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A New Formalism of the Einstein Equations for Relativistic Rotating Systems

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Combining Geroch's and the ADM formalisms, we give a new formalism for treating the dynamical problems of space-time having a rotational Killing vector. We have found that the basic equations for a rotating system strongly resemble the Einstein and the Maxwell equations for a non-rotating system.

The numerical approach to the relativistic gravitational collapse has been studied by several authors.[1),2)] However, none of them have succeeded in calculating the collapse of a rotating star though Nakamura, Maeda, Miyama and Sasaki[1)] succeeded in constructing the initial data. For a non-rotating star, Nakamura, Maeda, Miyama and Sasaki[3)] have calculated the formation of black holes. But if one tries to develop their code to a rotating case, one thinks that the basic equations will be terribly complicated. In this short note, we give a new formalism in which the basic equations are simpler and which has some nice features.

In the axially symmetric space-time, there is a rotational Killing vector. This implies we can divide out the Killing direction using Geroch's three-dimensional formalism for space-time with one Killing vector.[4)] Applying this formalism to an axially symmetric system, we get the 3-dimensional "Einstein" equations with respect to $h_{\mu\nu}$ plus some new equations for λ and ω_μ, that is,

$$\begin{aligned}{}^{(3)}R_{\mu\nu} &= (2\lambda^4)^{-1}[\omega_\mu\omega_\nu - h_{\mu\nu}\omega^\rho\omega_\rho] + \lambda^{-1}D_\mu D_\nu\lambda \\ &\quad + 8\pi[\mathcal{T}_{\mu\nu} - (1/2)h_{\mu\nu}(\mathcal{T}_\rho{}^\rho + \lambda^{-2}\mathcal{T})],\end{aligned} \tag{1}$$

$$\begin{aligned}\lambda^{-1}D^\rho D_\rho\lambda &= -(2\lambda^4)^{-1}\omega^\alpha\omega_\alpha \\ &\quad -4\pi(\lambda^{-2}\mathcal{T} - \mathcal{T}_\alpha{}^\alpha),\end{aligned} \tag{2}$$

$$D_{[\mu}\omega_{\nu]} = 8\pi\lambda\varepsilon_{\mu\nu\rho}\mathcal{T}^\rho, \tag{3}$$

$$D^\rho[\lambda^{-3}\omega_\rho] = 0\,, \tag{4}$$

where

$$\lambda^2 = \xi_\mu\xi^\mu, \tag{5}$$

$$h_{\mu\nu} = g_{\mu\nu} - \lambda^{-2}\xi_\mu\xi_\nu\,, \tag{6}$$

$$\omega_\mu = \varepsilon_{\mu\nu\rho\sigma}\xi^\nu\nabla^\rho\xi^\sigma, \tag{7}$$

$$\varepsilon_{\mu\nu\rho} = \lambda^{-1}\xi^\sigma\varepsilon_{\mu\nu\rho\sigma}\,,$$

and ξ^μ, ∇^μ, D^μ and ${}^{(3)}R_{\mu\nu}$ represent a rotational Killing vector, the 4-dimensional covariant differentiation, the 3-dimensional covariant differentiation and the 3-dimensional Ricci tensor, respectively.[4)] $\mathcal{T}$, $\mathcal{T}_\mu$ and $\mathcal{T}_{\mu\nu}$ are defined as

$$\mathcal{T} = T_{\mu\nu}\xi^\mu\xi^\nu,$$

$$\mathcal{T}^\rho = h^{\rho\mu}\xi^\nu T_{\mu\nu}$$

and

$$\mathcal{T}_{\rho\sigma} = h_\rho{}^\mu h_\sigma{}^\nu T_{\mu\nu}\,.$$

We now reduce Eqs. (1)$\sim$(4) to the canonical Hamiltonian form applying the ADM formalism.[5)] We first define a new projection tensor[*)]

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*) Small Latin indices refer to the range 0, 1, 2, and capital Latin indices to the range 1, 2.

$$H^{ab}=h^{ab}+n^a n^b, \tag{8}$$

where n_a is a unit normal vector of hypersurface $t=$constant. Carrying out the projection of all the tensors appeared in Eqs. (1)～(4) by using n_a and H_{ab}, we obtain

$$\chi^2-\chi^{AB}\chi_{AB}+{}^{(2)}R=2\lambda^{-1}({}^{(2)}\Delta\lambda-\chi\kappa)+(1/2)\lambda^{-4}(\Omega_A\Omega^A+\Omega_H^2)+16\pi\rho_H, \tag{9·a}$$

$$\lambda^{-1}(\lambda\chi_A{}^B)_{\Vert B}-\lambda^{-1}\partial_A\kappa-\partial_A\chi=8\pi J_A-(1/2)\lambda^{-4}\Omega_H\Omega_A, \tag{9·b}$$

$$\begin{aligned}\partial_0\chi_{AB}-\eta^C\chi_{AB\Vert C}+\alpha[{}^{(2)}R_{AB}+\chi\chi_{AB}]&-2\alpha\chi_A{}^C\chi_{CB}-\alpha_{\Vert AB}+(\chi_{AC}\eta^C_{\Vert B}+\chi_{BC}\eta^C_{\Vert A})\\&-\alpha\lambda^{-1}\lambda_{\Vert AB}+\alpha\lambda^{-1}\kappa\chi_{AB}-(1/2)\alpha\lambda^{-4}\\&\times[\Omega_A\Omega_B-H_{AB}(\Omega_C\Omega^C-\Omega_H^2)]\\&-8\pi\alpha[S_{AB}+(1/2)H_{AB}\\&\times(\rho_H-S_C{}^C-\lambda^{-2}\mathcal{T})],\end{aligned} \tag{10·a}$$

$$\partial_0 H_{AB}=-2\alpha\chi_{AB}+\eta_{A\Vert B}+\eta_{B\Vert A}, \tag{10·b}$$

$$\partial_0\lambda-\eta^A\partial_A\lambda=-\alpha\kappa, \tag{11·a}$$

$$\begin{aligned}\partial_0\kappa-\eta^A\partial_A\kappa&=\alpha\chi\kappa-H^{AB}(\partial_A\alpha)(\partial_B\lambda)-\alpha{}^{(2)}\Delta\lambda\\&-(1/2)\alpha\lambda^{-3}(\Omega_A\Omega^A-\Omega_H^2)-4\pi\alpha\lambda\\&\times[\rho_H-S_A{}^A+\lambda^{-2}\mathcal{T}],\end{aligned} \tag{11·b}$$

$$\Omega_{A\Vert B}-\Omega_{B\Vert A}=-16\pi\lambda\varepsilon_{AB}J_\phi, \tag{12·a}$$

$$\partial_0\Omega_A-\eta^B\Omega_{A\Vert B}=\eta^B_{\Vert A}\Omega_B+\partial_A(\alpha\Omega_H)+16\pi\varepsilon_{AB}\alpha\lambda S^B, \tag{12·b}$$

$$\partial_0\Omega_H-\eta^A\partial_A\Omega_H=(\alpha\Omega^A)_{\Vert A}+\alpha\chi\Omega_H-3\alpha\lambda^{-1}[\Omega_H\kappa+\Omega^A\partial_A\lambda], \tag{12·c}$$

where α and η_A are defined by

$$ds^2=h_{ab}dx^a dx^b=-\alpha^2 dt^2+H_{AB}\times(dx^A+\eta^A dt)(dx^B+\eta^B dt),$$

and χ_{AB} are the extrinsic curvatures with respect to H_{AB}, and

$$\chi=\chi^A A,\ \kappa=-n^\mu\partial_\mu\lambda,$$

$$\Omega_A=H_A{}^b\omega_b,\ \Omega_H=n_a\omega^a,$$
$$\varepsilon_{ab}=n^c\varepsilon_{cab},$$
$$\rho_H=n_a n_b\mathcal{T}^{ab},\ J_\phi=-n_a\mathcal{T}^a,$$
$$J^A=-n_a H_b{}^A\mathcal{T}^{ab},\ S^A=H_b{}^A\mathcal{T}^b$$

and

$$S_{AB}=H_{Aa}H_{Bb}\mathcal{T}^{ab}.$$

In Eqs. (9)～(12), ${}^{(2)}\Delta$, $\Vert$, ${}^{(2)}R$ and ${}^{(2)}R_{AB}$ are the Laplacian, the covariant differentiation, the scalar curvature and the Ricci tensor with respect to H_{AB}, respectively. Equations (9) and (10) are the constraint equations and the evolution equations of two-dimensional space metric, reduced from the 3-dimensional "Einstein" equations. Equations (11) and (12) resemble the Maxwell equations if we consider $(1/2)\xi_\mu$ as the vector potential, that is, if we define the "electromagnetic" field as

$$B_\phi=\Omega_H/2,\ B^A=\varepsilon^{AB}\partial_B\lambda,$$
$$E_\phi=\kappa\lambda \quad\text{and}\quad E^A=\varepsilon^{AC}\Omega_c/2\lambda.$$

Using these "electromagnetic" fields, we have the hydrodynamics equations as follows:[*)]

(i) the energy equation

$$\begin{aligned}\partial_0(\lambda\sqrt{H}\rho_H)+\partial_B(U^B\lambda\sqrt{H}\rho_H)&=-\partial_B(\alpha\lambda\sqrt{H}pV^B)\\&+\alpha\lambda\sqrt{H}p(\chi+\lambda^{-1}\kappa)\\&+\alpha\lambda\sqrt{H}(p+\rho_H)\{-a_BV^B+V^AV^B\chi_{AB}\}\\&+\alpha\lambda\sqrt{H}\lambda^{-2}J_\phi(\lambda^{-2}E_\phi V_\phi+2E_BV^B),\end{aligned} \tag{13}$$

(ii) the conservation of angular momentum

$$\partial_0(\lambda\sqrt{H}J_\phi)+\partial_B(U^B\lambda\sqrt{H}J_\phi)=0, \tag{14}$$

(iii) the Euler equation

$$\begin{aligned}\partial_0(\lambda\sqrt{H}J_A)+\partial_B(U^B\lambda\sqrt{H}J_A)&=-\alpha\lambda\sqrt{H}[\partial_A p+(p+\rho_H)a_A]\\&+\alpha\lambda\sqrt{H}(p+\rho_H)[(1/2)(\partial_A H_{BC})\\&\times V^BV^C+\alpha^{-1}V_C\partial_A\eta^C]\end{aligned}$$

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*) The method of deriving the hydrodynamics equations is the same as Ref. 1).

$$+\alpha\lambda\sqrt{H}\,\lambda^{-2}J_\phi[2E_A+\varepsilon_{AC}$$
$$\times(2\lambda^{-1}B_\phi V^C-\lambda^{-1}B^C V_\phi)] \qquad (15)$$

and

(iv) the equation of continuity

$$\partial_0\left(\frac{\lambda\sqrt{H}n}{\sqrt{1-V^2}}\right)+\partial_B\left(U^B\frac{\lambda\sqrt{H}n}{\sqrt{1-V^2}}\right)=0\,, \qquad (16)$$

where

$$H=\det(H_{AB}),\ a_B=\partial_B\ln\alpha\,,$$
$$V^B=(p+\rho_H)^{-1}J^B,\ V_\phi=(p+\rho_H)^{-1}J_\phi\,,$$
$$V^2=V_BV^B+\lambda^{-2}V_\phi^{\,2}\ \text{and}\ U^A=\alpha V^A-\eta^A.$$

p and n are the pressure and the proper number density of matter, respectively.

We can see that the "electromagnetic" fields give Joule's heat-like and the Lorentz force-like terms into the hydrodynamics equations. Note that the angular momentum density behaves like charge both in Eqs. (12) and (14).

We want to call the present new formalism the [(2+1)+1]-dimensional representation of the Einstein equations. On the other hand the ADM formalism is called the (3+1)-dimensional representation of the Einstein equations. Comparing these two for a rotating system, we find that the [(2+1)+1]-dimensional formalism has the following merits:

(1) Since the ϕ-component of the shift vector does not appear in the basic equations, we do not need to consider the coordinate condition about ϕ-component.

(2) In the (3+1)-dimensional formalism, for a rotating case we should treat $g_{A\phi}$ and $\dot{g}_{A\phi}$ whose behaviours are unknown. In the [(2+1)+1]-dimensional formalism, new variables which are needed for a rotating case can be considered the electromagnetic fields. So we can understand their behaviours easily. These facts will enable us to utilize our code for a non-rotating collapse almost as it is.

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