MODULES OF G-DIMENSION ZERO OVER LOCAL RINGS OF DEPTH TWO

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Abstract. Let R be a commutative noetherian local ring. Denote by $\mathrm{mod}R$ the category of finitely generated R-modules, and by $\mathcal{G}(R)$ the full subcategory of $\mathrm{mod}R$ consisting of all R-modules of G-dimension zero. Suppose that R is henselian and non-Gorenstein, and that there is a non-free R-module in $\mathcal{G}(R)$. Then it is known that $\mathcal{G}(R)$ is not contravariantly finite in $\mathrm{mod}R$ if R has depth at most one. In this paper, we prove that the same statement holds if R has depth two.

1. Introduction

Throughout the present paper, we assume that all rings are commutative noetherian rings and all modules are finitely generated modules.

Auslander [1] has introduced a homological invariant for modules which is called Gorenstein dimension, or G-dimension for short. This invariant has a lot of properties similar to those of projective dimension. For example, it is well-known that the finiteness of projective dimension characterizes the regular property of the base ring: any module over a regular local ring has finite projective dimension, and a local ring whose residue class field has finite projective dimension is regular. The finiteness of G-dimension characterizes the Gorenstein property of the base ring.

Over a Gorenstein local ring, a module has G-dimension zero if and only if it is a maximal Cohen-Macaulay module. Hence it is natural to expect that modules of G-dimension zero over an arbitrary local ring may behave similarly to maximal Cohen-Macaulay modules over a Gorenstein local ring.

A Cohen-Macaulay local ring is called to be of finite Cohen-Macaulay representation type if it has only finitely many non-isomorphic indecomposable maximal Cohen-Macaulay modules. Under a few assumptions, Gorenstein local rings of finite Cohen-Macaulay representation type have been classified completely, and it is known that all non-isomorphic indecomposable maximal Cohen-Macaulay modules over them can be described concretely; see [7] for the details.

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Thus we are interested in non-Gorenstein local rings which have only finitely many non-isomorphic indecomposable modules of G-dimension zero, especially interested in determining all non-isomorphic indecomposable modules of G-dimension zero over such rings.

Now, we form the following conjecture:

Conjecture 1.1. Let R be a non-Gorenstein local ring. Suppose that there exists a non-free R-module of G-dimension zero. Then there exist infinitely many non-isomorphic indecomposable R-modules of G-dimension zero.

The above conjecture is against our expectation that modules of G-dimension zero over an arbitrary local ring behave similarly to maximal Cohen-Macaulay modules over a Gorenstein local ring. Indeed, let S be a d-dimensional non-regular Gorenstein local ring of finite Cohen-Macaulay representation type. (Such a ring does exist; see [7].) Then the dth syzygy module of the residue class field of S is a non-free maximal Cohen-Macaulay S-module. Hence the above conjecture does not necessarily hold without the assumption that R is non-Gorenstein.

For a local ring R, we denote by $\mathrm{mod} R$ the category of finitely generated R-modules, and by $\mathcal{G}(R)$ the full subcategory of $\mathrm{mod} R$ consisting of all R-modules of G-dimension zero. We guess that even the following statement that is stronger than Conjecture 1.1 is true. (It is seen from the proof of [5, Theorem 2.9] that Conjecture 1.2 implies Conjecture 1.1.)

Conjecture 1.2. Let R be a non-Gorenstein local ring. Suppose that there exists a non-free R-module in $\mathcal{G}(R)$. Then the category $\mathcal{G}(R)$ is not contravariantly finite in $\mathrm{mod} R$.

In [4] and [5], it is proved that Conjecture 1.2 is true if R is henselian and has depth at most one:

Theorem 1.3. [4, Theorem 1.2][5, Theorem 2.8] *Let $(R, \mathfrak{m}, k)$ be a henselian non-Gorenstein local ring. Suppose that there exists a non-free R-module in $\mathcal{G}(R)$. If the depth of R is zero (resp. one), then k (resp. $\mathfrak{m}$) does not admit a $\mathcal{G}(R)$-precover, and hence $\mathcal{G}(R)$ is not contravariantly finite in* $\mathrm{mod} R$.

The purpose of this paper is to prove that Conjecture 1.2 is true if R is henselian and has depth two:

Theorem 1.4. *Let R be a henselian non-Gorenstein local ring of depth two. Suppose that there exists a non-free R-module in $\mathcal{G}(R)$. Then the category $\mathcal{G}(R)$ is not contravariantly finite in* $\mathrm{mod} R$.

Under the assumption in Theorem 1.4, take a non-split exact sequence

$$0 \to R \to M \to \mathfrak{m} \to 0,$$

where $\mathfrak{m}$ is the unique maximal ideal of R. (Such an exact sequence exists because $\mathrm{Ext}_R^1(\mathfrak{m}, R) \neq 0$.) Then, it can be proved that the R-module M does not admit a $\mathcal{G}(R)$-precover, and hence $\mathcal{G}(R)$ is not contravariantly finite in mod$R$.

In the next section, we will state some definitions and results which are necessary to prove the theorem. We will spend the whole of the last section on giving a proof of the theorem.

2. Background material

In this section, we provide some background material. Throughout this section, let $(R, \mathfrak{m}, k)$ be a commutative noetherian local ring. All R-modules in this section are assumed to be finitely generated.

First of all, we recall the definition of G-dimension. We denote by mod$R$ the category of finitely generated R-modules. Put $M^* = \mathrm{Hom}_R(M, R)$ for an R-module M.

Definition 2.1. (1) We denote by $\mathcal{G}(R)$ the full subcategory of mod$R$ consisting of all R-modules M satisfying the following three conditions.

i) The natural homomorphism $M \to M^{**}$ is an isomorphism.
ii) $\mathrm{Ext}^i_R(M, R) = 0$ for every $i > 0$.
iii) $\mathrm{Ext}^i_R(M^*, R) = 0$ for every $i > 0$.

(2) Let M be an R-module. If n is a non-negative integer such that there is an exact sequence

$$0 \to G_n \to G_{n-1} \to \cdots \to G_1 \to G_0 \to M \to 0$$

of R-modules with $G_i \in \mathcal{G}(R)$ for every i, $0 \le i \le n$, then we say that M has *G-dimension at most* n, and write $\text{G-dim}_R M \le n$. If such an integer n does not exist, then we say that M has *infinite G-dimension*, and write $\text{G-dim}_R M = \infty$.

Of course, if an R-module M has G-dimension at most n but does not have at most $n-1$, then we say that M has G-dimension n and write $\text{G-dim}_R M = n$.

Let M be an R-module. We denote by $\Omega^n M$ the nth syzygy module of M, and set $\Omega M = \Omega^1 M$. If $F_1 \xrightarrow{\partial} F_0 \to M \to 0$ is the minimal free presentation of M, then we denote by $\mathrm{Tr} M$ the cokernel of the dual homomorphism $\partial^* : F_0^* \to F_1^*$. G-dimension is a homological invariant for modules sharing a lot of properties with projective dimension. We state here just the properties that will be used later.

Proposition 2.2. (1) *The following conditions are equivalent.*

i) *R is Gorenstein.*
ii) $\text{G-dim}_R M < \infty$ *for any R-module M.*
iii) $\text{G-dim}_R k < \infty$.

(2) *Let M, N be R-modules. Then* $\text{G-dim}_R(M \oplus N) = \sup\{\text{G-dim}_R M, \text{G-dim}_R N\}$.

(3) *If an R-module M belongs to $\mathcal{G}(R)$, then so do M^*, ΩM, $\mathrm{Tr} M$, and any direct summand of M.*

(4) *Let $0 \to L \to M \to N \to 0$ be an exact sequence of R-modules. If L and N belong to $\mathcal{G}(R)$, then so does M.*

The proof of this proposition and other properties of G-dimension are stated in detail in [2, Chapter 3,4] and [3, Chapter 1].

Now we introduce the notion of a cover of a module.

Definition 2.3. Let $\mathcal{X}$ be a full subcategory of $\mathrm{mod}R$.

(1) Let $\phi : X \to M$ be a homomorphism from $X \in \mathcal{X}$ to $M \in \mathrm{mod}R$.
  i) We call ϕ an *$\mathcal{X}$-precover* of M if for any homomorphism $\phi' : X' \to M$ with $X' \in \mathcal{X}$ there exists a homomorphism $f : X' \to X$ such that $\phi' = \phi f$.
  ii) Assume that ϕ is an $\mathcal{X}$-precover of M. We call ϕ an *$\mathcal{X}$-cover* of M if any endomorphism f of X with $\phi = \phi f$ is an automorphism.

(2) The category $\mathcal{X}$ is said to be *contravariantly finite* if every $M \in \mathrm{mod}R$ has an $\mathcal{X}$-precover.

An $\mathcal{X}$-precover (resp. an $\mathcal{X}$-cover) is often called a right $\mathcal{X}$-approximation (resp. a right minimal $\mathcal{X}$-approximation).

Proposition 2.4. [5, Remark 2.6] *Let $\mathcal{X}$ be a full subcategory of* $\mathrm{mod}R$ *which is closed under direct summands, and let*

$$0 \to N \xrightarrow{\psi} X \xrightarrow{\phi} M$$

be an exact sequence of R-modules where ϕ is an $\mathcal{X}$-precover of M. Suppose that R is henselian. Then there exists a direct summand L of N satisfying the following conditions:

i) *$\psi(L)$ is a direct summand of X.*

ii) *Let N' (resp. X') be the complement of L (resp. $\psi(L)$) in N (resp. X), and let*

$$0 \to N' \xrightarrow{\psi'} X' \xrightarrow{\phi'} M$$

be the induced exact sequence. Then ϕ' is an $\mathcal{X}$-cover of M.

For R-modules M, N, we define a homomorphism

$$\lambda_M(N) : M \otimes_R N \to \mathrm{Hom}_R(M^*, N)$$

of R-modules by $\lambda_M(N)(m \otimes n)(f) = f(m)n$ for $m \in M,\ n \in N$ and $f \in M^*$.

3. Proof of the theorem

Now, let us prove our theorem.

Proof of Theorem 1.4. Let $(R, \mathfrak{m}, k)$ be a henselian non-Gorenstein local ring of depth two. Then, since $\mathrm{Ext}^1_R(\mathfrak{m}, R) \cong \mathrm{Ext}^2_R(k, R) \neq 0$, we have a non-split exact sequence

$$\sigma : 0 \to R \to M \to \mathfrak{m} \to 0. \tag{1}$$

Dualizing this, we obtain an exact sequence

$$0 \to \mathfrak{m}^* \to M^* \to R^* \xrightarrow{\eta} \operatorname{Ext}^1_R(\mathfrak{m}, R).$$

Note from definition that the connecting homomorphism η sends $\mathrm{id}_R \in R^*$ to the element $s \in \operatorname{Ext}^1_R(\mathfrak{m}, R)$ corresponding to the exact sequence σ. Since σ does not split, s is a non-zero element of $\operatorname{Ext}^1_R(\mathfrak{m}, R)$. Hence η is a non-zero map. Noting that $\operatorname{Ext}^1_R(\mathfrak{m}, R) \cong \operatorname{Ext}^2_R(k, R)$, we see that the image of η is annihilated by $\mathfrak{m}$. Also noting that $\mathfrak{m}^* \cong R^* \cong R$, we get an exact sequence

$$0 \to R \to M^* \to \mathfrak{m} \to 0. \tag{2}$$

Claim 1. *The modules* $\operatorname{Hom}_R(G, M)$ *and* $\operatorname{Hom}_R(G, M^*)$ *belong to* $\mathcal{G}(R)$ *for every non-free indecomposable module* $G \in \mathcal{G}(R)$.

Proof. Applying the functor $\operatorname{Hom}_R(G, -)$ to the exact sequence (1) gives an exact sequence

$$0 \to G^* \to \operatorname{Hom}_R(G, M) \to \operatorname{Hom}_R(G, \mathfrak{m}) \to \operatorname{Ext}^1_R(G, R).$$

Since G is non-free and indecomposable, any homomorphism from G to R factors through $\mathfrak{m}$, and hence $\operatorname{Hom}_R(G, \mathfrak{m}) \cong G^*$. Also, since $G \in \mathcal{G}(R)$, we have $\operatorname{Ext}^1_R(G, R) = 0$. Thus Proposition 2.2.4 implies that $\operatorname{Hom}_R(G, M) \in \mathcal{G}(R)$. The same argument for the exact sequence (2) shows that $\operatorname{Hom}_R(G, M^*) \in \mathcal{G}(R)$. $\square$

We shall prove that the module M can not have a $\mathcal{G}(R)$-precover. Suppose that M has a $\mathcal{G}(R)$-precover. Then M has a $\mathcal{G}(R)$-cover $\pi : X \to M$ by Proposition 2.4. Since $R \in \mathcal{G}(R)$, any homomorphism from R to M factors through π. Hence π is a surjective homomorphism. Setting $N = \operatorname{Ker} \pi$, we get an exact sequence

$$0 \to N \xrightarrow{\theta} X \xrightarrow{\pi} M \to 0, \tag{3}$$

where θ is the inclusion. We see from Proposition 2.2.3, 2.2.4, and Wakamatsu's Lemma [6, Lemma 2.1.1] that $\operatorname{Ext}^i_R(G, N) = 0$ for any $G \in \mathcal{G}(R)$ and any $i > 0$. Dualizing the exact sequence (3), we obtain an exact sequence

$$0 \to M^* \xrightarrow{\pi^*} X^* \xrightarrow{\theta^*} N^*.$$

Put $C = \operatorname{Im}(\theta^*)$ and let $\mu : X^* \to C$ be the surjection induced by θ^*.

Claim 2. *The homomorphism* μ *is a* $\mathcal{G}(R)$*-precover of* C.

Proof. Fix a non-free indecomposable module $G \in \mathcal{G}(R)$. Applying the functors $G \otimes_R -$ and $\operatorname{Hom}_R(G^*, -)$ to the exact sequence (3) yields a commutative diagram

$$\begin{CD}
@. 0 \\
@. @VVV \\
G\otimes_R N @>{\lambda_G(N)}>> \mathrm{Hom}_R(G^*,N) \\
@V{G\otimes_R\theta}VV @VV{\mathrm{Hom}_R(G^*,\theta)}V \\
G\otimes_R X @>{\lambda_G(X)}>> \mathrm{Hom}_R(G^*,X) \\
@V{G\otimes_R\pi}VV @VV{\mathrm{Hom}_R(G^*,\pi)}V \\
G\otimes_R M @>{\lambda_G(M)}>> \mathrm{Hom}_R(G^*,M) \\
@VVV @VVV \\
0 @. \mathrm{Ext}^1_R(G^*,N)
\end{CD}$$

with exact columns. Noting that $\mathrm{Tr}G \in \mathcal{G}(R)$ by Proposition 2.2.3, we see from [2, Proposition (2.6)] that $\mathrm{Ker}\,\lambda_G(N) = \mathrm{Ext}^1_R(\mathrm{Tr}G, N) = 0$ and $\mathrm{Coker}\,\lambda_G(N) = \mathrm{Ext}^2_R(\mathrm{Tr}G, N) = 0$. This means that $\lambda_G(N)$ is an isomorphism. It follows from the commutativity of the above diagram that the homomorphism $G \otimes_R \theta$ is injective. Also, we have $\mathrm{Ext}^1_R(G^*, N) = 0$ because $G^* \in \mathcal{G}(R)$ by Proposition 2.2.3. Thus we obtain a commutative diagram

$$\begin{CD}
0 @>>> G\otimes_R N @>{G\otimes_R\theta}>> G\otimes_R X @>>> G\otimes_R M @>>> 0 \\
@. @V{\lambda_G(N)}V{\cong}V @VVV @VVV @. \\
0 @>>> \mathrm{Hom}_R(G^*,N) @>>> \mathrm{Hom}_R(G^*,X) @>>> \mathrm{Hom}_R(G^*,M) @>>> 0
\end{CD}$$

with exact rows. Dualizing this diagram induces a commutative diagram

$$\begin{CD}
\mathrm{Hom}_R(G^*,X)^* @>>> \mathrm{Hom}_R(G^*,N)^* @>>> \mathrm{Ext}^1_R(\mathrm{Hom}_R(G^*,M),R) \\
@VVV @V{(\lambda_G(N))^*}V{\cong}V @VVV \\
(G\otimes_R X)^* @>{(G\otimes_R\theta)^*}>> (G\otimes_R N)^* @>>> \mathrm{Ext}^1_R(G\otimes_R M,R)
\end{CD}$$

with exact rows. Since $\mathrm{Hom}_R(G^*, M) \in \mathcal{G}(R)$ by Claim 1, we have $\mathrm{Ext}^1_R(\mathrm{Hom}_R(G^*, M), R) = 0$. From the above commutative diagram, it is seen that $(G \otimes_R \theta)^*$ is a surjective homomorphism. Note that there is a natural commutative diagram

$$\begin{CD}
(G\otimes_R X)^* @>{(G\otimes_R\theta)^*}>> (G\otimes_R N)^* \\
@V{\cong}VV @V{\cong}VV \\
\mathrm{Hom}_R(G,X^*) @>{\mathrm{Hom}_R(G,\theta^*)}>> \mathrm{Hom}_R(G,N^*)
\end{CD}$$

with isomorphic vertical maps. Therefore the homomorphism $\mathrm{Hom}_R(G, \theta^*)$ is also surjective, and so is the homomorphism $\mathrm{Hom}_R(G, \mu) : \mathrm{Hom}_R(G, X^*) \to \mathrm{Hom}_R(G, C)$. It is easy to see from this that μ is a $\mathcal{G}(R)$-precover of C. $\square$

According to Claim 2 and Proposition 2.4, we have direct sum decompositions $M^* = Y \oplus L,\ X^* = \pi^*(Y) \oplus Z$, and an exact sequence

$$0 \to L \to Z \xrightarrow{\nu} C \to 0$$

where ν is a $\mathcal{G}(R)$-cover of C. Since Y is isomorphic to the direct summand $\pi^*(Y)$ of X^*, Proposition 2.2.3 implies that $Y \in \mathcal{G}(R)$. Wakamatsu's Lemma yields $\mathrm{Ext}_R^1(G, L) = 0$ for any $G \in \mathcal{G}(R)$.

Claim 3. *The module* $\mathrm{Hom}_R(G, Y)$ *belongs to* $\mathcal{G}(R)$ *for any* $G \in \mathcal{G}(R)$.

Proof. We may assume that G is non-free and indecomposable. The module $\mathrm{Hom}_R(G, Y)$ is isomorphic to a direct summand of $\mathrm{Hom}_R(G, M^*)$. Since the module $\mathrm{Hom}_R(G, M^*)$ is an object of $\mathcal{G}(R)$ by Claim 1, so is the module $\mathrm{Hom}_R(G, Y)$ by Proposition 2.2.3. $\square$

Here, by the assumption of the theorem, we have a non-free indecomposable module $W \in \mathcal{G}(R)$. There is an exact sequence

$$0 \to \Omega W \to F \to W \to 0$$

such that F is a free module. Applying the functor $\mathrm{Hom}_R(-, Y)$ to this exact sequence, we get an exact sequence

$$0 \to \mathrm{Hom}_R(W, Y) \to \mathrm{Hom}_R(F, Y) \to \mathrm{Hom}_R(\Omega W, Y) \to \mathrm{Ext}_R^1(W, Y) \to 0.$$

Since $\mathrm{Hom}_R(W, Y)$, $\mathrm{Hom}_R(F, Y)$, and $\mathrm{Hom}_R(\Omega W, Y)$ belong to $\mathcal{G}(R)$ by Claim 3, the R-module $\mathrm{Ext}_R^1(W, Y)$ has G-dimension at most two, in particular it has finite G-dimension.

On the other hand, there are isomorphisms

$$\begin{aligned} \mathrm{Ext}_R^1(W, Y) &\cong \mathrm{Ext}_R^1(W, Y) \oplus \mathrm{Ext}_R^1(W, L) \\ &\cong \mathrm{Ext}_R^1(W, M^*) \\ &\cong \mathrm{Ext}_R^1(W, \mathfrak{m}), \end{aligned}$$

where the last isomorphism is induced by the exact sequence (2). Applying the functor $\mathrm{Hom}_R(W, -)$ to the exact sequence

$$0 \to \mathfrak{m} \to R \to k \to 0$$

and noting that $\mathrm{Hom}_R(W, \mathfrak{m}) \cong W^*$ because W is a non-free indecomposable module, we obtain an isomorphism $\mathrm{Ext}_R^1(W, \mathfrak{m}) \cong \mathrm{Hom}_R(W, k)$, and hence $\mathrm{Ext}_R^1(W, Y)$ is a non-zero k-vector space. Therefore Proposition 2.2.1 and 2.2.2 say that R is Gorenstein, contrary to the assumption of our theorem. This contradiction proves that the R-module M does not have a $\mathcal{G}(R)$-precover, which establishes our theorem. $\square$

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