調和写像としてのアインシュタイン方程式

山田澄生 (学習院大)

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The Einstein Equation

Collaboration (2017–) with Gilbert Weinstein, Marcus Khuri and Martin Reiris

Spacetime: a Lorentzian n-manifold $(N^n, \mathbf{g})$ satisfying the Einstein equations (1916)

$$R_{ab} - \frac{1}{2} R\, \mathbf{g}_{ab} = T_{ab}$$

Here T is the energy-momentum-stress tensor of the matter fields.
Here we assume it is identically zero: vacuum.
the Euler-Lagrange equation for the Hilbert-Einstein functional;

$$\mathcal{H}(\mathbf{g}) = \int_N \{R_{\mathbf{g}} + L\}\, d\mu_{\mathbf{g}}$$

with L: the Lagrangian for non-gravitational fields (if any).

Vacuum = Ricci-flat

Recall the following rigidity result from Riemannian/Lorentzian geometry;

$$R_{abcd} = 0 \Rightarrow \text{ the space is flat; e.g. } \mathbb{R}^n, \mathbb{R}^{n,1}$$

When $L = 0$ ($\Leftrightarrow T_{ab} = 0$), the vacuum Einstein equation (VEE) $R_{ab} - \frac{1}{2} R\, \mathbf{g}_{ab} = 0$ implies $R_{\mathbf{g}} = 0$, and hence

$$R_{ab} = 0;$$

the second simplest curvature equation.

Gravitational Solitons

Non-trivial static, geodesically complete solutions of the Einstein equations (without blackholes).

- 4D asymptotically flat vacuum gravitational soliton $\Rightarrow$ the Minkowsky spacetime and its quotients (Lichnerowitz, Anderson)
- 4D spherically symmetric Einstein-$SO(2)$ Yang-Mills equation: Bartik-McKinnon (1988) McLeod-Smoller-Wasserman-Yau (1991)
- 4D asymptotically Kasner vacuum with ∞ many black holes: Myers (1987), Korotkin-Nicolai (1994)
- Higher dimensional vacuum solitons? ($\exists$ super-symmetric examples)

Gravitational Soliton

Main Theorem: There exist vacuum solitons in $d \geq 5$.

Corollary: 4D Gravitational instantons: i.e. Riemannian Ricci-flat complete manifolds with generic holonomy, of infinite topological types (infinite 2nd Betti number), of Kasner asymptotics.

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Weyl Coordinates for $3+1$ spacetime

H. Weyl wrote down (1916) the Schwarzschild metric (1915)

$$ds^2 = -(1-\frac{m}{r})dt^2 + (1-\frac{m}{r})^{-1}dr^2 + r^2 d\omega^2_{S^2}.$$

as

$$ds^2 = -e^u \rho^2 dt^2 + e^u d\phi^2 + e^\alpha (d\rho^2 + dz^2).$$

for $u = u(\rho, z)$. Then, $R_{ij} = 0$ (VEE) becomes

$$\frac{1}{\rho}\frac{\partial u}{\partial \rho}\Big(\rho \frac{\partial u}{\partial \rho}\Big) + \frac{\partial^2 u}{\partial z^2} = 0$$

$$\alpha_\rho = \frac{\rho}{2}\rho(u_\rho^2 - u_z^2 - \frac{2}{\rho}u_z), \quad \alpha_z = \frac{\rho}{2}(2u_z u_\rho - \frac{2}{\rho}u_z)$$

and the harmonicity $\triangle_{\mathbb{R}^3} u = 0$ is the integrability condition for the quadrature

$$d\alpha = \frac{1}{2}\rho\,(\mathrm{Re}|\partial \mathrm{u}|^2)\,\mathrm{d}\xi \qquad (\xi = \rho + \mathrm{iz}).$$

The asymptotic flatness: $u \to 2\log\rho$ at ∞.

Ernst Reduction and the 3+1 Kerr Metric

The axisymmetry of the $3+1$ Kerr spacetime :$\frac{\partial}{\partial t}$ and $\frac{\partial}{\partial \phi}$.

X : $\mathbf{g}$-norm of $\eta := \frac{\partial}{\partial \phi}$

Y : the potential function of the one-form $*(\eta \wedge d\eta)$.

(X, Y) completely determines the 2×2 metrix G_{ij} of the Lorentz metric with the symmetries

$$\mathbf{g} = G_{ij}(\rho, z) dx^i \otimes dx^j + e^{\alpha(\rho, z)}(d\rho^2 + dz^2).$$

Namely; the Weyl-Papapetrou form for $3+1$ dimension is

$$\mathbf{g} = [-\frac{\rho^2}{X} dt^2 + X(d\phi + \omega dt)^2] + e^{\alpha}(d\rho^2 + dz^2).$$

Note $\rho = \sqrt{|\det G_{ij}|}$ where

$$G_{ij} = \begin{pmatrix} -\rho^2/X + X\omega^2 & X\omega \\ X\omega & X \end{pmatrix}$$

with $x^0 = t, x^1 = \phi$.

And $\mathbf{g}$ satisfies the VEE is equivalent to the fact that

$$(\rho, z, \phi) \mapsto (X, Y)$$

$$[G] = \frac{1}{X} \begin{pmatrix} 1 & Y \\ Y & X^2 + Y^2 \end{pmatrix}$$

is a harmonic map (B. Carter)

$$\mathbb{R}^3 \setminus \{\rho = 0\} \to SL(2, \mathbb{R})/SO(2)$$

being a critical point of

$$\int \left(\frac{|\nabla X|^2 + |\nabla Y|^2}{X^2} \right) d\mu_{\mathbb{R}^3}.$$

Bi-axisymmetric stationary solutions

Consider a spacetime $(N^5, \mathbf{g})$ with three mutually commuting Killing fields

$$\frac{\partial}{\partial t}, \quad \frac{\partial}{\partial \phi_1}, \quad \frac{\partial}{\partial \phi_2}$$

which are the generators of the Lie group $\mathbb{R} \times U(1)^2$.
If stationary spacetime $(N^5, \mathbf{g})$ has a BH horizon, its connected component is diffeomorphic to 3-fold Σ^3 with positive Yamabe constant (i.e. it admits a metric of positive scalar curvature. [Galloway-Schoen]) Under the additional symmetry condition $\mathbb{R} \times U(1)^2$, the list of the topological types are restricted [Hollands-Yazadjiev] to

$$S^3, S^1 \times S^2, \mathbb{R}P^3 (= L(2,1)), L(p,q).$$

Hawking: $3+1$ stationary BH horizons $\cong^{\text{diffeo.}} S^2$

The examples include

- $\mathbb{R}^{4,1}$
- Myers-Perry spacetime (S^3 horizon, 5D Kerr, VEE)
- Emperan-Reall ($S^2 \times S^1$ horizon, VEE)
- Kunduri-Lucietti ($\mathbb{R}P^3$ horizon, supersymmetry)
- Nozawa-Tomizawa ($L(p,1)$ horizon, supersymmetry)

The Weyl-Papapetrou Coordinates

The Einstein metric $\mathbf{g}$ on N^{4+1} can be modelled on

$$[\mathbb{R} \times U(1)^2] \times \{(\rho, z) \mid \rho \in \mathbb{R}, \quad z > 0\}$$

so that

$$\mathbf{g} = G_{ij} dx^i \otimes dx^j + e^{2\nu}(d\rho^2 + dz^2)$$

where

$$\rho = \sqrt{|\det G_{ij}|}, \quad \nu = \nu(\rho, z), \quad G_{ij} = G_{ij}(\rho, z)$$

defined on $\mathbb{R}^5 \setminus \{\rho = 0\}$.

The equation G_{ij} satisfies is a harmonic map into the symmetric space $SL(3,\mathbb{R})/SO(3)$

$$u : (\rho, z, \phi) \mapsto [G]$$

where $[G]$ is a point in $SL(3,\mathbb{R})/SO(3)$, determined by G and its first derivatives(1-jet), being the critical point of

$$\int \left(\frac{1}{4}\frac{df^2}{f^2} + \frac{1}{4} f^{ij} f^{kl} df_{ik} df_{jl} + \frac{1}{2} f^{ij} \frac{dv_i dv_j}{f} \right) \, d\mu_{\mathbb{R}^3}.$$

Here v_i is the Ernst potential for the symmetry generated by $\frac{\partial}{\partial \phi^i}$.

The identification between G and $[G]$ given by Maison is: for

$$G_{ij}dx^i dx^j = -\frac{\rho^2}{f}dt^2 + \sum_{1\le i,j,\le 2}^{2} f_{ij}(d\phi^i + \omega^i dt)(d\phi^j + \omega^j dt),$$

a point $[G]$ is taken to be (for $f = \det f_{ij}$)

$$[G] = \frac{1}{f}\begin{pmatrix} 1 & -v_1 & -v_2 \\ -v_1 & ff_{11} + (v_1)^2 & ff_{12} + v_1 v_2 \\ -v_2 & ff_{21} + v_2 v_1 & ff_{22} + (v_2)^2 \end{pmatrix}$$

a positive definite, symmetric, $\det[G] = 1$ matrix, representing a point in $SL(3,\mathbb{R})/SO(3)$.

Rod Structure

We have so far set things up so that

- $N^5/(\mathbb{R} \times U(1)^2)$ is the ρz-half plane ($\rho > 0$)
- $G_{ij}dx^i dx^j$ is the Lorentzian metric on each fiber $\mathbb{R} \times (S^1)^2$.
- $G(\rho, z)$ is C^∞ and nonsingular 3×3 matrix, with real eigenvalues.
- $\rho := \sqrt{|\det G|} \to 0 \quad \Rightarrow \quad \dim\ker(G(0,z)) \geq 1$ and $\dim \geq 2$ only at isolated values $-\infty = a_0 < a_1 < \cdots < a_N < a_{N+1} = \infty$ on the z-axis.

Let $V(z)$ be the eigenvectors of $G(0,z)$ with zero eigenvalues

$$G(0,z)V(z) = 0$$

where for relatively prime $\beta, \gamma \in \mathbb{N}$

$$V = \alpha\frac{\partial}{\partial t} + \beta\frac{\partial}{\partial \phi_1} + \gamma\frac{\partial}{\partial \phi_2}$$

For z being a rank 2 point of the 3×3 matrix $G(0, z)$, there are two cases:

- $V(z)$ lightlike $\Rightarrow$ the rod $(a_n, a_{n+1}) \ni z$ represents a horizon, a union of tori generated by $\frac{\partial}{\partial \phi_1}$ and $\frac{\partial}{\partial \phi_2}$. (a_n, a_{n+1}) is a horizon rod.
- $V(z)$ spacelike $\Rightarrow$ the rod $(a_m, a_{m+1}) \ni z$ represents the set of points where the $U(1)^2$ action on N^5 is not free.

Definition Rod structure:

$$\{\rho = 0\} = \cup_{i=0}^{N}(a_i, a_{i+1}) \text{ together with the eigenvectors } V_i.$$

Main Theorem

Theorem[Khuri-Weinstein-Y. 2018] Given the rod structure and the angular momenta J_1, J_2, there exists a unique harmonic map, which would then produce a stationary solution for the $4+1$ VEE.

Remark There may be singularities along the z-axis, which needs be removed by adjusting the parameters $m, J_1 \neq 0, J_2 \neq 0$.
(cf. Weyl-Bach, Emperan-Reall)

The simplest solution to the VEE is given by

$$\begin{aligned}
\mathbf{g} &= -dt^2 + dr^2 + r^2 d\omega_{S^3} \\
&= -dt^2 + dr^2 + r^2[d\theta^2 + \sin^2\theta(d\phi^1)^2 + \cos^2\theta(d\phi^2)^2)] \\
&= \{-dt^2 + r^2\sin^2\theta(d\phi^1)^2 + r^2\cos^2\theta(d\phi^2)^2)\} + dr^2 + r^2 d\theta^2 \\
&= G_{ij}dx^i dx^j + dr^2 + r^2 d\theta^2 \qquad (0 \le \theta \le \pi/2) \\
&= G_{ij}(\rho, z)dx^i dx^j + \frac{1}{\sqrt{\rho^2 + z^2}}(d\rho^2 + dz^2)
\end{aligned}$$

defined on $N^{4+1} = \mathbb{R} \times \mathbb{R}^4$. The change of the variables are given by

$$\rho = \frac{1}{2}r^2 \sin 2\theta, \quad z = \frac{1}{2}r^2 \cos 2\theta$$

(the Riemann mapping: $z \mapsto z^2$, quadrant $\to$ ρz half-plane $(\mathbb{R} \times \mathbb{R}^4)/(\mathbb{R} \times U(1)^2) = \mathbb{R}_{\ge 0} \times \mathbb{R}_{\ge 0}$)

The Minkowski metric is

$$
\begin{aligned}
ds^2 &= G_{ij}(\rho, z)dx^i dx^j + \frac{1}{\sqrt{\rho^2+z^2}}(d\rho^2 + dz^2) \\
&= -\frac{\rho^2}{e^{u_1+u_2}}dt^2 + e^{u_1}d\phi_1^2 + e^{u_2}d\phi_2^2 + e^{\alpha}(d\rho^2 + dz^2) \\
&= -dt^2 + e^{u_1}d\phi_1^2 + e^{u_2}d\phi_2^2 + e^{\alpha}(d\rho^2 + dz^2)
\end{aligned}
$$

where $u_1 = \log(r+z)$ blowing up on the $(0,1)$ rod, and $u_2 = \log(r-z)$ blowing up on the $(1,0)$ rod, with $r = \sqrt{\rho^2+z^2}$. Note that

$$
G_{ij} = \begin{pmatrix} -1 & 0 & 0 \\ 0 & e^{u_1} & 0 \\ 0 & 0 & e^{u_2} \end{pmatrix}, \quad [G] = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \frac{r+z}{\rho} & 0 \\ 0 & 0 & \frac{r-z}{\rho} \end{pmatrix}
$$

The $3+1$ Schwarzschild metric is

$$ds^2 = -\frac{\rho^2}{e^u}dt^2 + e^u d\phi^2 + e^\alpha(d\rho^2 + dz^2)$$

where $u = 2\log\rho - u_0$ blowing up on the $\{z > m\}$ rod, and $\{z < -m\}$ rod. Namely u_0 is the potential for the charged rod $[-m, m]$.
Note that

$$G_{ij} = \begin{pmatrix} -\frac{\rho^2}{e^u} & \\ 0 & e^u \end{pmatrix}, \quad [G] = \begin{pmatrix} e^{-u} & 0 \\ 0 & e^u \end{pmatrix}$$

Regularity of the rod Γ:

$$\frac{\text{Circumference}}{2\pi\text{Radius}} = \frac{2\pi e^{u/2}}{2\pi\int_0^\rho e^{\alpha/2}d\rho} \sim \frac{e^{u/2}}{\rho e^{\alpha/2}} \to 1$$

Weyl superposed **two** $3+1$ Schwarzschild metrics

$$ds^2 = -\frac{\rho^2}{e^u} dt^2 + e^u d\phi^2 + e^\alpha (d\rho^2 + dz^2)$$

where $u = 2\log\rho - \overline{u}_0$ and $\overline{u}_0$ is the potential for the **two** charged rod $[-3m, -m]$ and $[m, 3m]$.
Consider the finite axis rod $\Gamma_0 := [-m, m]$. Weyl and Bach calculated (1923) that
Cone Singularities of the rod Γ_0:

$$\frac{\text{Circumference}}{2\pi\text{Radius}} = \frac{2\pi e^{u/2}}{2\pi \int_0^\rho e^{\alpha/2} d\rho} \sim \frac{e^{u/2}}{\rho e^{\alpha/2}} \to 1 + \varepsilon > 1.$$

Myers (1987) and Korotkin-Nikolai (1994) superposed **infinitely many** $3+1$ Schwarzschild metrics

$$ds^2 = -\frac{\rho^2}{e^u}dt^2 + e^u d\phi^2 + e^\alpha(d\rho^2 + dz^2)$$

where $u = 2\log\rho - \tilde{u}_0$ is the harmonic function, with

$$\tilde{u}_0 = \lim_{m\to\infty}\Big(\sum_{|j|\le m} u_j - c\log m\Big).$$

convergent. u_j is the potential of the j-th horizon. The renormalization is due to $u_j(\rho, z) \sim \frac{c}{n}$ for fixed (ρ, z).
No conical singularities as there is a balancing among the attractive forces from the infinitely many black holes.
Not asymptotically flat but Kasner: $u - 2\log\rho \to \frac{2m}{2L}2\log\rho$.

Gravitational Solitons

Consider a $4+1$ dimensional setting with z-periodic spacetime, with no horizons, and with alternating axis rods $(1,0)$ and $(0,1)$.

$$ds^2 = -\frac{\rho^2}{e^u}dt^2 + e^{u_1}d\phi_1^2 + e^{u_2}d\phi_2^2 + e^{\alpha}(d\rho^2 + dz^2)$$

where $u = u_1 + u_2, u_1, u_2$ are all harmonic functions, with

$$u_1 = \lim_{m\to\infty}\Big(\sum_{|j|\le m} u_{[2j]} - c\log m\Big).$$

and

$$u_2 = \lim_{m\to\infty}\Big(\sum_{|j|\le m} u_{[2j+1]} - c\log m\Big).$$

convergent. Here $u_{[2j]}$ is the potential of the $2j$-th axis rod with its rod structure $(1,0)$, and $u_{[2j]+1}$ with $(0,1)$. Recall α can be obtained by solving a quadrature $d\alpha = F(u_1, u_2, \partial u_1, \partial u_2)$ once u_1 and u_2 are specified.

Note $\frac{\rho^2}{e^u} = 1$ as there is no horizon, and the total effect of the axis rods constitutes the "full" $2 \log \rho$;

$$ds^2 = -dt^2 + e^{u_1} d\phi_1^2 + e^{u_2} d\phi_2^2 + e^{\alpha}(d\rho^2 + dz^2)$$

and thus $u_1 + u_2 = 2 \log \rho$.
No conical singularities as α, u_1, u_2 can be adjusted with constants so that

$$\frac{\text{Circumference}}{2\pi\text{Radius}} =\sim \frac{e^{u_i/2}}{\rho e^{\alpha/2}} \to 1 \quad (i = 1, 2)$$

Not asymptotically flat but **Kasner**: as $\rho \to \infty$, it is asymptotic to a complete metric;

$$ds^2 = -dt^2 + \rho d\phi_1^2 + \rho d\phi_2^2 + \frac{1}{\sqrt{\rho}}(d\rho^2 + dz^2)$$

Not asymptotically flat but **Kasner**:

$$\overline{u} = \frac{1}{2L}\int_{-L}^{L} u dz \Rightarrow \overline{u} = a + b\log\rho$$

a (2D-radial) harmonic function. Check that $a = 0$ and $b = -\frac{1}{4}$. The quadrature $d\alpha = F(u_1, u_2 \partial u_1, \partial u_2)$ implies

$$e^{\alpha} = \frac{1}{\sqrt{\rho}}$$

so that

$$ds^2 = -dt^2 + \rho d\phi_1^2 + \rho d\phi_2^2 + \frac{1}{\sqrt{\rho}}(d\rho^2 + dz^2)$$

The $(\rho \to \infty)$-end is a 3-torus:$\mathbb{T}^3 = (S^1)^2 \times S^1 \ni (\phi_1, \phi_2, z)$ where the 2-torus is expanding, and the circle is shrinking.

Gravitational Solitons

Note that for the soliton metric,

$$ds^2 = -\frac{\rho^2}{e^u}dt^2 + e^{u_1}d\phi_1^2 + e^{u_2}d\phi_2^2 + e^{\alpha}(d\rho^2 + dz^2)$$

the part of the metric spanned by the three Killing vectors $\partial_t, \partial_{\phi_1}, \partial_{\phi_2}$,

$$G_{ij} = \begin{pmatrix} -1 & 0 & 0 \\ 0 & e^{u_1} & 0 \\ 0 & 0 & e^{u_2} \end{pmatrix}, \quad [G] = \begin{pmatrix} \frac{1}{\rho^2} & 0 & 0 \\ 0 & e^{u_1} & 0 \\ 0 & 0 & e^{u_2} \end{pmatrix}$$

and the image of harmonic map in the symmetric space $SL(3)/SO(3)$ lies in a 2D flat ($\cong \mathbb{R}^2$), hence harmonic functions rather than a harmonic map. (cf. Iwasawa decomposition $G = KAN$)

Topology of 5D Gravitational Solitons

The $4+1$ dimensional setting with z-periodic spacetime, with no horizons, and $(1,0)$ and $(0,1)$ axis rods alternating.

$$\mathbf{g} = -dt^2 + e^{u_1} d\phi_1^2 + e^{u_2} d\phi_2^2 + e^{\alpha}(d\rho^2 + dz^2)$$

The fundamental region consisting of

- 2 consecutive axis rods $(1,0),(0,1) \Rightarrow S^4 \setminus (T^2 \times B^2)$
- 4 consecutive axis rods $(1,0),(0,1),(1,0),(0,1) \Rightarrow (S^2 \times S^2) \setminus (T^2 \times B^2)$

Space-periodic 5D Black Hole Spacetimes

The $4+1$ dimensional setting

$$\mathbf{g} = -dt^2 + e^{u_1} d\phi_1^2 + e^{u_2} d\phi_2^2 + e^{\alpha}(d\rho^2 + dz^2)$$

with z-periodic spacetime, with **horizons**.

- String of spheres (Schwarzschild-Tangherlini)
- String of double spheres
- String of Black Saturns

Gravitational Instantons

Given Myers-Korotkin-Nikolai $3+1$ Schwarzschild metrics

$$ds^2 = -\frac{\rho^2}{e^u}dt^2 + e^u d\phi^2 + e^\alpha(d\rho^2 + dz^2)$$

Wick-rotate $t \mapsto \sqrt{-1}t$ and obtain 4 dimensional Riemannian setting with z-periodic spacetime, with no horizons, and $(1,0)$ and $(0,1)$ axis rods alternating.

$$\begin{aligned} g &= +\tfrac{\rho^2}{e^u}dt^2 + e^u d\phi^2 + e^\alpha(d\rho^2 + dz^2) \\ &= e^{u_1}d\phi_1^2 + e^{u_2}d\phi_2^2 + e^\alpha(d\rho^2 + dz^2) \end{aligned}$$

where $\frac{\rho^2}{e^u} =: u_1$ and $u =: u_2$ are the harmonic functions with $u_1 + u_2 = 2\log\rho$. This coincides with the spacelike part of our gravitational soliton $\mathbf{g}$

$$\mathbf{g}\Big|_{t=0} := e^{u_1}d\phi_1^2 + e^{u_2}d\phi_2^2 + e^\alpha(d\rho^2 + dz^2)$$

Gravitational Instantons

The Riemannian metric g: Ricci-flat, the holonomy is of generic type, i.e. **not hyper-Kähler**. (cf. Gibbons-Hawking Asatz) by the Ambrose-Singer Theorem,
The fundamental regions consisting of

- 2 consecutive axis rods $(1,0),(0,1) \Rightarrow S^4 \setminus (T^2 \times B^2)$
- 4 consecutive axis rods $(1,0),(0,1),(1,0),(0,1) \Rightarrow (S^2 \times S^2) \setminus (T^2 \times B^2)$
- the whole spaceimte $\#^\infty(S^2 \times S^2)$ so that $b_2 = \infty$

Gravitational Instantons

Rmk: Gromov: For a manifold (M^n, g) of non-negative K_g

$$\exists C_n \text{ such that } \sum_{i=0}^{n} b_i < C_n$$

Q. What about non-negative Ricci curvature? (e.g. Sha–Yang examples of $b_* = \infty$ with $\mathrm{Ric}_g > 0$, JDG(1989))
A. No such C_n $(n \geq 4)$ exists, as [KRWY] says the gravitational instanton satisfies $b_2 = \infty$ for $\mathrm{Ric}_g = 0$.

Summary

- Einstein equation with time symmetry $\leftrightarrow$ Harmonic maps to symmetric spaces
- Harmonic map as a solution of asymptotic Dirichlet problems (boundary condition: spatial axial symmetries cf. Toric varieties)
- Static, space-periodic solutions: harmonic functions (Image $\subset$ flats of the symmetric space)
- New constructions of gravitational solitons, and gravitational instantons of various topologies.