mathrm{depth}_R M$.

Proposition 4.2.2 *Let C be a semi-dualizing R-module. Then the following properties hold.*

(1) *The R-module C is faithful.*

(2) *A sequence of elements of R is R-regular if and only if it is C-regular. In particular,* $\operatorname{depth}_R C = \operatorname{depth} R$.

(3) *One has the equality* $\mu_R(C)\, r_R(C) = r_R(R)$.

Now let us state the definitions of Cohen-Macaulay dimension and Gorenstein dimension. We denote by $\Omega_R^n M$ the n-th syzygy module of an R-module M.

Definition 4.2.3 Let M be a finitely generated R-module.

(1) If there is a faithfully flat homomorphism $R \to S$ of local rings together with a semi-dualizing S-module C such that the module $\Omega_S^n(M \otimes_R S)$ is totally C-reflexive, then we say that the *Cohen-Macaulay dimension* (abbr. CM-dimension) of M is not larger than n, and denote $\text{CM-dim}_R M \leq n$. If there does not exist such an integer n, we say that the CM-dimension of M is infinite, and denote $\text{CM-dim}_R M = \infty$.

(2) If the module $\Omega_R^n M$ is totally R-reflexive, then we say that the *Gorenstein dimension* (abbr. G-dimension) of M is not larger than n, and denote $\text{G-dim}_R M \leq n$. If there does not exist such an integer n, we say that the G-dimension of M is infinite, and denote $\text{G-dim}_R M = \infty$.

Remark 4.2.4 In [26] Gerko defines CM-dimension in two different ways. Compare [26, Definition 3.2] with [26, Definition 3.2']. We have adopted the latter one in the above definition. Since he proves in his paper that both coincide with each other, the following conditions are equivalent for a finitely generated R-module M.

(1) $\text{CM-dim}_R M < \infty$.

(2) There exist a faithfully flat homomorphism $R \to S$ and a surjective homomorphism $T \to S$ of local rings such that $\text{G-dim}_T S = \operatorname{grade}_T S$ ($< \infty$) and that $\text{G-dim}_T(M \otimes_R S) < \infty$.

G-dimension and CM-dimension satisfy many properties as follows. Notice that these properties are quite similar to those of projective dimension. For their proofs, see [4], [7], [19], [26], [36], and [57]

Proposition 4.2.5 (1) *If* $\text{G-dim}_R M < \infty$ *for a finitely generated* R*-module* M*, then* $\text{G-dim}_R M = \operatorname{depth} R - \operatorname{depth}_R M$*.*

(2) *The following conditions are equivalent.*

   i) *R is Gorenstein.*

   ii) $\text{G-dim}_R M < \infty$ *for any finitely generated* R*-module* M*.*

   iii) $\text{G-dim}_R k < \infty$*.*

(3) *Let* N *be a direct summand of an* R*-module* M*. Then* $\text{G-dim}_R N < \infty$ *whenever* $\text{G-dim}_R M < \infty$*.*

(4) *Let* $0 \to L \to M \to N \to 0$ *be a short exact sequence of finitely generated* R*-modules. If any two of* L, M, N *have finite G-dimension, then so do the all.*

(5) *The following conditions are equivalent for a finitely generated* R*-module.*

   i) $\text{G-dim}_R M < \infty$*.*

   ii) *The natural morphism* $M \to \mathbf{R}\mathrm{Hom}_R(\mathbf{R}\mathrm{Hom}_R(M, R), R)$ *is an isomorphism in* $\mathcal{D}^b(R)$*.*

(6) *If* $\text{CM-dim}_R M < \infty$ *for a finitely generated* R*-module* M*, then* $\text{CM-dim}_R M = \operatorname{depth} R - \operatorname{depth}_R M$*.*

(7) *The following conditions are equivalent.*

   i) *R is Cohen-Macaulay.*

   ii) $\text{CM-dim}_R M < \infty$ *for any finitely generated* R*-module* M*.*

   iii) $\text{CM-dim}_R k < \infty$*.*

(8) *Let* N *be a direct summand of a finitely generated* R*-module* M*. Then* $\text{CM-dim}_R N < \infty$ *whenever* $\text{CM-dim}_R M < \infty$*.*

(9) $\text{CM-dim}_R M \leq \text{G-dim}_R M$ *for any finitely generated* R*-module* M*. In particular, a module of finite G-dimension is also of finite CM-dimension.*

4.3 A characterization of Cohen-Macaulay rings

First of all, we introduce an invariant of a local ring.

Definition 4.3.1 Let R be a local ring with maximal ideal $\mathfrak{m}$. Take a maximal R-regular sequence $\boldsymbol{x} = x_1, x_2, \cdots, x_t$ where $t = \operatorname{depth} R$. Then, note that the 0-th local cohomology module $H^0_{\mathfrak{m}}(R/(\boldsymbol{x}))$ of $R/(\boldsymbol{x})$ with respect to $\mathfrak{m}$ is a non-trivial submodule of $R/(\boldsymbol{x})$ of finite length, and hence it follows from Artin-Rees lemma that $H^0_{\mathfrak{m}}(R/(\boldsymbol{x})) \cap \mathfrak{m}^n(R/(\boldsymbol{x})) = 0$ for any sufficiently large n. We define the integral valued invariant $\nu(R)$ to be the smallest integer n satisfying $H^0_{\mathfrak{m}}(R/(\boldsymbol{x})) \cap \mathfrak{m}^n(R/(\boldsymbol{x})) = 0$ for some maximal R-regular sequence $\boldsymbol{x}$.

Remark 4.3.2 (1) Several invariants similar to the above invariant has been investigated in [34].

(2) Note from definition that R is a regular local ring if and only if $\nu(R) = 1$. Furthermore, if $(R, \mathfrak{m})$ is a Cohen-Macaulay local ring, then $\nu(R)$ is the smallest integer n such that $\mathfrak{m}^n$ is contained in some parameter ideal of R. Hence it is not bigger than the multiplicity of R with respect to $\mathfrak{m}$.

(3) Let $(R, \mathfrak{m}) \to (S, \mathfrak{n})$ be a faithfully flat homomorphism of local rings. Assume that its closed fiber is artinian, and let l be the smallest integer with $\mathfrak{n}^l \subseteq \mathfrak{m}S$. Then we have the inequality $\nu(S) \leq l \cdot \nu(R)$.

In fact, take a maximal R-regular sequence $\boldsymbol{x}$ with $H^0_{\mathfrak{m}}(R/(\boldsymbol{x})) \cap \mathfrak{m}^{\nu(R)}(R/(\boldsymbol{x})) = 0$. Then, since the induced map $R/(\boldsymbol{x}) \to S/\boldsymbol{x}S$ is also faithfully flat, it is easy to see that $H^0_{\mathfrak{n}}(S/\boldsymbol{x}S) = H^0_{\mathfrak{m}}(R/(\boldsymbol{x}))(S/\boldsymbol{x}S)$ and that

$$\begin{aligned} H^0_{\mathfrak{n}}(S/\boldsymbol{x}S) \cap \mathfrak{n}^{l\cdot\nu(R)}(S/\boldsymbol{x}S) &\subseteq H^0_{\mathfrak{m}}(R/(\boldsymbol{x}))(S/\boldsymbol{x}S) \cap \mathfrak{m}^{\nu(R)}(S/\boldsymbol{x}S) \\ &= \left(H^0_{\mathfrak{m}}(R/(\boldsymbol{x})) \cap \mathfrak{m}^{\nu(R)}(R/(\boldsymbol{x}))\right)(S/\boldsymbol{x}S) \\ &= 0. \end{aligned}$$

Hence we have $\nu(S) \leq l \cdot \nu(R)$.

Now we shall prove the main lemma of this section. It is rather easy to prove the lemma, but it has a lot of applications as we see later.

Lemma 4.3.3 *Let $(R, \mathfrak{m}, k) \to (S, \mathfrak{n}, l)$ be a homomorphism of local rings. Suppose that $\mathfrak{m}S \subseteq \mathfrak{n}^{\nu(S)}$. Then, there is an S-regular sequence $\boldsymbol{y}$ such that k is isomorphic to a direct summand of $S/(\boldsymbol{y})$ as an R-module.*

PROOF Take a maximal S-regular sequence $\boldsymbol{y}$ with $H^0_{\mathfrak{n}}(S/(\boldsymbol{y})) \cap \mathfrak{n}^{\nu(S)}(S/(\boldsymbol{y})) = 0$. Set $\overline{S} = S/(\boldsymbol{y})$, and let $\theta : H^0_{\mathfrak{n}}(\overline{S}) \to \overline{S}$ and $\pi : \overline{S} \to \overline{S}/\mathfrak{m}\overline{S}$ be natural homomorphisms. Since $\mathfrak{m}S \subseteq \mathfrak{n}^{\nu(S)}$, we have $H^0_{\mathfrak{n}}(\overline{S}) \cap \mathfrak{m}\overline{S} = 0$. Hence the composite map $\pi\theta$ is injective. Note that $\overline{S}/\mathfrak{m}\overline{S}$ is a k-vector space, and hence so is $H^0_{\mathfrak{n}}(\overline{S})$. Therefore $\pi\theta$ is a split-monomorphism of R-modules, and hence so is θ. Since $H^0_{\mathfrak{n}}(\overline{S})$ is a non-zero k-vector space, the assertion follows. $\square$

Applying the above lemma to the Frobenius map, one obtains the following result for local rings of prime characteristic.

Corollary 4.3.4 *Let $(R, \mathfrak{m}, k)$ be a local ring of characteristic p, and let e be an integer with $p^e \geq \nu(R)$. Then k is isomorphic to a direct summand of ${}^eR/(\boldsymbol{x})$ as an R-module for some maximal ${}^eR$-regular sequence $\boldsymbol{x}$ in ${}^e\mathfrak{m}$.*

PROOF Let $a_1, a_2, \cdots, a_r$ be a system of generators of $\mathfrak{m}$. Then the ideal $\mathfrak{m}{}^eR$ of ${}^eR$ is generated by $a_1^{p^e}, a_2^{p^e}, \cdots, a_r^{p^e}$. On the other hand, the elements $a_1, a_2, \cdots, a_r$ in ${}^eR$ generate the ideal ${}^e\mathfrak{m}$. Since $p^e \geq \nu(R) = \nu({}^eR)$, one has $\mathfrak{m}{}^eR \subseteq ({}^e\mathfrak{m})^{\nu({}^eR)}$. Thus the assertion follows from Lemma 4.3.3. $\square$

Corollary 4.3.5 *Let $P^\bullet$ be a property of local rings, and let $(R, \mathfrak{m}, k)$ be an F-finite local ring of characteristic p. Consider a numerical invariant λ of a finitely generated R-module (that is, $\lambda(0) = -\infty$ and $\lambda(X) \in \mathbb{N} \cup \{\infty\}$ for a finitely generated R-module X) satisfying the following conditions.*

a) Let $R \to S$ be a finite homomorphism of local rings, and let $y \in S$ be a non-zero-divisor on S. If $\lambda(S) < \infty$, then $\lambda(S/(y)) < \infty$.

b) Let N be a direct summand of a finitely generated R-module M. If $\lambda(M) < \infty$, then $\lambda(N) < \infty$.

c) If $\lambda(k) < \infty$, then R satisfies $P^\bullet$.

Under these circumstances, suppose that $\lambda({}^eR) < \infty$ for some integer e with $p^e \geq \nu(R)$. Then R satisfies $P^\bullet$.

Before proving this corollary, we should note that there are many numerical invariants satisfying the conditions a)-c) for certain properties of local rings. For instance, projective dimension satisfies them for the regular property. We will give several examples below the proof of this corollary and below Theorem 4.3.6.

PROOF OF COROLLARY 4.3.5 Assume that $\lambda({}^eR) < \infty$ for some integer e with $p^e \geq \nu(R)$. We have from Corollary 4.3.4 that k is a direct summand of

${}^eR/(\boldsymbol{x})$ as an R-module, for some ${}^eR$-sequence $\boldsymbol{x}$. It follows from the condition a) that $\lambda({}^eR/(\boldsymbol{x})) < \infty$. Hence we see that $\lambda(k) < \infty$ by b), and R satisfies the property $P^\bullet$ by c). $\square$

Now we shall observe that CM-dimension is a numerical invariant satisfying the conditions a)-c) in the above corollary for the Cohen-Macaulay property. Let us take CM-dimension as λ, and the Cohen-Macaulay property as $P^\bullet$ in the corollary. Then it follows from Proposition 4.2.5(7) and 4.2.5(8) that CM-dimension satisfies the condition b) and c) for the Cohen-Macaulay property. Furthermore, it is easy to see that CM-dimension also satisfies the condition a). Indeed, let $R \to S$ be a finite homomorphism of local rings, and let $y \in S$ be a non-zero-divisor on S. Suppose that $\text{CM-dim}_R S < \infty$. Then, by Remark 4.2.4, there exist a faithfully flat homomorphism $R \to R'$ and a surjective homomorphism $T \to R'$ of local rings such that $\text{G-dim}_T R' = \text{grade}_T R' < \infty$ and that $\text{G-dim}_T(S \otimes_R R') < \infty$. On the other hand, we have a short exact sequence

$$0 \to S \xrightarrow{y} S \to S/(y) \to 0.$$

Since R' is faithfully flat, the tensored sequence

$$0 \to S \otimes_R R' \xrightarrow{y} S \otimes_R R' \to (S/(y)) \otimes_R R' \to 0$$

is also exact. Hence we see from Proposition 4.2.5(4) that $\text{G-dim}_T((S/(y)) \otimes_R R') < \infty$, and we have $\text{CM-dim}_R S/(y) < \infty$, as desired.

Corollary 4.3.5 and the above remark induce the implication (4) $\Rightarrow$ (1) in the following theorem. The implications (2) $\Rightarrow$ (3) and (3) $\Rightarrow$ (4) are obvious while (1) $\Rightarrow$ (2) follows from Proposition 4.2.5(7).

Theorem 4.3.6 *Let R be an F-finite local ring of characteristic p. Then the following conditions are equivalent.*

(1) *R is Cohen-Macaulay.*

(2) $\text{CM-dim}_R {}^eR < \infty$ *for all integers e.*

(3) $\text{CM-dim}_R {}^eR < \infty$ *for infinitely many integers e.*

(4) $\text{CM-dim}_R {}^eR < \infty$ *for some integer e with $p^e \geq \nu(R)$.*

Projective dimension, CI-dimension, and G-dimension also satisfy the conditions a)-c) in Corollary 4.3.5 for the associated properties of local rings. Hence an F-finite local ring R of characteristic p is a regular ring (resp. a complete intersection, a Gorenstein ring) if ${}^eR$ has finite projective dimension (resp. finite CI-dimension, finite G-dimension) for some integer e with $p^e \geq \nu(R)$. Compare this with Theorem 4.1.1, Theorem 4.1.2, and Theorem 4.6.2.

Moreover, since upper G-dimension G*-dim and lower CI-dimension CI$_*$-dim also satisfy those conditions, an F-finite local ring R is a Gorenstein ring (resp. a complete intersection) if ${}^eR$ has finite upper G-dimension (resp. finite lower CI-dimension) for some integer e with $p^e \geq \nu(R)$. (For the details of these homological dimensions, see [7], [26], and [54].)

We end this section with proposing a natural conjecture.

Conjecture 4.3.7 Let R be an F-finite local ring. Suppose that the R-module ${}^eR$ has finite CM-dimension for *some* integer $e > 0$. Then R would be Cohen-Macaulay.

The authors believe that it would be true, but have no proof of it at this moment. Compare this with Theorem 4.6.2 to see that the corresponding statement about G-dimension is true.

4.4 A generalization of a theorem of Herzog

Herzog proved in his paper [30] the following theorem as a generalization of a theorem due to Kunz [35].

Theorem 4.4.1 (Herzog) *Let R be a local ring of characteristic p, and let M be a finitely generated R-module.*

(1) *If* $\mathrm{Tor}_i^R(M, {}^eR) = 0$ *for any $i > 0$ and infinitely many e, then* $\mathrm{pd}_R M < \infty$.

(2) *Suppose that R is F-finite. If* $\mathrm{Ext}_R^i({}^eR, M) = 0$ *for any $i > 0$ and infinitely many e, then* $\mathrm{id}_R M < \infty$.

According to the method of Herzog, the first assertion of this theorem is proved by observing the structure of the complex $F_\bullet \otimes_R {}^eR$ where $F_\bullet$ is a minimal R-free resolution of the module M, and the second assertion is proved by considering the Matlis duality and using the first assertion. However, Lemma 4.3.3 actually enables us to prove easily the assertions of the Herzog's theorem in a more general setting.

To do this, we introduce the following result of Avramov and Foxby.

Lemma 4.4.2 *[10, Proposition 5.5] Let $(R, \mathfrak{m}, k) \to (S, \mathfrak{n}, l)$ be a local homomorphism of local rings, and let N be a finitely generated S-module. Then one has the following equalities.*

(1) $\mathrm{fd}_R N = \sup\{\, i \mid \mathrm{Tor}_i^R(N,k) \neq 0 \,\}$.

(2) $\mathrm{id}_R N = \sup\{\, i \mid \mathrm{Ext}_R^i(k,N) \neq 0 \,\}$.

Combining this lemma with Lemma 4.3.3, we obtain the following proposition.

Proposition 4.4.3 *Let $(R,\mathfrak{m},k) \to (R',\mathfrak{m}',k')$ and $(R,\mathfrak{m},k) \to (S,\mathfrak{n},l)$ be local homomorphisms of local rings, and let N be a finitely generated S-module. Suppose that $\mathfrak{m}R' \subseteq \mathfrak{m}'^{\nu(R')}$.*

(1) *If $\mathrm{Tor}_i^R(N,R') = 0$ for any sufficiently large integer i, then $\mathrm{fd}_R N < \infty$.*

(2) *If $\mathrm{Ext}_R^i(R',N) = 0$ for any sufficiently large integer i, then $\mathrm{id}_R N < \infty$.*

PROOF First of all, we note from Lemma 4.3.3 that there exists a maximal R'-regular sequence $\boldsymbol{x}$ such that k is a direct summand of $R'/(\boldsymbol{x})$ as an R-module. We easily see from the assumption that $\mathrm{Tor}_i^R(N,R'/(\boldsymbol{x})) = 0$ or $\mathrm{Ext}_R^i(R'/(\boldsymbol{x}),N) = 0$ for $i \gg 0$. Hence we have $\mathrm{Tor}_i^R(N,k) = 0$ and $\mathrm{Ext}_R^i(k,N) = 0$ for $i \gg 0$. Thus the proposition follows from Lemma 4.4.2. $\square$

Remark 4.4.4 Let $(R,\mathfrak{m}) \to (S,\mathfrak{n})$ be a homomorphism of local rings such that $\mathfrak{m}S \subseteq \mathfrak{n}^{\nu(S)}$ and $\mathrm{fd}_R S < \infty$. Then R is regular.

In fact, since $\mathrm{Tor}_i^R(k,S) = 0$ for $i \gg 0$, it follows from Proposition 4.4.3 that $(\mathrm{pd}_R k =)\ \mathrm{fd}_R k < \infty$. Hence the local ring R is regular.

It is easy to see the following theorem follows from Proposition 4.4.3. Notice that the second assertion in the theorem holds without F-finiteness of the ring R.

Theorem 4.4.5 *Let $R \to S$ be a local homomorphism of local rings, and let N be a finitely generated S-module. Suppose that R has characteristic p. Let e be an integer with $p^e \geq \nu(R)$.*

(1) *If $\mathrm{Tor}_i^R(N,{}^eR) = 0$ for any sufficiently large integer i, then $\mathrm{fd}_R N < \infty$.*

(2) *If $\mathrm{Ext}_R^i({}^eR,N) = 0$ for any sufficiently large integer i, then $\mathrm{id}_R N < \infty$.*

It is needless to say that this theorem implies Theorem 4.4.1.

4.5 Another characterization of Cohen-Macaulay rings

Let R be a local ring of characteristic p. In Section 3 we have given necessary and sufficient conditions for R to be Cohen-Macaulay in terms of the CM-dimension of the R-module ${}^eR$, in the case that the ring R is *F-finite*. In this section, we would like to characterize Cohen-Macaulay local rings of prime characteristic which are not necessarily F-finite.

Proposition 4.5.1 *Let R be a local ring with maximal ideal $\mathfrak{m}$. Assume that there exists a non-zero injective R-module which is separated in its $\mathfrak{m}$-adic topology. Then R is artinian.*

Proof Let M be a non-zero injective R-module which is $\mathfrak{m}$-adically separated. Since any submodule of a separated module is also separated, we may assume that M is the injective hull $E_R(R/\mathfrak{p})$ of $R/\mathfrak{p}$ for some prime ideal $\mathfrak{p}$ of R. Then, since $M = E_R(R/\mathfrak{p})$ is an $R_\mathfrak{p}$-module and since M is $\mathfrak{m}$-adically separated, we must have $\mathfrak{p} = \mathfrak{m}$. Thus M is an artinian R-module. Then that $\bigcap_n \mathfrak{m}^n M = 0$ implies $\mathfrak{m}^n M = 0$ for some integer n. Since the module $\mathrm{End}_R(M)$ is isomorphic to the $\mathfrak{m}$-adic completion $\widehat{R}$ of R, we finally have that $\mathfrak{m}^n = 0$, and hence R is an artinian ring. $\square$

In the following proposition, if the module N is finitely generated over the ring R, it is a well-known result due to Bass [16].

Proposition 4.5.2 *Let $(R, \mathfrak{m}, k) \to (S, \mathfrak{n}, l)$ be a local homomorphism of local rings, and let N be a non-zero finitely generated S-module with $\mathrm{id}_R N < \infty$. Then $\mathrm{id}_R N = \operatorname{depth} R$.*

Proof Set $s = \mathrm{id}_R N$ and $t = \operatorname{depth} R$, and take a maximal R-regular sequence $\boldsymbol{x} = x_1, x_2, \cdots, x_t$. Since $\operatorname{depth} R/(\boldsymbol{x}) = 0$, there is a short exact sequence

$$0 \to k \to R/(\boldsymbol{x}) \to C \to 0$$

of R-modules. Hence we have an exact sequence

$$\mathrm{Ext}_R^s(R/(\boldsymbol{x}), N) \to \mathrm{Ext}_R^s(k, N) \to \mathrm{Ext}_R^{s+1}(C, N) = 0.$$

Lemma 4.4.2 implies that $\mathrm{Ext}_R^s(k, N) \neq 0$, and hence $\mathrm{Ext}_R^s(R/(\boldsymbol{x}), N) \neq 0$. Since the Koszul complex $K_\bullet(\boldsymbol{x})$ of $\boldsymbol{x}$ over R is an R-free resolution of $R/(\boldsymbol{x})$ of length t, we see that $s \leq t$. On the other hand, it follows from Nakayama's lemma that $\mathrm{Ext}_R^t(R/(\boldsymbol{x}), N) \cong N/\boldsymbol{x}N \neq 0$. Therefore $s = t$, as desired. $\square$

Now we are ready to prove a theorem which characterizes the Cohen-Macaulay property for local rings of prime characteristic that are not necessarily F-finite.

Theorem 4.5.3 *Let $(R, \mathfrak{m}, k)$ be a local ring of characteristic p. Then the following conditions are equivalent.*

(1) *R is Cohen-Macaulay.*

(2) *There exist a local ring homomorphism $R \to S$ and a finitely generated S-module N satisfying the following two conditions.*

i) *There is a maximal R-regular sequence that is also N-regular.*

ii) *There is an integer e such that $p^e \geq \nu(R)$ and that $\mathrm{Ext}^i_R({}^eR, N) = 0$ for any sufficiently large integer i.*

Proof (1) $\Rightarrow$ (2): Let S be the $\mathfrak{m}$-adic completion of the Cohen-Macaulay local ring R, and let N be the canonical module of S. Note that S is faithfully flat over R, and that N is a maximal Cohen-Macaulay S-module. Hence it is easy to see that every R-regular sequence is also S-regular, and that every S-regular sequence is also N-regular. Noting that N is of finite injective dimension over S, one has $\mathrm{Ext}^i_R({}^eR, N) \cong \mathrm{Ext}^i_S({}^eR \otimes_R S, N) = 0$ for any $e > 0$ and $i \gg 0$.

(2) $\Rightarrow$ (1): Set $t = \operatorname{depth} R$. By the assumption, we have a sequence $\boldsymbol{x} = x_1, x_2, \cdots, x_t$ of elements of R which is both R-regular and N-regular. Theorem 4.4.5 and Proposition 4.5.2 imply that $\mathrm{id}_R N = t$. Especially we have $\mathrm{Ext}^i_R(k, N) = 0$ for $i > t$. Hence we also have $\mathrm{Ext}^i_{R/(\boldsymbol{x})}(k, N/\boldsymbol{x}N) = 0$ for $i > 0$. Thus it follows from Lemma 4.4.2 that $\mathrm{id}_{R/(\boldsymbol{x})} N/\boldsymbol{x}N = 0$, i.e. $N/\boldsymbol{x}N$ is $R/(\boldsymbol{x})$-injective. On the other hand, since the induced homomorphism $R/(\boldsymbol{x}) \to S/\boldsymbol{x}S$ is local and since $N/\boldsymbol{x}N$ is a finitely generated $S/\boldsymbol{x}S$-module, $N/\boldsymbol{x}N$ is $\mathfrak{m}/(\boldsymbol{x})$-adically separated. Therefore it follows from Proposition 4.5.1 that $R/(\boldsymbol{x})$ is artinian, and hence R is Cohen-Macaulay. $\square$

As a corollary of the above theorem, we obtain the characterization of F-finite Cohen-Macaulay local rings as it is stated in Section 3. We give here an alternative proof of the implication (4) $\Rightarrow$ (1) of Theorem 4.3.6 by using the above theorem.

Alternative Proof of Theorem 4.3.6 (4) $\Rightarrow$ (1) Since $\text{CM-dim}_R {}^eR < \infty$, one has $\text{CM-dim}_R {}^eR = \operatorname{depth} R - \operatorname{depth}_R {}^eR$ by Proposition 4.2.5(6). Noting that a sequence $x_1, x_2, \cdots, x_n$ of elements of R is ${}^eR$-regular if and only if $x_1^{p^e}, x_2^{p^e}, \cdots, x_n^{p^e}$ is an R-regular sequence, one sees that $\operatorname{depth}_R {}^eR = \operatorname{depth} R$, and hence $\text{CM-dim}_R {}^eR = 0$. So it follows from definition that there exist a faithfully flat homomorphism $R \to S$ of local rings and a semi-dualizing S-module C such that $\mathrm{Ext}^i_S({}^eR \otimes_R S, C) = 0$ for $i > 0$. Since S is faithfully flat over R, one has from Proposition 4.2.2(2) that every R-regular sequence is C-regular, and $\mathrm{Ext}^i_R({}^eR, C) \cong \mathrm{Ext}^i_S({}^eR \otimes_R S, C) = 0$ for $i > 0$. Thus Theorem 4.5.3 implies that R is Cohen-Macaulay. $\square$

4.6 A characterization of Gorenstein rings

We shall prove in this section a result concerning G-dimension, which is similar to Theorem 4.3.6 but in a slightly stronger form. We begin with stating a proposition that follows directly from Theorem 4.4.5(2).

Proposition 4.6.1 *Let R be a local ring of characteristic p. Then the following conditions are equivalent.*

(1) *R is Gorenstein.*

(2) *There is an integer e such that $p^e \geq \nu(R)$ and that $\mathrm{Ext}^i_R({}^eR, R) = 0$ for any sufficiently large integer i.*

Now we can prove the main result of this section.

Theorem 4.6.2 *The following conditions are equivalent for an F-finite local ring R of prime characteristic p.*

(1) *R is Gorenstein.*

(2) $\text{G-dim}_R {}^eR < \infty$ *for every integer $e > 0$.*

(3) $\text{G-dim}_R {}^eR < \infty$ *for infinitely many integers $e > 0$.*

(4) $\text{G-dim}_R {}^eR < \infty$ *for some integer e with $p^e \geq \nu(R)$.*

(5) $\text{G-dim}_R {}^eR < \infty$ *for some integer $e > 0$.*

Proof The implications (1) $\Rightarrow$ (2) $\Rightarrow$ (3) $\Rightarrow$ (4) $\Rightarrow$ (5) are obvious from Proposition 4.2.5(2), and the implication (4) $\Rightarrow$ (1) follows from Proposition 4.6.1.

It remains to prove the implication (5) $\Rightarrow$ (4). For this, it is enough to show the following claim.

Claim *If* $\text{G-dim}_R {}^eR < \infty$ *for some integer e, then* $\text{G-dim}_R {}^{2e}R < \infty$.

Suppose that $\text{G-dim}_R {}^eR < \infty$ for some integer e. Then we have $\text{G-dim}_R {}^eR = \operatorname{depth} R - \operatorname{depth}_R {}^eR = 0$, and hence $\mathrm{Hom}_R({}^eR, R) \cong$

$\mathbf{R}\mathrm{Hom}_R({}^eR, R)$ in $\mathcal{D}^b(R)$. Now put $C = \mathrm{Hom}_R({}^eR, R)$, and we see from Proposition 4.2.5(5) that

$$\begin{aligned}\mathbf{R}\mathrm{Hom}_{{}^eR}(C, C) &\cong \mathbf{R}\mathrm{Hom}_{{}^eR}(\mathbf{R}\mathrm{Hom}_R({}^eR, R), \mathbf{R}\mathrm{Hom}_R({}^eR, R)) \\ &\cong \mathbf{R}\mathrm{Hom}_R(\mathbf{R}\mathrm{Hom}_R({}^eR, R), R) \\ &\cong {}^eR.\end{aligned}$$

Therefore C is a semi-dualizing ${}^eR$-module.

We would like to show that C is isomorphic to ${}^eR$ as an ${}^eR$-module. To do this, let ${}^ek$ be the residue field of ${}^eR$, and put $t = \operatorname{depth} R$. Since $\mathbf{R}\mathrm{Hom}_{{}^eR}({}^ek, C) \cong \mathbf{R}\mathrm{Hom}_{{}^eR}({}^ek, \mathbf{R}\mathrm{Hom}_R({}^eR, R)) \cong \mathbf{R}\mathrm{Hom}_R({}^ek, R)$, we have

$$\mathrm{Ext}^t_{{}^eR}({}^ek, C) \cong \mathrm{Ext}^t_R({}^ek, R).$$

Note that $\mathrm{depth}_{{}^eR} C = \operatorname{depth} {}^eR = t$ by Proposition 4.2.2(2). Hence comparing the k-dimension of the both sides of the above isomorphism, we have

$$\begin{aligned}r_{{}^eR}(C) \cdot \dim_k {}^ek &= \dim_k \mathrm{Ext}^t_{{}^eR}({}^ek, C) \\ &= \dim_k \mathrm{Ext}^t_R({}^ek, R) \\ &= r_R(R) \cdot \dim_k {}^ek.\end{aligned}$$

Therefore we obtain $r_{{}^eR}(C) = r_R(R) = r_{{}^eR}({}^eR)$. It then follows from Proposition 4.2.2(3) that $\mu_{{}^eR}(C) = 1$, that is, C is a cyclic ${}^eR$-module. But Proposition 4.2.2(1) implies that $C \cong {}^eR$, as desired.

Since we have shown that there is an isomorphism $\mathbf{R}\mathrm{Hom}_R({}^eR, R) \cong {}^eR$ in $\mathcal{D}^b({}^eR)$, we have ismorphisms

$$\begin{aligned}\mathbf{R}\mathrm{Hom}_{{}^eR}(X, {}^eR) &\cong \mathbf{R}\mathrm{Hom}_{{}^eR}(X, \mathbf{R}\mathrm{Hom}_R({}^eR, R)) \\ &\cong \mathbf{R}\mathrm{Hom}_R(X, R)\end{aligned}$$

for any obeject X in $\mathcal{D}({}^eR)$. Thus we have an isomorphism

$$\mathbf{R}\mathrm{Hom}_{{}^eR}(\mathbf{R}\mathrm{Hom}_{{}^eR}({}^{2e}R, {}^eR), {}^eR) \cong \mathbf{R}\mathrm{Hom}_R(\mathbf{R}\mathrm{Hom}_R({}^{2e}R, R), R).$$

Noting that $\text{G-dim}_{{}^eR} {}^{2e}R = \text{G-dim}_{{}^eR} {}^e({}^eR) = \text{G-dim}_R {}^eR < \infty$, we see from Proposition 4.2.5(5) that the left hand side is isomorphic to ${}^{2e}R$. Then it follows from Proposition 4.2.5(5) again that $\text{G-dim}_R {}^{2e}R < \infty$, and the proof is completed. $\square$

5 Gorenstein dimension and the Peskine-Szpiro intersection theorem

The contents of this chapter are entirely contained in the author's own paper [49].

A commutative noetherian local ring admitting a Cohen-Macaulay module of finite projective dimension is Cohen-Macaulay. This result follows directly from the Peskine-Szpiro intersection theorem. With relation to this result, there is a conjecture that a commutative noetherian local ring admitting a Cohen-Macaulay module of finite Gorenstein dimension would be Cohen-Macaulay.

In this chapter, we shall characterize Gorenstein local rings by the existence of special modules of finite Gorenstein dimension. The main theorem of this chapter (Theorem 5.2.3) not only characterizes the Gorenstein property of a ring, but also proves a special case of the above conjecture.

5.1 Introduction

Throughout this chapter, R always denotes a commutative noetherian local ring with unique maximal ideal $\mathfrak{m}$ and residue class field $k = R/\mathfrak{m}$. All modules considered in this chapter are finitely generated.

Gorenstein dimension (abbr. G-dimension), which was defined by Auslander [3], has played an important role in the classification of modules and rings together with projective dimension. Let us recall the definition of G-dimension.

Definition 5.1.1 Let M be an R-module.

(1) If the following conditions hold, then we say that the G-dimension of M is zero, and write $\text{G-dim}_R M = 0$.

  i) The natural homomorphism $M \to \mathrm{Hom}_R(\mathrm{Hom}_R(M, R), R)$ is isomorphic.

  ii) $\mathrm{Ext}_R^i(M, R) = 0$ for every $i > 0$.

  iii) $\mathrm{Ext}_R^i(\mathrm{Hom}_R(M, R), R) = 0$ for every $i > 0$.

(2) Let n be a non-negative integer. If the G-dimension of the nth syzygy module $\Omega_R^n M$ of M is zero, then we say that the G-dimension of M is not bigger than n, and write $\text{G-dim}_R M \leq n$.

G-dimension has a lot of properties that are similar to those of projective dimension. We state here several properties of G-dimension. For the proofs, we refer to [4], [19], [36], and [57].

Proposition 5.1.2 *Let R be a local ring with residue field k.*

(1) *Let M be a non-zero R-module. If* $\text{G-dim}_R M < \infty$*, then* $\text{G-dim}_R M = \operatorname{depth} R - \operatorname{depth}_R M$.

(2) *The following conditions are equivalent.*

  i) *R is Gorenstein.*

  ii) $\text{G-dim}_R M < \infty$ *for any R-module M.*

  iii) $\text{G-dim}_R k < \infty$.

(3) *For any R-module M,* $\text{G-dim}_R M \leq \operatorname{pd}_R M$*. Hence any module of finite projective dimension is also of finite G-dimension.*

(4) *Let $0 \to L \to M \to N \to 0$ be a short exact sequence of R-modules. If two of L, M, N are of finite G-dimension, then so is the third.*

(5) *Let M be an R-module and let $\boldsymbol{x} = x_1, \cdots, x_n$ be a sequence of elements of R. Then the following hold.*

  i) *If $\boldsymbol{x}$ is an M-sequence, then* $\text{G-dim}_R M/\boldsymbol{x}M = \text{G-dim}_R M + n$.

  ii) *If $\boldsymbol{x}$ is an R-sequence in* $\operatorname{Ann}_R M$*, then* $\text{G-dim}_{R/(\boldsymbol{x})} M = \text{G-dim}_R M - n$.

The Peskine-Szpiro intersection theorem is one of the main results in commutative algebra in the 1980's.

The Peskine-Szpiro Intersection Theorem *Let R be a commutative noetherian local ring, and let M, N be finitely generated R-modules such that $M \otimes_R N$ has finite length. Then*

$$\dim_R N \leq \operatorname{pd}_R M.$$

This was proved by Peskine and Szpiro [42] in the positive characteristic case, by Hochster [31] in the equicharacteristic case, and by Roberts [44] in the general case. The following theorem is directly given as a corollary of the Peskine-Szpiro intersection theorem.

Theorem 5.1.3 *[45, Proposition 6.2.4] Let R be a commutative noetherian local ring. Suppose that there exists a Cohen-Macaulay R-module of finite projective dimension. Then the local ring R is Cohen-Macaulay.*

As we have observed in Proposition 5.1.2, G-dimension shares many properties with projective dimension. So we are naturally leaded to the following conjecture:

Conjecture 5.1.4 Let R be a commutative noetherian local ring. Suppose that there exists a Cohen-Macaulay R-module M of finite G-dimension. Then the local ring R is Cohen-Macaulay.

If this conjecture is proved, as the assertion is itself very interesting, its proof might give another easier proof of the Peskine-Szpiro intersection theorem. However, nothing has known about this conjecture until now.

In the next section, we shall give several conditions in terms of G-dimension that are equivalent to the condition that R is Gorenstein. The main result of this chapter is Theorem 5.2.3. This theorem says that the above conjecture is true if the type of R or M is one.

5.2 Criteria for the Gorenstein property

In this section, we consider when the local ring R is Gorenstein, by using G-dimension. For an R-module M, denote by $\mu_R^i(M)$ the ith Bass number of M and by $r_R(M)$ the type of M, i.e., $\mu_R^i(M) = \dim_k \mathrm{Ext}_R^i(k, M)$ and $r_R(M) = \mu_R^t(M)$ where $t = \mathrm{depth}_R M$. We begin with stating properties of type.

Proposition 5.2.1 *Let M be an R-module and let $\boldsymbol{x} = x_1, \cdots, x_n$ be an M-sequence. Then $r_R(M/\boldsymbol{x}M) = r_R(M)$. If, in addition, $\boldsymbol{x}$ is also an R-sequence, then $r_{R/(\boldsymbol{x})}(M/\boldsymbol{x}M) = r_R(M)$.*

We omit the proof of the above proposition because it is standard. Refer to the proof of [18, Lemma 1.2.4].

In order to prove our main theorem, we prepare the following lemma.

Lemma 5.2.2 *[21, Theorem 1.1] Let R be a commutative noetherian local ring with residue class field k, and M be a finitely generated R-module. Then $\mathrm{Ext}_R^i(k, M) \neq 0$ for all i, $\mathrm{depth}_R M \leq i \leq \mathrm{id}_R M$.*

Here, we remark that the assertion of the lemma holds without finiteness of the injective dimension of M. In other words, if the R-module M is of infinite injective dimension, then one has $\mathrm{Ext}^i_R(k, M) \neq 0$ for all $i \geq \mathrm{depth}_R M$.

We denote by $l_R(M)$ the length of an R-module M, and by $\nu_R(M)$ the minimal number of generators of an R-module M, that is to say, $\nu_R(M) = \dim_k(M \otimes_R k)$. We shall state the main result of this chapter.

Theorem 5.2.3 *The following conditions are equivalent.*

i) *R is Gorenstein.*

ii) *R admits an ideal I of finite G-dimension such that the factor ring R/I is Gorenstein.*

iii) *R admits a Cohen-Macaulay module of type one and of finite G-dimension.*

iv) *R is a local ring of type one admitting a Cohen-Macaulay module of finite G-dimension.*

Proof i) $\Rightarrow$ ii): Both the zero ideal and the maximal ideal of R satisfy the condition ii) by Proposition 5.1.2(2).

ii) $\Rightarrow$ iii): Let $\boldsymbol{x}$ be a sequence of elements of R which forms a maximal R/I-sequence. Then the factor ring $R/I + (\boldsymbol{x})$ is an artinian Gorenstein ring, so one has $\mathrm{Hom}_R(k, R/I + (\boldsymbol{x})) \cong \mathrm{Hom}_{R/I+(\boldsymbol{x})}(k, R/I + (\boldsymbol{x})) \cong k$. Hence one sees from Proposition 5.2.1 that $r_R(R/I) = r_R(R/I + (\boldsymbol{x})) = 1$. On the other hand, Proposition 5.1.2(4) yields that $\text{G-dim}_R R/I < \infty$. Thus the R-module R/I is a Cohen-Macaulay module which has type one and finite G-dimension.

iii) $\Rightarrow$ iv): Let M be a Cohen-Macaulay module of type 1 and of finite G-dimension. Taking a maximal M-sequence $\boldsymbol{x}$, we see from Proposition 5.2.1 and Proposition 5.1.2(5) that $M/\boldsymbol{x}M$ is an R-module of finite length, of type 1, and of finite G-dimension. Hence, replacing M by $M/\boldsymbol{x}M$, one may assume that the length of M is finite. Then since the ideal $\mathrm{Ann}_R M$ is $\mathfrak{m}$-primary, one can choose an R-sequence $\boldsymbol{y} = y_1, \cdots, y_t$ in $\mathrm{Ann}_R M$, where $t = \mathrm{depth}\, R$. Noting that $\mathrm{Hom}_{R/(\boldsymbol{y})}(k, M) \cong \mathrm{Hom}_R(k, M)$, one sees that $r_{R/(\boldsymbol{y})}(M) = r_R(M) = 1$. On the other hand, Proposition 5.1.2(5) implies that $\text{G-dim}_{R/(\boldsymbol{y})} M < \infty$. Therefore, replacing R by $R/(\boldsymbol{y})$, we may assume that $\mathrm{depth}\, R = 0$. Thus we see from Proposition 5.1.2(1) that the G-dimension of the R-module M is zero, which especially yields that $M^{**} \cong M$.

Hence we have the following isomorphisms.

$$\begin{array}{rcl}\mathrm{Hom}_R(k, M) & \cong & \mathrm{Hom}_R(k, M^{**}) \\ & \cong & \mathrm{Hom}_R(k \otimes_R M^*, R) \\ & \cong & \mathrm{Hom}_R((k \otimes_R M^*) \otimes_k k, R) \\ & \cong & \mathrm{Hom}_k(k \otimes_R M^*, \mathrm{Hom}_R(k, R)).\end{array}$$

It follows from these isomorphisms that $1 = r_R(M) = \nu_R(M^*)\, r_R(R)$. Hence $r_R(R) = 1$.

iv) $\Rightarrow$ i): Let M be a Cohen-Macaulay R-module of finite G-dimension, and let $\boldsymbol{x}$ be a maximal M-sequence. Then we see that the residue R-module $M/\boldsymbol{x}M$ is of finite length and of finite G-dimension by Proposition 5.1.2(5). Hence, replacing M by $M/\boldsymbol{x}M$, we may assume that $l_R(M) < \infty$. Set $t = \operatorname{depth} R$. Since $\mathrm{Ann}_R M$ is an $\mathfrak{m}$-primary ideal, one can take an R-sequence $\boldsymbol{x} = x_1, \cdots, x_t$ in $\mathrm{Ann}_R M$. It then follows from Proposition 5.1.2(5) that $\text{G-dim}_{R/(\boldsymbol{x})} M = \text{G-dim}_R M - t < \infty$. Since $r_{R/(\boldsymbol{x})}(R/(\boldsymbol{x})) = r_R(R) = 1$ by Proposition 5.2.1, replacing R by $R/(\boldsymbol{x})$, we may assume that $\operatorname{depth} R = 0$. Suppose that R is not Gorenstein. Then the R-module R has infinite injective dimension. Hence Lemma 5.2.2 especially yields that $\mathrm{Ext}^1_R(k, R) \neq 0$. Put $s = \dim_k \mathrm{Ext}^1_R(k, R) (> 0)$ and let

$$0 = M_0 \subset M_1 \subset M_1 \subset \cdots \subset M_n = M$$

be a composition series of M. Decompose this series to short exact sequences

$$0 \to M_{i-1} \to M_i \to k \to 0$$

for $1 \le i \le n$. Since $r(R) = 1$, one has $k^* \cong k$. Hence, applying the R-dual functor $(-)^* = \mathrm{Hom}_R(-, R)$ to these sequences, one obtains exact sequences

$$\begin{cases} 0 \to k \to M_i^* \to M_{i-1}^* \ \text{ for } \ 1 \le i \le n-1, \ \text{ and} \\ 0 \to k \to M^* \to M_{n-1}^* \to k^s \to \mathrm{Ext}^1_R(M, R). \end{cases}$$

Here, as $\text{G-dim}_R M = 0$ by Proposition 5.1.2(1), it follows from definition that $\mathrm{Ext}^1_R(M, R) = 0$. Therefore we get

$$\begin{cases} l_R(M_i^*) \le l_R(M_{i-1}^*) + 1 \ \text{ for } \ 1 \le i \le n-1, \ \text{ and} \\ l_R(M^*) = l_R(M_{n-1}^*) + 1 - s. \end{cases}$$

Since $s > 0$, we have the following inequalities.

$$\begin{array}{rcl} l_R(M^*) & < & l_R(M_{n-1}^*) + 1 \\ & \le & l_R(M_{n-2}^*) + 2 \\ & \le & \cdots \\ & \le & l_R(M_0^*) + n \\ & = & n. \end{array}$$

That is to say, $l_R(M^*) < l_R(M)$. Because M^* is also of G-dimension zero by definition, the same argument for M^* shows that $l_R(M^{**}) < l_R(M^*)$. However, since $\text{G-dim}_R M = 0$, one has $M^{**} \cong M$. Thus, we obtain $l_R(M) < l_R(M^*) < l_R(M)$, which is contradiction. Hence R is Gorenstein. $\square$

Remark 5.2.4 Each of the criteria for the Gorenstein property in the above theorem requires that the Bass number of a certain module is one. With relation to this, we should remark that the local ring R is Gorenstein if and only if $\mu_R^d(R) = 1$ where $d = \dim R$. This result is due to Foxby and Roberts; see [18, Corollary 9.6.3, Remark 9.6.4].

Now, let us study the following proposition, which will be used as a lemma to give the other characterizations of Gorenstein local rings. This proposition says that, the existence of a module M of finite G-dimension with exact sequence $0 \to W \to M \to V \to 0$ where V, W are annihilated by $\mathfrak{m}$, determines the higher Bass numbers of the base ring. Yoshino [60], observing this, gives a characterization of artinian local rings of low Loewy length admitting modules of finite G-dimension.

Proposition 5.2.5 *Let $0 \to W \to M \to V \to 0$ be a short exact sequence of R-modules. Suppose that R is not Gorenstein,* $\operatorname{depth} R = 0$*, $M \neq 0$,* $\text{G-dim}_R M < \infty$ *and V, W are annihilated by the maximal ideal $\mathfrak{m}$ of R. Set $m = \dim_k V$, $n = \dim_k W$, and $r = r_R(R)$. Then the following hold.*

i) $n = rm$.

ii) $\mu_R^i(R) = \begin{cases} r & \text{for } i = 0, \\ r^{i+1} - r^{i-1} & \text{for } i > 0. \end{cases}$

iii) $l_R(M^*) = l_R(M) = m + n$.

Proof First of all, note from Proposition 5.1.2(1) that $\text{G-dim}_R M = 0$. Since $V \cong k^m$ and $W \cong k^n$, we have a short exact sequence

$$0 \to k^n \to M \to k^m \to 0. \tag{5.1}$$

Put $s_i = \mu_R^i(R)$ and $s = s_1$. As R is non-Gorenstein and M is non-zero, Proposition 5.1.2(2) and Lemma 5.2.2 imply that $m, n \neq 0$ and $s_i \neq 0$ for every $i > 0$. Note from definition that $\operatorname{Ext}_R^i(M, R) = 0$ for every $i > 0$. Applying the R-dual functor $(-)^* = \operatorname{Hom}_R(-, R)$ to this sequence, we obtain an exact sequence $0 \to k^{mr} \to M^* \to k^{nr} \to k^{ms} \to 0$ and isomorphisms $k^{ns_i} \cong k^{ms_{i+1}}$ for all $i > 0$. Hence one gets a short exact sequence

$$0 \to k^{mr} \to M^* \to k^{nr-ms} \to 0, \tag{5.2}$$

and equalities

$$ns_i = ms_{i+1} \tag{5.3}$$

for all $i > 0$. Note by definition that $M^{**} \cong M$ and that $\mathrm{Ext}^i_R(M^*, R) = 0$ for every $i > 0$. Applying the R-dual functor to the sequence (5.2), one gets an exact sequence

$$0 \to k^{(nr-ms)r} \to M \to k^{mr^2} \to k^{(nr-ms)s} \to 0, \tag{5.4}$$

and isomorphisms $k^{mrs_i} \cong k^{(nr-ms)s_{i+1}}$ for all $i > 0$. Therefore one obtains equalities

$$mrs_i = (nr - ms)s_{i+1} \tag{5.5}$$

for all $i > 0$. It follows from (5.3) and (5.5) that

$$s = \frac{n^2 - m^2}{nm} r. \tag{5.6}$$

On the other hand, the sequences (5.4) and (5.1) yield the following equalities.

$$\begin{aligned} (nr - ms)r + mr^2 &= l_R(k^{(nr-ms)r}) + l_R(k^{mr^2}) \\ &= l_R(M) + l_R(k^{(nr-ms)s}) \\ &= l_R(k^n) + l_R(k^m) + l_R(k^{(nr-ms)s}) \\ &= n + m + (nr - ms)s. \end{aligned} \tag{5.7}$$

From the equalities (5.6) and (5.7), we easily see that $n = rm$ and $s = r^2 - 1$. Hence we obtain $s_i = r^{i-1}s = r^{i+1} - r^{i-1}$ for any $i > 0$ by (5.3), and $l_R(M^*) = n + m = l_R(M)$ by (5.2) and (5.1), as desired. □

Using the above proposition, let us induce another characterization of Gorensteinness, which is a corollary of Theorem 5.2.3.

Corollary 5.2.6 *The following conditions are equivalent.*

i) *R is Gorenstein.*

ii) *R admits a non-zero module of length ≤ 2 and of finite G-dimension.*

iii) *R admits a non-zero module M satisfying $\mathfrak{m}^2 M = 0$, $l_R(M) \leq 2\,\nu_R(M)$, and $\text{G-dim}_R M < \infty$.*

PROOF i) $\Rightarrow$ ii): Proposition 5.1.2(2) implies that the simple R-module k satisfies the condition ii).

ii) $\Rightarrow$ iii): Let $M \neq 0$ be an R-module with $l_R(M) \leq 2$ and $\text{G-dim}_R M < \infty$. Then $\mathfrak{m}M$ is a non-trivial submodule of M by Nakayama's lemma. Since $l_R(M) \leq 2$, we see that the submodule $\mathfrak{m}^2 M$ of M must be the zero module. On the other hand, we have $l_R(M) \leq 2 \leq 2\,\nu_R(M)$.

iii) $\Rightarrow$ i): Since the length of the R-module M is finite, the ideal $\text{Ann}_R M$ is $\mathfrak{m}$-primary. Hence there exists an R-sequence $\boldsymbol{x} = x_1, \cdots, x_t$ in $\text{Ann}_R M$, where $t = \operatorname{depth} R$. By Proposition 5.1.2(5), the $R/(\boldsymbol{x})$-module M is of finite G-dimension. So, replacing R by $R/(\boldsymbol{x})$, we may assume that the depth of R is zero. Consider the natural short exact sequence

$$0 \to \mathfrak{m}M \to M \to M/\mathfrak{m}M \to 0.$$

By the assumption, the module $\mathfrak{m}M$ is annihilated by $\mathfrak{m}$, as well as $M/\mathfrak{m}M$. Suppose that R is non-Gorenstein. Then we see from Proposition 5.2.5 that

$$\begin{array}{rcl} 2\,\nu_R(M) & \geq & l_R(M) \\ & = & l_R(\mathfrak{m}M) + \nu_R(M) \\ & = & r_R(R)\,\nu_R(M) + \nu_R(M) \\ & = & (r_R(R) + 1)\,\nu_R(M). \end{array}$$

Therefore we get $2 \geq r_R(R) + 1$, that is, $r_R(R) \leq 1$, and hence $r_R(R) = 1$. Thus, Theorem 5.2.3 implies that R is Gorenstein, which contradicts the present assumption. This contradiction proves that R is Gorenstein, and we are done. $\square$

6 The category of modules of Gorenstein dimension zero

The contents of this chapter are entirely contained in the author's own papers [50], [51], and [52].

Let R be a commutative noetherian non-Gorenstein local ring. In this chapter, we consider the following problem:

(A) There exist infinitely many isomorphism classes of indecomposable R-modules of Gorenstein dimension zero provided there exists at least a non-free R-module of Gorenstein dimension zero.

We shall work on this problem from a categorical point of view. Denote by $\mathrm{mod}R$ the category of finitely generated R-modules, and by $\mathcal{G}(R)$ the full subcategory of $\mathrm{mod}R$ consisting of all modules of Gorenstein dimension zero. The problem (A) is resolved if we can prove the following:

(B) The category $\mathcal{G}(R)$ is not contravariantly finite in $\mathrm{mod}R$ provided that the ring R has a non-free module in $\mathcal{G}(R)$.

It is proved in this chapter that the problem (B) holds if R is a henselian local ring of depth at most two.

6.1 Introduction

Throughout this chapter, we assume that all rings are commutative noetherian rings and all modules are finitely generated modules.

Gorenstein dimension (G-dimension for short), which is a homological invariant for modules, was defined by Auslander [3] and was deeply studied by him and Bridger [4]. With that as a start, G-dimension has been studied by a lot of algebraists until now.

The notion of modules of finite G-dimension is a common generalization of that of modules of finite projective dimension and that of modules over Gorenstein local rings: over an arbitrary local ring all modules of finite projective dimension are also of finite G-dimension, and all modules over a Gorenstein local ring are of finite G-dimension. (Conversely, a local ring whose residue class field has finite G-dimension is Gorenstein. In the next section, we will introduce several properties of G-dimension.)

Over a Gorenstein local ring, a module has G-dimension zero if and only if it is a maximal Cohen-Macaulay module. Hence it is natural to expect that modules of G-dimension zero over an arbitrary local ring may behave similarly to maximal Cohen-Macaulay modules over a Gorenstein local ring.

A Cohen-Macaulay local ring is called to be of finite Cohen-Macaulay representation type if it has only finitely many isomorphism classes of indecomposable maximal Cohen-Macaulay modules. This kind of rings have been well researched for a long time. Under a few assumptions, Gorenstein local rings of finite Cohen-Macaulay representation type have been classified completely, and it is known that all isomorphism classes of indecomposable maximal Cohen-Macaulay modules over them can be described concretely; see [58] for the details.

Thus we are interested in non-Gorenstein local rings which have only finitely many isomorphism classes of indecomposable modules of G-dimension zero, especially interested in determining all isomorphism classes of indecomposable modules of G-dimension zero over such rings.

Here, it is natural to ask whether such a ring in fact exists or not. Such a ring does exist. For example, let $(R, \mathfrak{m})$ be a non-Gorenstein local ring with $\mathfrak{m}^2 = 0$. Then every indecomposable R-module of G-dimension zero is isomorphic to R (cf. [60, Proposition 2.4]).

Thus, we would like to know whether there exists a non-Gorenstein local ring which has a non-free module of G-dimension zero and only finitely many isomorphism classes of indecomposable modules of G-dimension zero. Our guess is that such a ring can not exist:

Conjecture 6.1.1 Let R be a non-Gorenstein local ring. Suppose that there exists a non-free R-module of G-dimension zero. Then there exist infinitely many isomorphism classes of indecomposable R-modules of G-dimension zero.

Indeed, over a certain artinian local ring having a non-free module of G-dimension zero, Yoshino [60] has constructed a family of modules of G-dimension zero with continuous parameters.

The above conjecture is against our expectation that modules of G-dimension zero over an arbitrary local ring behave similarly to maximal Cohen-Macaulay modules over a Gorenstein local ring. Indeed, let S be a d-dimensional non-regular Gorenstein local ring of finite Cohen-Macaulay representation type. (Such a ring does exist; see [58].) Then the dth syzygy module of the residue class field of S is a non-free maximal Cohen-Macaulay

S-module. Hence the above conjecture does not necessarily hold without the assumption that R is non-Gorenstein.

For a local ring R, we denote by $\mathrm{mod}R$ the category of finitely generated R-modules, and by $\mathcal{G}(R)$ the full subcategory of $\mathrm{mod}R$ consisting of all R-modules of G-dimension zero. We guess that even the following statement that is stronger than Conjecture 6.1.1 is true. (It will be seen from Proposition 6.2.9 that Conjecture 6.1.2 implies Conjecture 6.1.1.)

Conjecture 6.1.2 Let R be a non-Gorenstein local ring. Suppose that there exists a non-free R-module in $\mathcal{G}(R)$. Then the category $\mathcal{G}(R)$ is not contravariantly finite in $\mathrm{mod}R$.

The purpose of this chapter is to prove that Conjecture 6.1.2 is true if R is a henselian local ring of depth at most two:

Theorem 6.1.3 *Let R be a henselian non-Gorenstein local ring of depth at most two. Suppose that there exists a non-free R-module in $\mathcal{G}(R)$. Then the category $\mathcal{G}(R)$ is not contravariantly finite in* $\mathrm{mod}R$.

We should remark that the above theorem especially asserts that Conjecture 6.1.2 holds if R is an artinian local ring and hence that the above theorem extends [60, Theorem 6.1].

In Section 2, we introduce several notions for later use. In Section 3, 4, and 5, we shall prove Theorem 6.1.3 when R has depth zero, one, and two, respectively.

6.2 Preliminaries

In this section, we recall the definitions of G-dimension and a (pre)cover, and give several preliminary lemmas involving Wakamatsu's Lemma, which plays a key role for proving Theorem 6.1.3.

Throughout this section, let R be a commutative noetherian local ring with unique maximal ideal $\mathfrak{m}$. Let $k = R/\mathfrak{m}$ be the residue class field of R. All R-modules in this section are assumed to be finitely generated.

First of all, we recall the definition of G-dimension. Put $M^* = \mathrm{Hom}_R(M, R)$ for an R-module M.

Definition 6.2.1 Let M be an R-module.

(1) If the following conditions hold, then we say that M has *G-dimension zero*, and write $\text{G-dim}_R M = 0$.

i) The natural homomorphism $M \to M^{**}$ is an isomorphism.

ii) $\mathrm{Ext}_R^i(M, R) = 0$ for every $i > 0$.

iii) $\mathrm{Ext}_R^i(M^*, R) = 0$ for every $i > 0$.

(2) If n is a non-negative integer such that there is an exact sequence

$$0 \to G_n \to G_{n-1} \to \cdots \to G_1 \to G_0 \to M \to 0$$

of R-modules with $\text{G-dim}_R G_i = 0$ for every $i = 0, 1, \cdots, n$, then we say that M has *G-dimension at most* n, and write $\text{G-dim}_R M \leq n$. If such an integer n does not exist, then we say that M has *infinite G-dimension*, and write $\text{G-dim}_R M = \infty$.

(3) If M has G-dimension at most n but does not have G-dimension at most $n-1$, then we say that M has G-dimension n and write $\text{G-dim}_R M = n$.

For an R-module M, we denote by $\Omega^n M$ the nth syzygy module of M, and set $\Omega M = \Omega^1 M$. G-dimension is a homological invariant for modules sharing a lot of properties with projective dimension. We state here just the properties that will be used later.

Proposition 6.2.2 (1) *The following conditions are equivalent.*

i) *R is Gorenstein.*

ii) $\text{G-dim}_R M < \infty$ *for any R-module M.*

iii) $\text{G-dim}_R k < \infty$.

(2) *Let M be a non-zero R-module with* $\text{G-dim}_R M < \infty$. *Then*

$$\text{G-dim}_R M = \operatorname{depth} R - \operatorname{depth}_R M.$$

(3) *Let $0 \to L \to M \to N \to 0$ be a short exact sequence of R-modules. If two of L, M, N have finite G-dimension, then so does the third.*

(4) *Let M be an R-module. Then*

$$\text{G-dim}_R(\Omega^n M) = \sup\{\text{G-dim}_R M - n, 0\}$$

for any $n \geq 0$.

(5) *Let M, N be R-modules. Then*

$$\text{G-dim}_R(M \oplus N) = \sup\{\text{G-dim}_R M, \text{G-dim}_R N\}.$$

The proof of this proposition and other properties of G-dimension are stated in detail in [4, Chapter 3,4] and [19, Chapter 1].

We denote by mod$R$ the category of finitely generated R-modules, and by $\mathcal{G}(R)$ the full subcategory of mod$R$ consisting of all R-modules of G-dimension zero. For the minimal free presentation

$$F_1 \xrightarrow{\partial} F_0 \to M \to 0$$

of an R-module M, we denote by tr$M$ the cokernel of the dual homomorphism $\partial^* : F_0^* \to F_1^*$. The following result follows directly from Proposition 6.2.2.

Corollary 6.2.3 *Let M be an R-module. The category $\mathcal{G}(R)$ has the following properties.*

(1) *If M belongs to $\mathcal{G}(R)$, then so do M^*, ΩM, $\mathrm{tr}M$, and any direct summand of M.*

(2) *Let $0 \to L \to M \to N \to 0$ be an exact sequence of R-modules. If L and N belong to $\mathcal{G}(R)$, then so does M.*

Now we introduce the notion of a cover of a module.

Definition 6.2.4 Let $\mathcal{X}$ be a full subcategory of mod$R$.

(1) Let $\phi : X \to M$ be a homomorphism from $X \in \mathcal{X}$ to $M \in \mathrm{mod}R$.

  i) We call ϕ an *$\mathcal{X}$-precover* of M if for any homomorphism $\phi' : X' \to M$ with $X' \in \mathcal{X}$ there exists a homomorphism $f : X' \to X$ such that $\phi' = \phi f$.

  ii) Assume that ϕ is an $\mathcal{X}$-precover of M. We call ϕ an *$\mathcal{X}$-cover* of M if any endomorphism f of X with $\phi = \phi f$ is an automorphism.

(2) The category $\mathcal{X}$ is said to be *contravariantly finite* if every $M \in \mathrm{mod}R$ has an $\mathcal{X}$-precover.

An $\mathcal{X}$-precover (resp. an $\mathcal{X}$-cover) is often called a right $\mathcal{X}$-approximation (resp. a right minimal $\mathcal{X}$-approximation).

Proposition 6.2.5 *[59, Lemma (2.2)] Let $\mathcal{X}$ be a full subcategory of* mod$R$. *Suppose that R is henselian.*

(1) *Let $0 \to N \xrightarrow{\psi} X \xrightarrow{\phi} M$ be an exact sequence of R-modules where ϕ is an $\mathcal{X}$-precover. Then the following conditions are equivalent.*

i) ϕ *is not an* $\mathcal{X}$*-cover.*

ii) *There exists a non-zero submodule* L *of* N *such that* $\psi(L)$ *is a direct summand of* X*.*

(2) *The following conditions are equivalent for an* R*-module* M*.*

i) M *has an* $\mathcal{X}$*-precover.*

ii) M *has an* $\mathcal{X}$*-cover.*

PROOF (1) ii) $\Longrightarrow$ i): Let X' be the complement of $\psi(L)$ in X. Let $\theta : X' \to X$ (resp. $\pi : X \to X'$) be the natural inclusion (resp. the natural projection), and set $f = \theta\pi$. We easily see that $\phi = \phi f$. Suppose that ϕ is an $\mathcal{X}$-cover. Then the endomorphism f of X is an isomorphism, and hence θ and π are isomorphisms. Therefore we have $\psi(L) = 0$. Since ψ is an injection, we have $L = 0$, which is contradiction. Thus ϕ is not an $\mathcal{X}$-cover.

i) $\Longrightarrow$ ii): There exists a non-isomorphism $f \in \mathrm{End}_R X$ such that $\phi = \phi f$. Let $S = R[f]$ be the subalgebra of $\mathrm{End}_R X$ generated by f over R. Note that S is a commutative ring.

Assume that S is a local ring. Let $\mathfrak{n}$ be the unique maximal ideal of S. Noting that S is a finitely generated R-module, we see that the factor ring $S/\mathfrak{m}S$ is an artinian local ring with maximal ideal $\mathfrak{n}/\mathfrak{m}S$. Hence $n^r \subseteq \mathfrak{m}S$ for some integer r. Since $f \in \mathfrak{n}$, we have

$$f^r = a_0 + a_1 f + \cdots + a_s f^s$$

with $a_i \in \mathfrak{m}$, $0 \leq i \leq s$. Noting that $\phi = \phi f$, we get

$$\begin{aligned} \phi &= \phi f^r \\ &= (a_0 + a_1 + \cdots + a_s)\phi \\ &\in \mathfrak{m}\phi. \end{aligned}$$

It follows from Nakayama's Lemma that $\phi = 0$, i.e., the homomorphism $\psi : N \to X$ is isomorphic. Since f is not an isomorphism, we especially have $X \neq 0$, and hence $N \neq 0$. The module $L := N$ satisfies the condition ii).

Thus, it is enough to consider the case that S is not a local ring. Since R is henselian, the finite R-algebra S is a product of local rings, and hence there is a non-trivial idempotent e in S. Write

$$e = b_0 + b_1 f + \cdots + b_t f^t$$

with $b_i \in R$, $0 \leq i \leq t$, and put $b = b_0 + b_1 + \cdots + b_t$. Taking $1 - e$ instead of e when b is an element in $\mathfrak{m}$, we may assume that b is not an element in $\mathfrak{m}$, i.e., b is a unit of R.

It is easy to see that we have a direct sum decomposition

$$X = \mathrm{Ker}\, e \oplus \mathrm{Im}\, e.$$

Since e is not an isomorphism, we have $\mathrm{Ker}\, e \neq 0$. Noting that $\phi e = b\phi$ and that b is a unit of R, we obtain $\mathrm{Ker}\, e \subseteq \mathrm{Im}\, \psi$. Thus $L := \psi^{-1}(\mathrm{Ker}\, e)$ satisfies the condition ii).

(2) It is obvious that ii) implies i). Let $\phi : X \to M$ be an $\mathcal{X}$-precover of M. Putting $N = \mathrm{Ker}\, \phi$, we get an exact sequence

$$0 \to N \xrightarrow{\psi} X \xrightarrow{\phi} M,$$

where ψ is the natural inclusion. Suppose that ϕ is not an $\mathcal{X}$-cover. Then it follows from (1) that there exists a non-zero submodule L of N such that $\psi(L)$ is a direct summand of X. Note that L is a direct summand of N. Let N' (resp. X') be the complement of L (resp. $\psi(L)$) in N (resp. X). Then X' belongs to $\mathcal{X}$ by the assumption, and an exact sequence

$$0 \to N' \xrightarrow{\psi'} X' \xrightarrow{\phi'} M$$

is induced. It is easily seen that ϕ' is an $\mathcal{X}$-precover of M. Since the minimal number of generators of X' is smaller than that of X, repeating the same argument, we eventually obtain an $\mathcal{X}$-cover of M. $\square$

We say that a full subcategory $\mathcal{X}$ of mod$R$ is closed under direct summands if any direct summand of any R-module belonging to $\mathcal{X}$ also belongs to $\mathcal{X}$. Note by Corollary 6.2.3(1) that $\mathcal{G}(R)$ is closed under direct summands.

Remark 6.2.6 Under the assumption of the above proposition, suppose that $\mathcal{X}$ is closed under direct summands. Let $0 \to N \xrightarrow{\psi} X \xrightarrow{\phi} M$ be an exact sequence of R-modules where ϕ is an $\mathcal{X}$-precover. It is seen from the proof of the second statement in the above proposition that there exists a direct summand L of N satisfying the following conditions:

i) $\psi(L)$ is a direct summand of X.

ii) Let N' (resp. X') be the complement of L (resp. $\psi(L)$) in N (resp. X), and let $0 \to N' \xrightarrow{\psi'} X' \xrightarrow{\phi'} M$ be the induced exact sequence. Then ϕ' is an $\mathcal{X}$-cover of M.

We say that a full subcategory $\mathcal{X}$ of mod$R$ is closed under extensions provided that for any short exact sequence $0 \to L \to M \to N \to 0$ in mod$R$, if $L, N \in \mathcal{X}$ then $M \in \mathcal{X}$. Note by Corollary 6.2.3(2) that $\mathcal{G}(R)$ is closed under extensions. The lemma below is so-called Wakamatsu's Lemma, which plays an important role in the notion of a cover. For the proof, see [55] or [56, Lemma 2.1.1].

Lemma 6.2.7 (Wakamatsu) *Let $\mathcal{X}$ be a full subcategory of* $\mathrm{mod}R$ *which is closed under extensions, and let* $0 \to N \to X \xrightarrow{\phi} M$ *be an exact sequence of R-modules where ϕ is an $\mathcal{X}$-cover. Then* $\mathrm{Ext}^1_R(Y, N) = 0$ *for every* $Y \in \mathcal{X}$.

For R-modules M, N, we define a homomorphism $\lambda_M(N) : M \otimes_R N \to \mathrm{Hom}_R(M^*, N)$ of R-modules by $\lambda_M(N)(m \otimes n)(f) = f(m)n$ for $m \in M,\ n \in N$ and $f \in M^*$.

Proposition 6.2.8 *[4, Proposition 2.6] Let M be an R-module. There is an exact sequence*

$$\begin{array}{ccccccc} 0 & \longrightarrow & \mathrm{Ext}^1_R(\mathrm{tr}M, -) & \longrightarrow & M \otimes_R - & \xrightarrow{\lambda_M(-)} & \mathrm{Hom}_R(M^*, -) \\ & \longrightarrow & \mathrm{Ext}^2_R(\mathrm{tr}M, -) & \longrightarrow & 0 & & \end{array}$$

of functors from $\mathrm{mod}R$ *to itself.*

We close this section by showing the following proposition, which proves that Conjecture 6.1.2 implies Conjecture 6.1.1.

Proposition 6.2.9 *Suppose that there exist only finitely many isomorphism classes of indecomposable R-modules of G-dimension zero. Then $\mathcal{G}(R)$ is a contravariantly finite subcategory of* $\mathrm{mod}R$.

Proof We can show this by means of the idea appearing in the proof of [6, Proposition 4.2]. Fix an R-module M. Let X be the direct sum of the complete representatives of the isomorphism classes of indecomposable R-modules of G-dimension zero. Note that the R-module X is finitely generated. Taking a system of generators $\phi_1, \phi_2, \cdots, \phi_n$ of the R-module $\mathrm{Hom}_R(X, M)$, we easily see that the homomorphism $(\phi_1, \phi_2, \cdots, \phi_n) : X^n \to M$ is a $\mathcal{G}(R)$-precover of M. It follows that $\mathcal{G}(R)$ is contravariantly finite in $\mathrm{mod}R$. $\square$

6.3 The depth zero case

Throughout this section, R is always a local ring with maximal ideal $\mathfrak{m}$ and with residue class field k. We assume that all R-modules in this section are finitely generated.

Theorem 6.3.1 *Let R be a henselian non-Gorenstein local ring of depth zero. Suppose that there exists a non-free R-module in $\mathcal{G}(R)$. Then the residue class field of R does not admit a $\mathcal{G}(R)$-precover. In particular, the category $\mathcal{G}(R)$ is not contravariantly finite in* $\mathrm{mod}R$.

Proof Suppose that the residue field k of R has a $\mathcal{G}(R)$-precover as an R-module. We want to derive contradiction. Proposition 6.2.5(2) implies that k has a $\mathcal{G}(R)$-cover $\pi : Z \to k$. Since $R \in \mathcal{G}(R)$, every homomorphism $R \to k$ is factored as $R \to Z \xrightarrow{\pi} k$, that is, the homomorphism π is surjective. Hence there exists a short exact sequence

$$0 \to L \xrightarrow{\theta} Z \xrightarrow{\pi} k \to 0$$

of R-modules. Dualizing this sequence, we obtain an exact sequence

$$0 \to k^* \xrightarrow{\pi^*} Z^* \xrightarrow{\theta^*} L^*.$$

Set $C = \mathrm{Im}(\theta^*)$, and let $\alpha : Z^* \to C$ be the map induced by θ^* and $\beta : C \to L^*$ be the natural embedding.

We shall show that the surjective homomorphism $\alpha : Z^* \to C$ is a $\mathcal{G}(R)$-cover of C. Fix $X \in \mathcal{G}(R)$. To prove that any homomorphism $X \to C$ is factored as $X \to Z^* \xrightarrow{\alpha} C$, we may assume that X is non-free and indecomposable. Applying the functor $\mathrm{Hom}_R(X, -)$ to the above exact sequence, we get an exact sequence

$$0 \longrightarrow \mathrm{Hom}_R(X, k^*) \xrightarrow{\mathrm{Hom}_R(X,\pi^*)} \mathrm{Hom}_R(X, Z^*) \xrightarrow{\mathrm{Hom}_R(X,\theta^*)} \mathrm{Hom}_R(X, L^*).$$

Here we establish a claim.

Claim *The homomorphism* $\mathrm{Hom}_R(X, \theta^*)$ *is a split epimorphism.*

Proof Note from Corollary 6.2.3(1) that $\mathrm{tr}X$ and $\Omega\mathrm{tr}X$ belong to $\mathcal{G}(R)$. Applying Lemma 6.2.7, we see that $\mathrm{Ext}^1_R(\mathrm{tr}X, L) = 0$ and $\mathrm{Ext}^2_R(\mathrm{tr}X, L) \cong \mathrm{Ext}^1_R(\Omega\mathrm{tr}X, L) = 0$. Hence Proposition 6.2.8 shows that $\lambda_X(L)$ is an isomorphism. On the other hand, noting that X is non-free and indecomposable, we easily see that $\lambda_X(k)$ is the zero map. Thus we obtain a commutative diagram

$$\begin{array}{ccc}
 & & 0 \\
 & & \downarrow \\
X \otimes_R L & \xrightarrow[\cong]{\lambda_X(L)} & \mathrm{Hom}_R(X^*, L) \\
{\scriptstyle X\otimes_R\theta}\downarrow & & \downarrow{\scriptstyle \mathrm{Hom}_R(X^*,\theta)} \\
X \otimes_R Z & \xrightarrow{\lambda_X(Z)} & \mathrm{Hom}_R(X^*, Z) \\
{\scriptstyle X\otimes_R\pi}\downarrow & & \downarrow{\scriptstyle \mathrm{Hom}_R(X^*,\pi)} \\
X \otimes_R k & \xrightarrow[0]{\lambda_X(k)} & \mathrm{Hom}_R(X^*, k) \\
\downarrow & & \\
0 & &
\end{array}$$

with exact columns.

Since $\mathrm{Hom}_R(X^*, \pi) \cdot \lambda_X(Z) = 0$, there exists a homomorphism $\rho : X \otimes_R Z \to \mathrm{Hom}_R(X^*, L)$ such that $\mathrm{Hom}_R(X^*, \theta) \cdot \rho = \lambda_X(Z)$. Therefore we have $\rho \cdot (X \otimes_R \theta) = \lambda_X(L)$ because $\mathrm{Hom}_R(X^*, \theta)$ is an injection. Since $\lambda_X(L)$ is an isomorphism, $X \otimes_R \theta$ is a split monomorphism, and hence $(X \otimes_R \theta)^* : (X \otimes_R Z)^* \to (X \otimes_R L)^*$ is a split epimorphism. There is a commutative diagram

$$\begin{array}{ccc}
(X \otimes_R Z)^* & \xrightarrow{(X\otimes_R\theta)^*} & (X \otimes_R L)^* \\
\downarrow{\scriptstyle \cong} & & \downarrow{\scriptstyle \cong} \\
\mathrm{Hom}_R(X, Z^*) & \xrightarrow{\mathrm{Hom}_R(X,\theta^*)} & \mathrm{Hom}_R(X, L^*),
\end{array}$$

where the vertical maps are natural isomorphisms. It follows that $\mathrm{Hom}_R(X, \theta^*)$ is also a split epimorphism, and the claim is proved. $\square$

Since $\mathrm{Hom}_R(X, \theta^*) = \mathrm{Hom}_R(X, \beta) \cdot \mathrm{Hom}_R(X, \alpha)$ and $\mathrm{Hom}_R(X, \beta)$ is an injection, the above claim implies that $\mathrm{Hom}_R(X, \beta)$ is an isomorphism. Therefore $\mathrm{Hom}_R(X, \alpha)$ is a split epimorphism, and hence it is especially a surjection. This means that the homomorphism $\alpha : Z^* \to C$ is a $\mathcal{G}(R)$-precover of C.

Assume that α is not a $\mathcal{G}(R)$-cover. Then Proposition 6.2.5(1) shows that k^* has some non-zero summand whose image by π^* is a direct summand of Z^*. Since k^* is a k-vector space, the R-module Z^* has a summand isomorphic to the R-module k, and hence $k \in \mathcal{G}(R)$ by Corollary 6.2.3(1). It follows from Proposition 6.2.2(1) that R is Gorenstein, which contradicts the assumption of the theorem. Therefore α must be a $\mathcal{G}(R)$-cover of C.

Thus we can apply Lemma 6.2.7, and get $\mathrm{Ext}^1_R(Y, k^*) = 0$ for every $Y \in \mathcal{G}(R)$. Since R has depth zero, in other words, k^* is a non-zero k-vector space, every R-module in $\mathcal{G}(R)$ is free, which is contrary to the assumption of the theorem. This contradiction completes the proof of the theorem. $\square$

According to Proposition 6.2.9, we have the following result that gives a corollary of the above theorem:

Corollary 6.3.2 *Let R be a henselian non-Gorenstein local ring of depth zero. Suppose that there exists a non-free R-module of G-dimension zero. Then there exist infinitely many isomorphism classes of indecomposable R-modules of G-dimension zero.*

6.4 The depth one case

Throughout this section, R is always a local ring with maximal ideal $\mathfrak{m}$ and with residue class field k. We assume that all R-modules in this section are finitely generated.

We begin with proving the following proposition:

Proposition 6.4.1 *Suppose that there is a direct sum decomposition $\mathfrak{m} = I \oplus J$ where I, J are non-zero ideals of R and* $\text{G-dim}_R I < \infty$*. Then R is a Gorenstein local ring of dimension one.*

Proof We proceed the proof step by step.

Step 1 We show that $\operatorname{depth} R \le 1$. For this, according to [29, Proposition 2.1], it is enough to prove that the punctured spectrum $P = \operatorname{Spec} R - \{\mathfrak{m}\}$ of R is disconnected. For an ideal $\mathfrak{a}$ of R, let $V(\mathfrak{a})$ denote the set of all prime ideals of R containing $\mathfrak{a}$. Now we have $\mathfrak{m} = I + J$, $0 = I \cap J$, and $I, J \neq 0$. Hence P is the disjoint union of the two non-empty closed subsets $V(I) \cap P$ and $V(J) \cap P$. Therefore P is disconnected.

Step 2 We show that $\operatorname{depth} R = 1$. Suppose that $\operatorname{depth} R = 0$. Then note from Proposition 6.2.2(2) and 6.2.2(4) that $I, R/I \in \mathcal{G}(R)$ and $\operatorname{depth} R/I = 0$. Dualizing the natural exact sequence $0 \to I \to R \to R/I \to 0$, we obtain another exact sequence

$$0 \to (0 :_R I) \to R \to I^* \to 0.$$

Hence we have an isomorphism $I^* \cong R/(0 :_R I)$. It follows that

$$I \cong I^{**} \cong (R/(0 :_R I))^* \cong (0 :_R (0 :_R I)).$$

It is easy to see from this that $I = (0 :_R (0 :_R I))$. Since $(0 :_R I) \subseteq \mathfrak{m}$, we have $I = (0 :_R (0 :_R I)) \supseteq (0 :_R \mathfrak{m})$. On the other hand, note that $(0 :_{\mathfrak{m}/I} \mathfrak{m}) = (0 :_{R/I} \mathfrak{m}) \neq 0$. Since $J \cong \mathfrak{m}/I$, we see that $(0 :_J \mathfrak{m}) \neq 0$. However, we have $(0 :_J \mathfrak{m}) = (0 :_R \mathfrak{m}) \cap J \subseteq I \cap J = 0$, which is contradiction. This contradiction says that $\operatorname{depth} R = 1$, as desired.

Step 3 We show that $I = (0 :_R J)$ and $J = (0 :_R I)$. Noting that $IJ \subseteq I \cap J = 0$, we have $I \subseteq (0 :_R J)$ and $J \subseteq (0 :_R I)$. Hence $\mathfrak{m} = I + J \subseteq (0 :_R I) + (0 :_R J)$, and therefore $\mathfrak{m} = (0 :_R I) + (0 :_R J)$. Since $(0 :_R I) \cap (0 :_R J) = \operatorname{Soc} R = 0$, we obtain another decomposition

$$\mathfrak{m} = (0 :_R I) \oplus (0 :_R J)$$

of $\mathfrak{m}$. Consider the endomorphism

$$I \subseteq (0 :_R J) \subseteq \mathfrak{m} = I \oplus J \xrightarrow{\delta} I$$

of I, where δ is the projection onto I. It is easy to see that this endomorphism is the identity map of I, and hence I is a direct summand of $(0 :_R J)$. Similarly, J is a direct summand of $(0 :_R I)$. Write $(0 :_R J) = I \oplus I'$ and $(0 :_R I) = J \oplus J'$ for some ideals I', J', and we obtain

$$I \oplus J = \mathfrak{m} = (0 :_R I) \oplus (0 :_R J) = J \oplus J' \oplus I \oplus I'.$$

Thus we see that $I' = J' = 0$, and hence we have $I = (0 :_R J)$ and $J = (0 :_R I)$.

Step 4 We show that R is a Gorenstein local ring of dimension one. Dualizing the natural exact sequence $0 \to J \to R \to R/J \to 0$, we have an exact sequence

$$0 \to (0 :_R J) \to R \xrightarrow{\epsilon} J^*.$$

Since $\operatorname{depth}_R J^* \geq \inf\{2, \operatorname{depth} R\} > 0$ by [18, Exercise 1.4.19], the R-module $\operatorname{Im} \epsilon \cong R/I$ has positive depth. Therefore Proposition 6.2.2(2) implies that $R/I \in \mathcal{G}(R)$. It follows from this and Proposition 6.2.2(4) that $I \in \mathcal{G}(R)$ and $J = (0 :_R I) \cong (R/I)^* \in \mathcal{G}(R)$. Thus, $\mathfrak{m} = I \oplus J \in \mathcal{G}(R)$ by Proposition 6.2.2(5), and R is Gorenstein by Proposition 6.2.2(1) and 6.2.2(4). Hence we have $\dim R = \operatorname{depth} R = 1$. □

Now we can prove the main theorem of this section.

Theorem 6.4.2 *Let R be a henselian non-Gorenstein local ring of depth one. Suppose that there exists a non-free R-module in $\mathcal{G}(R)$. Then the maximal ideal of R does not admit a $\mathcal{G}(R)$-precover. In particular, the category $\mathcal{G}(R)$ is not contravariantly finite in* $\operatorname{mod} R$.

Proof Suppose that $\mathfrak{m}$ admits a $\mathcal{G}(R)$-precover. We want to derive contradiction. Proposition 6.2.5(2) implies that $\mathfrak{m}$ admits a $\mathcal{G}(R)$-cover $\pi : X \to \mathfrak{m}$. Since $R \in \mathcal{G}(R)$, any homomorphism from R to $\mathfrak{m}$ factors through π. Hence π is a surjective homomorphism. Setting $L = \operatorname{Ker}\pi$, we get an exact sequence

$$0 \to L \xrightarrow{\theta} X \xrightarrow{\pi} \mathfrak{m} \to 0, \tag{6.1}$$

where θ is the natural embedding. Lemma 6.2.7 says that $\operatorname{Ext}^1_R(G, L) = 0$ for every $G \in \mathcal{G}(R)$. According to Corollary 6.2.3(1), we have $\operatorname{Ext}^i_R(G, L) = 0$ for every $G \in \mathcal{G}(R)$ and every $i > 0$.

Fix $Y \in \mathcal{G}(R)$ which is non-free and indecomposable. Since $\mathrm{tr}Y \in \mathcal{G}(R)$ by Corollary 6.2.3(1), we have $\operatorname{Ker}\lambda_Y(L) = \operatorname{Ext}^1_R(\mathrm{tr}Y, L) = 0$ and $\operatorname{Coker}\lambda_Y(L) = \operatorname{Ext}^2_R(\mathrm{tr}Y, L) = 0$ by Proposition 6.2.8. This means that $\lambda_Y(L)$ is an isomorphism. Hence the composite map $\lambda_Y(X) \cdot (Y \otimes_R \theta) = \operatorname{Hom}_R(Y^*, \theta) \cdot \lambda_Y(L)$ is injective, and therefore so is the map $Y \otimes_R \theta$. Also, we have $\operatorname{Ext}^1_R(Y^*, L) = 0$ because $Y^* \in \mathcal{G}(R)$ by Corollary 6.2.3(1). Thus we obtain a commutative diagram

$$\begin{tikzcd}
0 \arrow[d] & 0 \arrow[d] \\
Y \otimes_R L \arrow[r, "\lambda_Y(L)", "\cong"'] \arrow[d, "Y\otimes_R\theta"'] & \operatorname{Hom}_R(Y^*, L) \arrow[d, "\operatorname{Hom}_R(Y^*,\theta)"] \\
Y \otimes_R X \arrow[r, "\lambda_Y(X)"] \arrow[d, "Y\otimes_R\pi"'] & \operatorname{Hom}_R(Y^*, X) \arrow[d, "\operatorname{Hom}_R(Y^*,\pi)"] \\
Y \otimes_R \mathfrak{m} \arrow[r, "\lambda_Y(\mathfrak{m})"] \arrow[d] & \operatorname{Hom}_R(Y^*, \mathfrak{m}) \arrow[d] \\
0 & 0
\end{tikzcd}$$

with exact columns, and this induces a commutative diagram

$$\begin{tikzcd}
\operatorname{Hom}_R(Y^*, X)^* \arrow[r] \arrow[d] & \operatorname{Hom}_R(Y^*, L)^* \arrow[r] \arrow[d, "\cong"] & \operatorname{Ext}^1_R(\operatorname{Hom}_R(Y^*, \mathfrak{m}), R) \arrow[d] \\
(Y \otimes_R X)^* \arrow[r, "\rho"] & (Y \otimes_R L)^* \arrow[r] & \operatorname{Ext}^1_R(Y \otimes_R \mathfrak{m}, R)
\end{tikzcd}$$

with exact rows.

Here, let us examine the module $\operatorname{Ext}^1_R(\operatorname{Hom}_R(Y^*, \mathfrak{m}), R)$. Since Y^* is a non-free indecomposable module, any homomorphism from Y^* to R factors through $\mathfrak{m}$. Therefore we have $\mathrm{Hom}_R(Y^*, \mathfrak{m}) \cong \mathrm{Hom}_R(Y^*, R) \cong Y$, and hence $\mathrm{Ext}^1_R(\mathrm{Hom}_R(Y^*, \mathfrak{m}), R) = 0$ because $Y \in \mathcal{G}(R)$. This means that the homomorphism ρ in the above diagram is surjective.

Dualizing the exact sequence (6.1), we obtain an exact sequence

$$0 \to \mathfrak{m}^* \xrightarrow{\pi^*} X^* \xrightarrow{\theta^*} L^*.$$

Set $C = \mathrm{Im}(\theta^*)$ and let $\sigma : X^* \to C$ be the surjection induced by θ^*. The surjectivity of ρ says that the homomorphism $\mathrm{Hom}_R(Y, \theta^*) : \mathrm{Hom}_R(Y, X^*) \to \mathrm{Hom}_R(Y, L^*)$ is also surjective since the two may be identified, and so is the homomorphism $\mathrm{Hom}_R(Y, \sigma) : \mathrm{Hom}_R(Y, X^*) \to \mathrm{Hom}_R(Y, C)$. This means that σ is a $\mathcal{G}(R)$-precover. According to Remark 6.2.6, we can take a direct summand Z of $\mathfrak{m}^*$ satisfying the following conditions:

i) $\pi^*(Z)$ is a direct summand of X^*.

ii) Let M (resp. W) be the complement of Z (resp. $\pi^*(Z)$) in $\mathfrak{m}^*$ (resp. X^*), and let $0 \to M \to W \xrightarrow{\tau} C \to 0$ be the induced exact sequence. Then τ is a $\mathcal{G}(R)$-cover.

Lemma 6.2.7 yields

$$\mathrm{Ext}^1_R(G, M) = 0 \tag{6.2}$$

for any $G \in \mathcal{G}(R)$.

Now, we prove that the maximal ideal $\mathfrak{m}$ is a reflexive ideal. Dualizing the natural exact sequence $0 \to \mathfrak{m} \xrightarrow{\zeta} R \to k \to 0$, we obtain an exact sequence

$$0 \to R \xrightarrow{\mu} \mathfrak{m}^* \to k^r \to 0, \tag{6.3}$$

where r is a positive integer because $\operatorname{depth} R = 1$. Dualizing this exact sequence again, we obtain an injection $\nu : \mathfrak{m}^{**} \to R$, which maps $\xi \in \mathfrak{m}^{**}$ to $\xi(\zeta) \in R$. Let $\eta : \mathfrak{m} \to \mathfrak{m}^{**}$ be the natural homomorphism. Then we easily see that $\zeta = \nu\eta$. Thus, we can regard $\mathfrak{m}^{**}$ as an ideal of R containing the maximal ideal $\mathfrak{m}$, and hence either ν or η is an isomorphism.

Assume that ν is an isomorphism. Then there exists $\xi \in \mathfrak{m}^{**}$ such that $\xi(\zeta) = 1$. This means that the composition $\xi\mu$ is the identity map of R, and hence the exact sequence (6.3) splits. Therefore we have $R \oplus k^r \cong \mathfrak{m}^* = Z \oplus M$. Noting that Z is isomorphic to $\pi^*(Z)$ which is a direct summand of X^*, we see from Corollary 6.2.3(1) that $X^* \in \mathcal{G}(R)$ and that $Z \in \mathcal{G}(R)$. Since $k \notin \mathcal{G}(R)$ by Proposition 6.2.2(1), we see from the Krull-Schmidt Theorem that $Z \cong R$ and $M \cong k^r$. According to (6.2), every R-module in $\mathcal{G}(R)$ is free, and we obtain contradiction. Hence η must be an isomorphism, which says that $\mathfrak{m}$ is a reflexive ideal, as desired.

Thus, we have $\mathfrak{m} \cong \mathfrak{m}^{**} \cong Z^* \oplus M^*$. It follows that the module $Z^* \in \mathcal{G}(R)$ can be regarded as a subideal of $\mathfrak{m}$. Since R is not Gorenstein, Proposition 6.4.1 implies that either $Z^* = 0$ or $M^* = 0$. But if $M^* = 0$, then $\mathfrak{m} \cong Z^* \in \mathcal{G}(R)$, which would imply that R was Gorenstein by Proposition 6.2.2(1) and 6.2.2(4). Thus, $Z^* = 0$. Hence $Z \cong Z^{**} = 0$, and therefore $M = \mathfrak{m}^*$. By (6.2) and (6.3), for every $G \in \mathcal{G}(R)$, we have $\mathrm{Ext}^1_R(G, k^r) = 0$, and see that G is a free R-module, which is contrary to the assumption of our theorem. This contradiction completes the proof of our theorem. $\square$

According to Proposition 6.2.9, we have the following result that gives a corollary of the above theorem:

Corollary 6.4.3 *Suppose that R is a henselian non-Gorenstein local ring of depth one and that there exists a non-free R-module of G-dimension zero. Then there exist infinitely many isomorphism classes of indecomposable R-modules of G-dimension zero.*

6.5 The depth two case

Throughout this section, R is always a local ring with maximal ideal $\mathfrak{m}$ and with residue class field k. We assume that all R-modules in this section are finitely generated.

Theorem 6.5.1 *Let R be a henselian non-Gorenstein local ring of depth two. Suppose that there exists a non-free R-module in $\mathcal{G}(R)$. Then the category $\mathcal{G}(R)$ is not contravariantly finite in* mod$R$.

Proof Since $\mathrm{Ext}^1_R(\mathfrak{m}, R) \cong \mathrm{Ext}^2_R(k, R) \neq 0$, we have a non-split exact sequence

$$\sigma : 0 \to R \to M \to \mathfrak{m} \to 0. \qquad (6.4)$$

Dualizing this, we obtain an exact sequence

$$0 \to \mathfrak{m}^* \to M^* \to R^* \xrightarrow{\eta} \mathrm{Ext}^1_R(\mathfrak{m}, R).$$

Note from definition that the connecting homomorphism η sends $\mathrm{id}_R \in R^*$ to the element $s \in \mathrm{Ext}^1_R(\mathfrak{m}, R)$ corresponding to the exact sequence σ. Since σ does not split, s is a non-zero element of $\mathrm{Ext}^1_R(\mathfrak{m}, R)$. Hence η is a non-zero map. Noting that $\mathrm{Ext}^1_R(\mathfrak{m}, R) \cong \mathrm{Ext}^2_R(k, R)$, we see that the image of η is annihilated by $\mathfrak{m}$. Also noting that $\mathfrak{m}^* \cong R^* \cong R$, we get an exact sequence

$$0 \to R \to M^* \to \mathfrak{m} \to 0. \qquad (6.5)$$

Claim 1 *The modules* $\mathrm{Hom}_R(G, M)$ *and* $\mathrm{Hom}_R(G, M^*)$ *belong to* $\mathcal{G}(R)$ *for every non-free indecomposable module* $G \in \mathcal{G}(R)$.

Proof Applying the functor $\mathrm{Hom}_R(G, -)$ to the exact sequence (6.4) gives an exact sequence

$$0 \to G^* \to \mathrm{Hom}_R(G, M) \to \mathrm{Hom}_R(G, \mathfrak{m}) \to \mathrm{Ext}^1_R(G, R).$$

Since G is non-free and indecomposable, any homomorphism from G to R factors through $\mathfrak{m}$, and hence $\mathrm{Hom}_R(G, \mathfrak{m}) \cong G^*$. Also, since $G \in \mathcal{G}(R)$, we have $\mathrm{Ext}^1_R(G, R) = 0$. Thus Corollary 6.2.3(2) implies that $\mathrm{Hom}_R(G, M) \in \mathcal{G}(R)$. The same argument for the exact sequence (6.5) shows that $\mathrm{Hom}_R(G, M^*) \in \mathcal{G}(R)$. $\square$

We shall prove that the module M can not have a $\mathcal{G}(R)$-precover. Suppose that M has a $\mathcal{G}(R)$-precover. Then M has a $\mathcal{G}(R)$-cover $\pi : X \to M$ by Proposition 6.2.5(2). Since $R \in \mathcal{G}(R)$, any homomorphism from R to M factors through π. Hence π is a surjective homomorphism. Setting $N = \mathrm{Ker}\,\pi$, we get an exact sequence

$$0 \to N \xrightarrow{\theta} X \xrightarrow{\pi} M \to 0, \tag{6.6}$$

where θ is the natural embedding. We see from Corollary 6.2.3 and Lemma 6.2.7 that $\mathrm{Ext}^i_R(G, N) = 0$ for any $G \in \mathcal{G}(R)$ and any $i > 0$. Dualizing the exact sequence (6.6), we obtain an exact sequence

$$0 \to M^* \xrightarrow{\pi^*} X^* \xrightarrow{\theta^*} N^*.$$

Put $C = \mathrm{Im}(\theta^*)$ and let $\mu : X^* \to C$ be the surjection induced by θ^*.

Claim 2 *The homomorphism* μ *is a* $\mathcal{G}(R)$*-precover of* C.

Proof Fix a non-free indecomposable module $G \in \mathcal{G}(R)$. Applying the functors $G \otimes_R -$ and $\mathrm{Hom}_R(G^*, -)$ to the exact sequence (6.6) yields a commutative diagram

$$\begin{tikzcd}
 & 0 \arrow[d] \\
G \otimes_R N \arrow[r, "\lambda_G(N)"] \arrow[d, "G\otimes_R\theta"'] & \mathrm{Hom}_R(G^*, N) \arrow[d, "\mathrm{Hom}_R(G^*,\theta)"'] \\
G \otimes_R X \arrow[r, "\lambda_G(X)"] \arrow[d, "G\otimes_R\pi"'] & \mathrm{Hom}_R(G^*, X) \arrow[d, "\mathrm{Hom}_R(G^*,\pi)"'] \\
G \otimes_R M \arrow[r, "\lambda_G(M)"] \arrow[d] & \mathrm{Hom}_R(G^*, M) \arrow[d] \\
0 & \mathrm{Ext}^1_R(G^*, N)
\end{tikzcd}$$

with exact columns. Noting that $\mathrm{tr}G \in \mathcal{G}(R)$ by Corollary 6.2.3(1), we see from Proposition 6.2.8 that $\mathrm{Ker}\,\lambda_G(N) = \mathrm{Ext}^1_R(\mathrm{tr}G, N) = 0$ and $\mathrm{Coker}\,\lambda_G(N) = \mathrm{Ext}^2_R(\mathrm{tr}G, N) = 0$. This means that $\lambda_G(N)$ is an isomorphism. It follows from the commutativity of the above diagram that the homomorphism $G \otimes_R \theta$ is injective. Also, we have $\mathrm{Ext}^1_R(G^*, N) = 0$ because $G^* \in \mathcal{G}(R)$ by Corollary 6.2.3(1). Thus we obtain a commutative diagram

$$\begin{tikzcd}
0 \arrow[d] & 0 \arrow[d] \\
G \otimes_R N \arrow[r, "\lambda_G(N)", "\cong"'] \arrow[d, "G\otimes_R\theta"'] & \mathrm{Hom}_R(G^*, N) \arrow[d] \\
G \otimes_R X \arrow[r] \arrow[d] & \mathrm{Hom}_R(G^*, X) \arrow[d] \\
G \otimes_R M \arrow[r] \arrow[d] & \mathrm{Hom}_R(G^*, M) \arrow[d] \\
0 & 0
\end{tikzcd}$$

with exact columns. Dualizing this diagram induces a commutative diagram

$$\begin{tikzcd}
\mathrm{Hom}_R(G^*, X)^* \arrow[r] \arrow[d] & \mathrm{Hom}_R(G^*, N)^* \arrow[r] \arrow[d, "(\lambda_G(N))^*"', "\cong"] & \mathrm{Ext}^1_R(\mathrm{Hom}_R(G^*, M), R) \arrow[d] \\
(G \otimes_R X)^* \arrow[r, "(G\otimes_R\theta)^*"] & (G \otimes_R N)^* \arrow[r] & \mathrm{Ext}^1_R(G \otimes_R M, R)
\end{tikzcd}$$

with exact rows. Since $\mathrm{Hom}_R(G^*, M) \in \mathcal{G}(R)$ by Claim 1, we have $\mathrm{Ext}^1_R(\mathrm{Hom}_R(G^*, M), R) = 0$. From the above commutative diagram, it is seen that $(G \otimes_R \theta)^*$ is a surjective homomorphism. Note that there is a natural commutative diagram

$$\begin{tikzcd}
(G \otimes_R X)^* \arrow[r, "(G\otimes_R\theta)^*"] \arrow[d, "\cong"] & (G \otimes_R N)^* \arrow[d, "\cong"] \\
\mathrm{Hom}_R(G, X^*) \arrow[r, "\mathrm{Hom}_R(G,\theta^*)"] & \mathrm{Hom}_R(G, N^*)
\end{tikzcd}$$

with isomorphic vertical maps. Therefore the homomorphism $\mathrm{Hom}_R(G, \theta^*)$ is also surjective, and so is the homomorphism $\mathrm{Hom}_R(G, \mu) : \mathrm{Hom}_R(G, X^*) \to \mathrm{Hom}_R(G, C)$. It is easy to see from this that μ is a $\mathcal{G}(R)$-precover of C. $\square$

According to Claim 2 and Remark 6.2.6, we have direct sum decompositions $M^* = Y \oplus L$, $X^* = \pi^*(Y) \oplus Z$, and an exact sequence

$$0 \to L \to Z \xrightarrow{\nu} C \to 0$$

where ν is a $\mathcal{G}(R)$-cover of C. Since Y is isomorphic to the direct summand $\pi^*(Y)$ of X^*, Corollary 6.2.3(1) implies that $Y \in \mathcal{G}(R)$. Lemma 6.2.7 yields $\mathrm{Ext}^1_R(G, L) = 0$ for any $G \in \mathcal{G}(R)$.

Claim 3 *The module* $\mathrm{Hom}_R(G, Y)$ *belongs to* $\mathcal{G}(R)$ *for any* $G \in \mathcal{G}(R)$.

Proof We may assume that G is non-free and indecomposable. The module $\mathrm{Hom}_R(G, Y)$ is isomorphic to a direct summand of $\mathrm{Hom}_R(G, M^*)$. Since the module $\mathrm{Hom}_R(G, M^*)$ is an object of $\mathcal{G}(R)$ by Claim 1, so is the module $\mathrm{Hom}_R(G, Y)$ by Corollary 6.2.3(1). $\square$

Here, by the assumption of the theorem, we have a non-free indecomposable module $W \in \mathcal{G}(R)$. There is an exact sequence

$$0 \to \Omega W \to F \to W \to 0$$

of R-modules such that F is a free module. Applying the functor $\mathrm{Hom}_R(-, Y)$ to this exact sequence, we get an exact sequence

$$0 \to \mathrm{Hom}_R(W, Y) \to \mathrm{Hom}_R(F, Y) \to \mathrm{Hom}_R(\Omega W, Y) \to \mathrm{Ext}^1_R(W, Y) \to 0.$$

Since $\mathrm{Hom}_R(W, Y)$, $\mathrm{Hom}_R(F, Y)$, and $\mathrm{Hom}_R(\Omega W, Y)$ belong to $\mathcal{G}(R)$ by Claim 3, the R-module $\mathrm{Ext}^1_R(W, Y)$ has G-dimension at most two, especially it has finite G-dimension.

On the other hand, there are isomorphisms

$$\begin{aligned}\mathrm{Ext}_R^1(W,Y) &\cong \mathrm{Ext}_R^1(W,Y)\oplus \mathrm{Ext}_R^1(W,L)\\ &\cong \mathrm{Ext}_R^1(W,M^*)\\ &\cong \mathrm{Ext}_R^1(W,\mathfrak{m}),\end{aligned}$$

where the last isomorphism is induced by the exact sequence (6.5). Applying the functor $\mathrm{Hom}_R(W,-)$ to the natural exact sequence

$$0 \to \mathfrak{m} \to R \to k \to 0$$

and noting that $\mathrm{Hom}_R(W,\mathfrak{m}) \cong W^*$ because W is a non-free indecomposable module, we obtain an isomorphism $\mathrm{Ext}_R^1(W,\mathfrak{m}) \cong \mathrm{Hom}_R(W,k)$, and hence $\mathrm{Ext}_R^1(W,Y)$ is a non-zero k-vector space. Therefore Proposition 6.2.2(1) and 6.2.2(5) say that R is Gorenstein, contrary to the assumption of our theorem. This contradiction proves that the R-module M does not have a $\mathcal{G}(R)$-precover, which establishes our theorem. □

According to Proposition 6.2.9, we have the following result that gives a corollary of the above theorem:

Corollary 6.5.2 *Suppose that R is a henselian non-Gorenstein local ring of depth two and that there exists a non-free R-module of G-dimension zero. Then there exist infinitely many isomorphism classes of indecomposable R-modules of G-dimension zero.*

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