Black hole の影と共役点の問題

椎野　克

CQG 38 025005(2020), PRD 106 044020 (2022)、arXiv/2305.08362

1　導入：Black hole shadow

2　wandering null geodesics

〇　定義

〇　存在・性質

3　Black room

〇　定義

〇　存在

まとめ

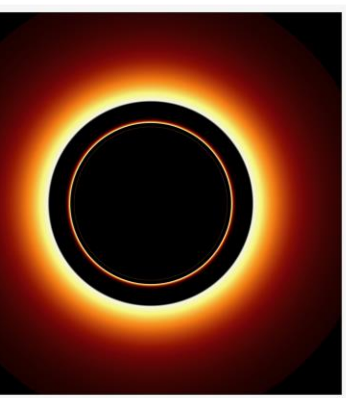
4　関係ない話

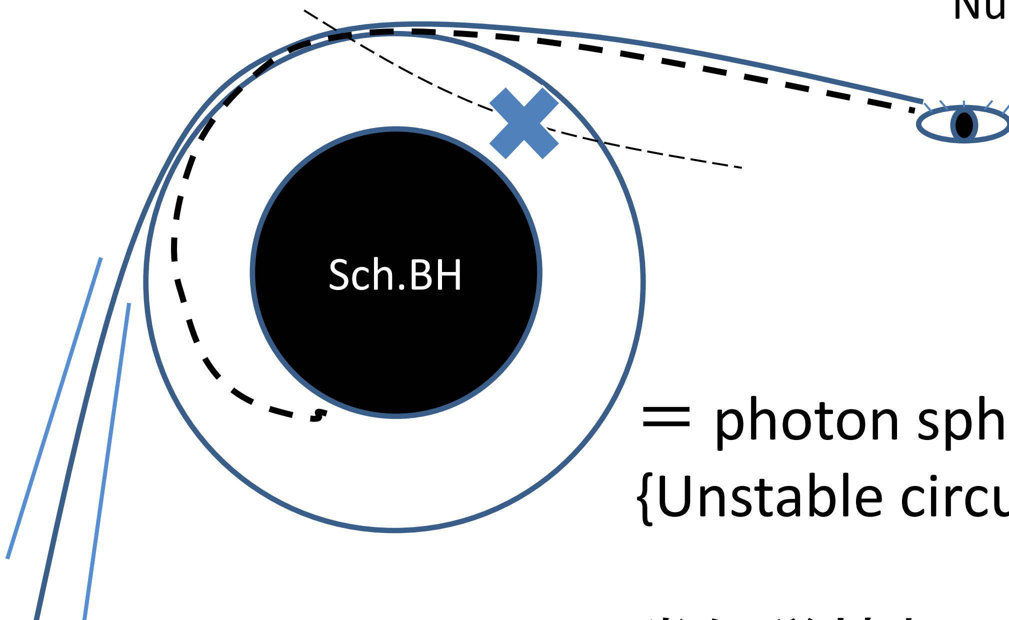
〇Cheeger理論

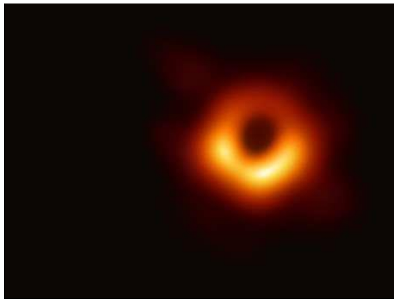
Black hole shadow by photon sphere

Rough sketch of black hole shadow　Schwarzschild BH

Null geodesic の集積と離散



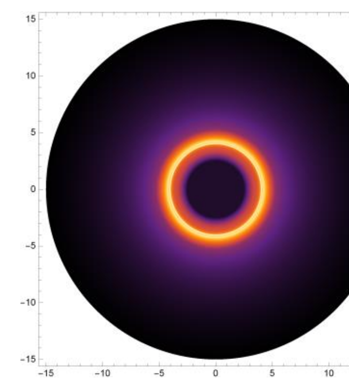




＝ photon sphere
{Unstable circular orbit　r＝3m}

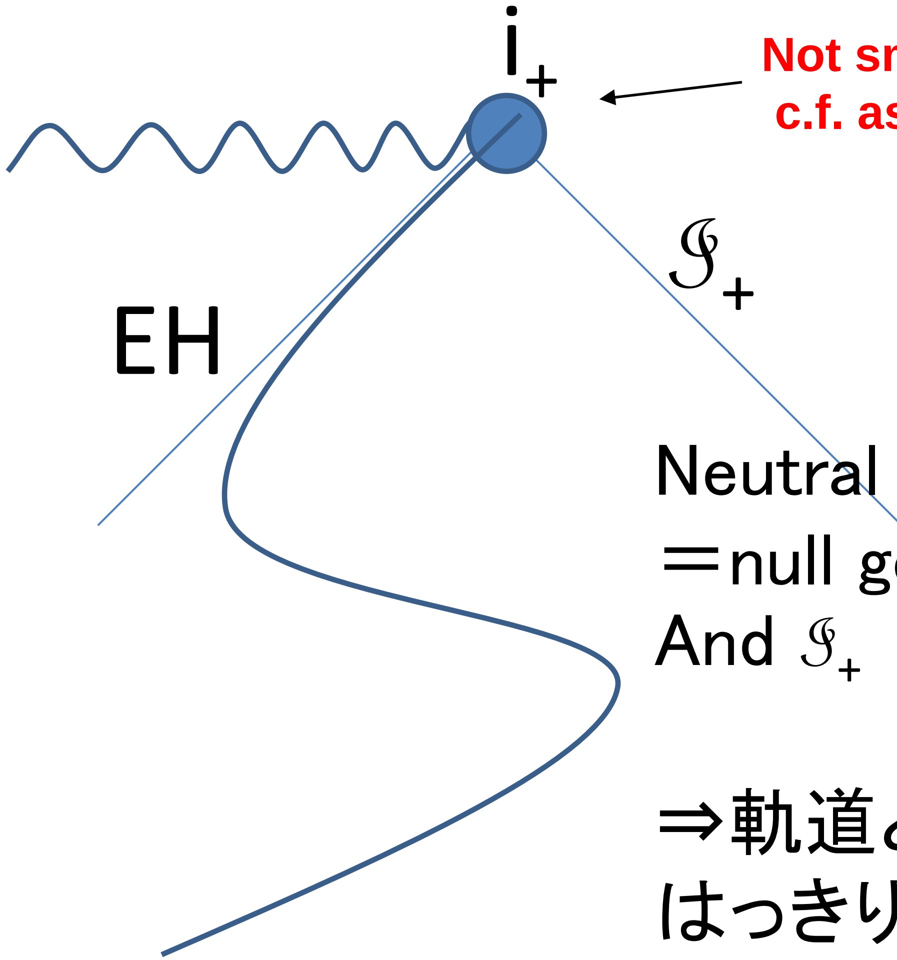
天体情報
○背景光源, 降着光源
Gralla,Holz,Wald
Narayan, Johnson,Gammie

幾何学情報
BH解、重力理論
非対称 Solution？....



No coordinate, No orbit

因果構造による
Generalized photon sphere



Not smooth
c.f. asymptotically simple

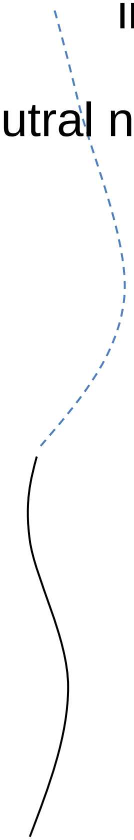
Neutral null geodesic
＝null geod. Not crossing EH
And $\mathscr{I}_+$

⇒軌道との関係が
はっきりしない（回っている？）
（affinely long??)

wandering null geod. congruence

i+ timelike infinity

Neutral null geodesic



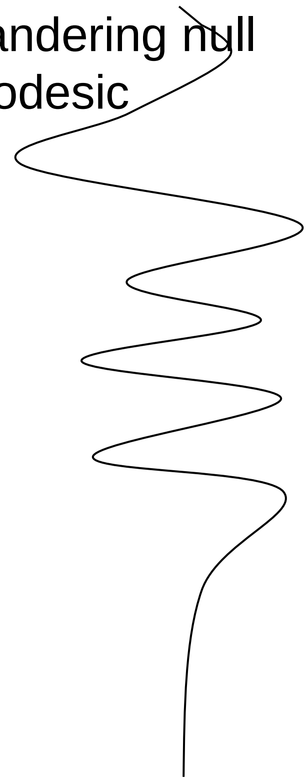
Riemann curvature
Raychaudhri eq.

$$\frac{d\theta}{ds} = -\frac{1}{2}\theta^2 - \hat{\sigma}_{ab}\hat{\sigma}^{ab} - R_{cd}k^c k^d$$

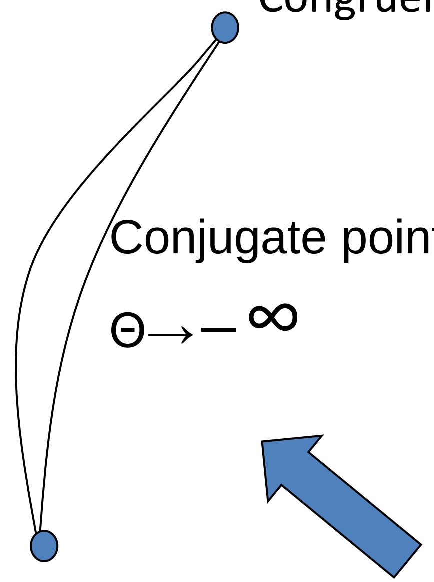
$$k^c\nabla_c\hat{\sigma}_{ab} = -\theta\hat{\sigma}_{ab} + \widehat{C_{cbad}k^c k^d},$$

i+

Wandering null geodesic



Congruence



Conjugate point

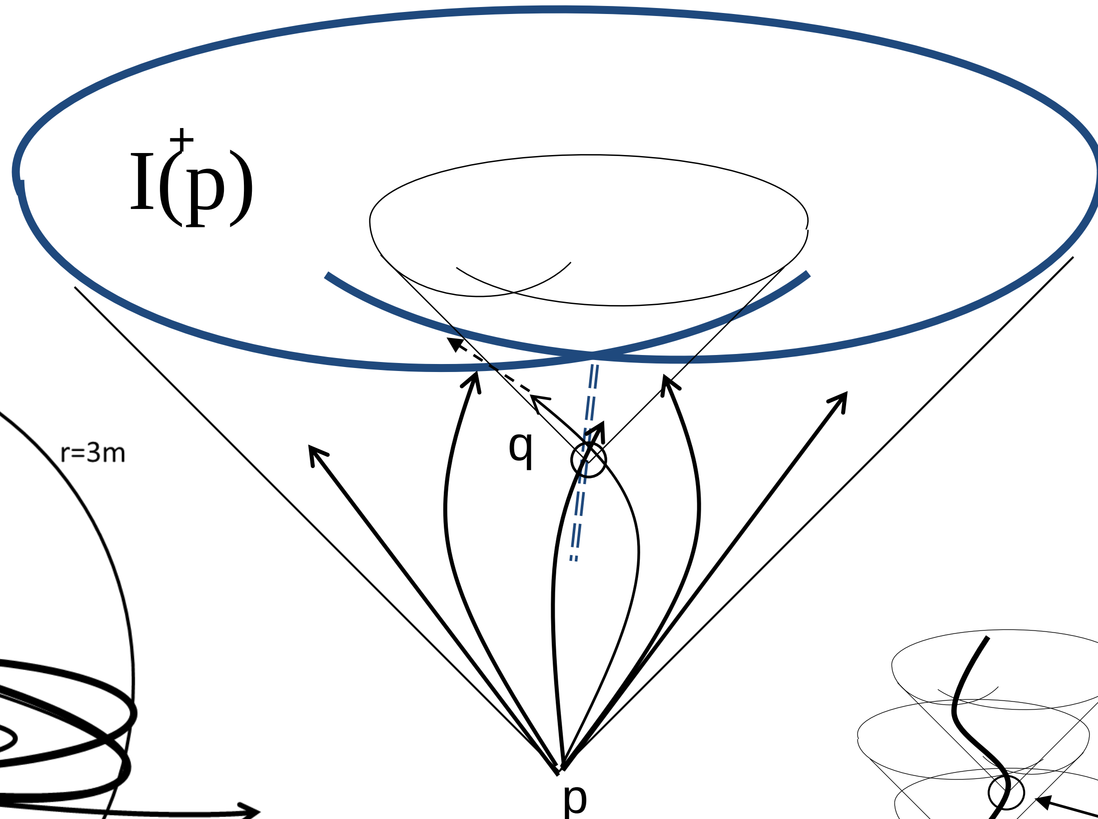
$\Theta \to -\infty$

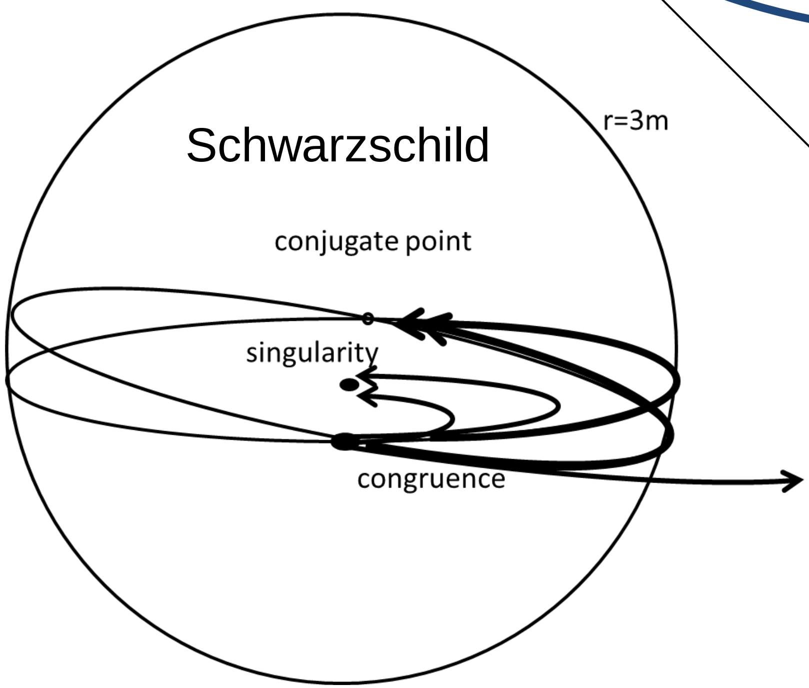
gravitation

Not crossing EH $\Rightarrow$ complete with infinite conjugate points
not going to $\mathscr{I}_+$
Affinely long

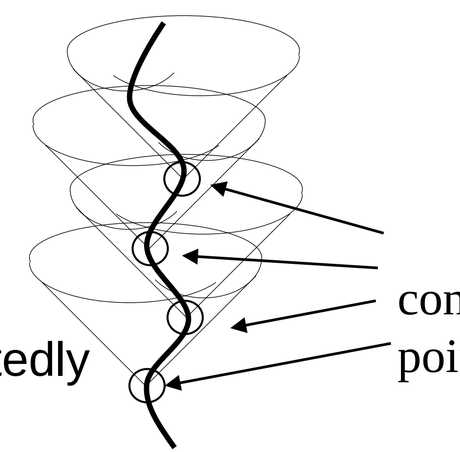
Conjugate point and wandering set (globally hyperbolic) CQG??

Theorem 9．3．11　 pp233 in Wald
q on boundary is connected to p by null geod.
without conjugate point: ∂J generator



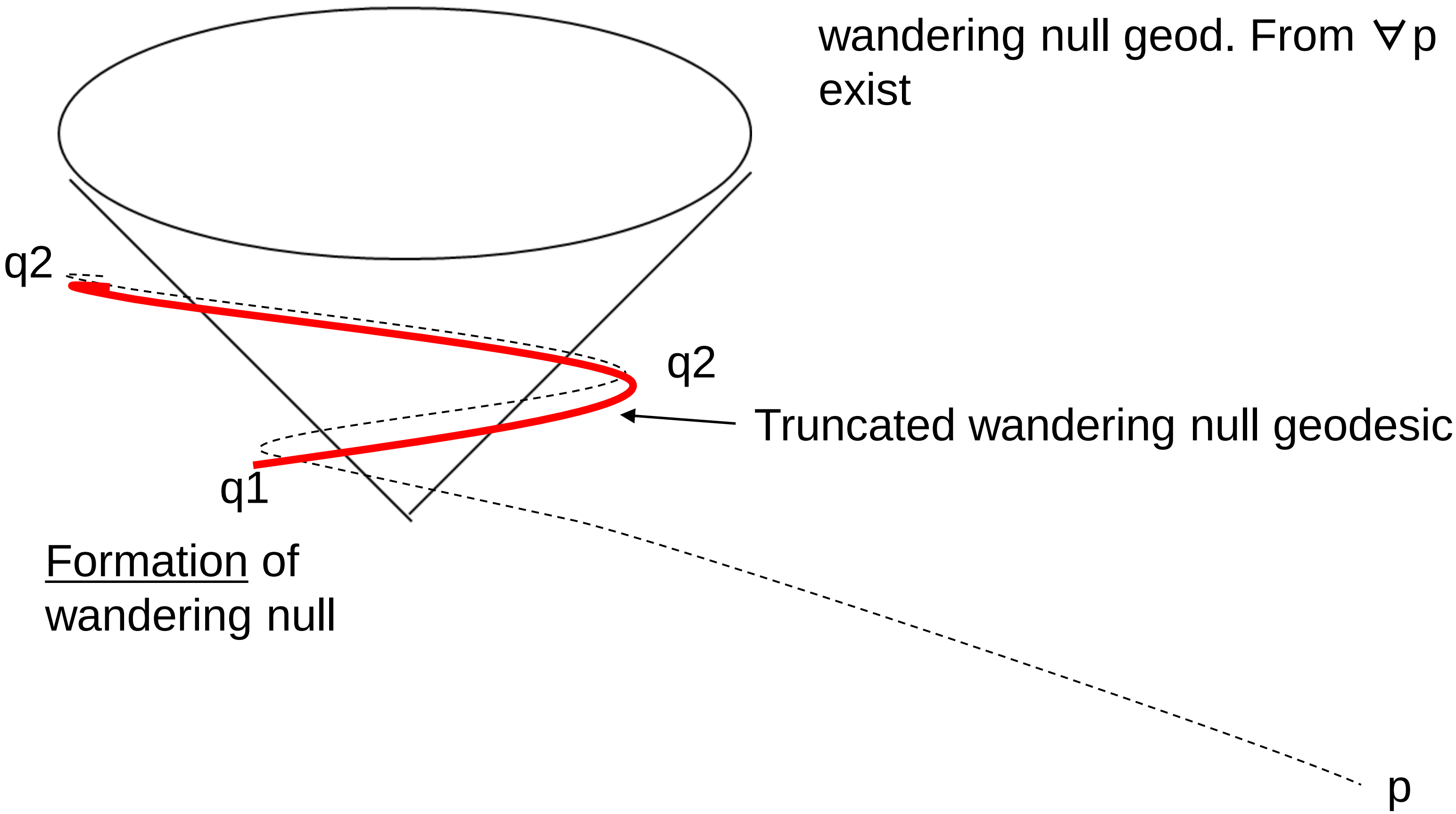


It appears repeatedly



Truncated wandering null

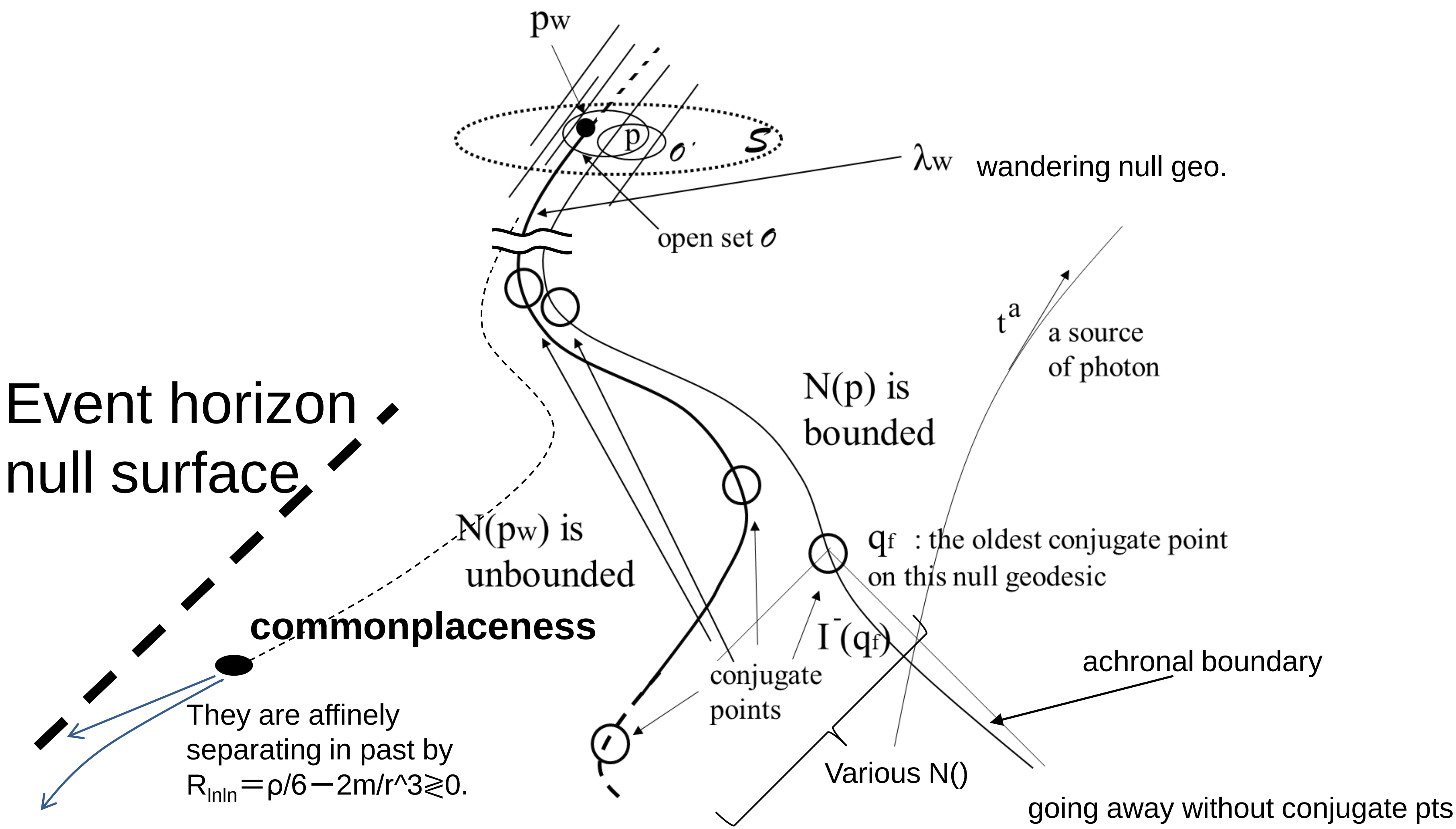
シュワルツシルトでは
wandering null geod. From ∀p
exist



Formation of wandering null

We should truncate p---q1 to discuss formation

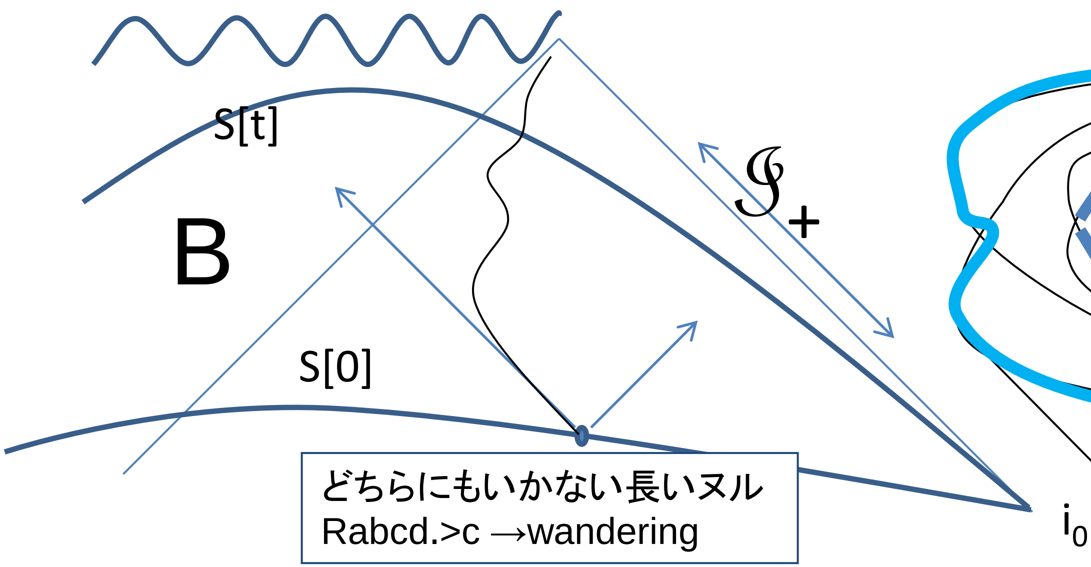
Wandering null geodesicの集積とシャドウ

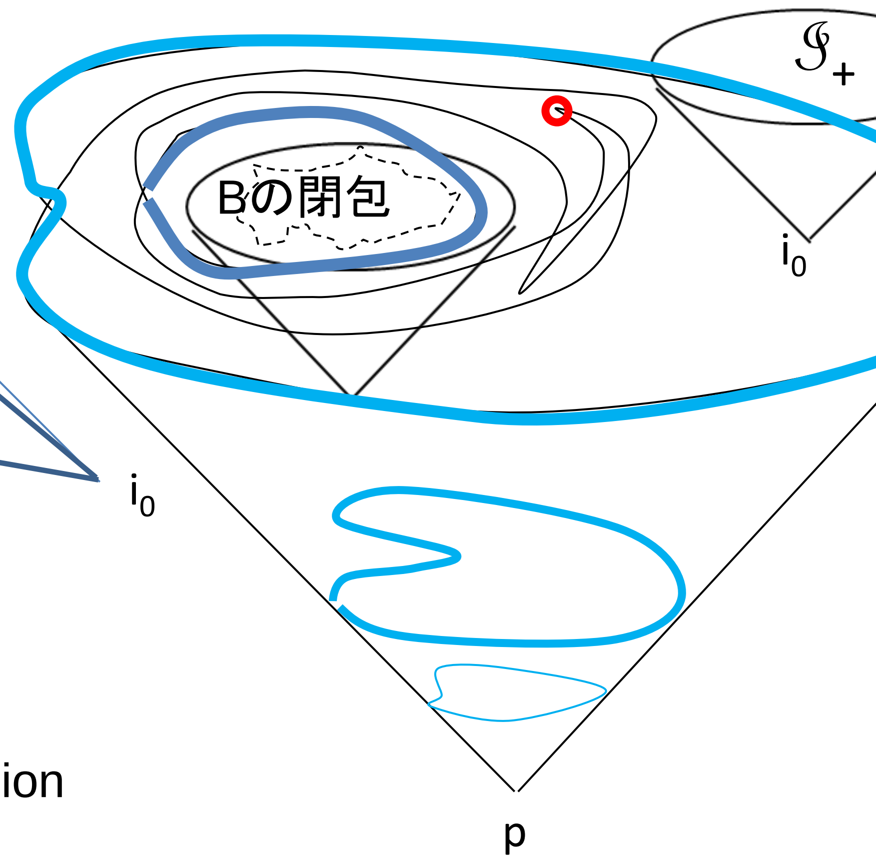


汎存(cosmopolitan)定理

M:asymptotic flat, globally hyperbolic, black hole spacetime

S[t] : singularity avoiding slice
十分大きいtにEHの外は含まれる





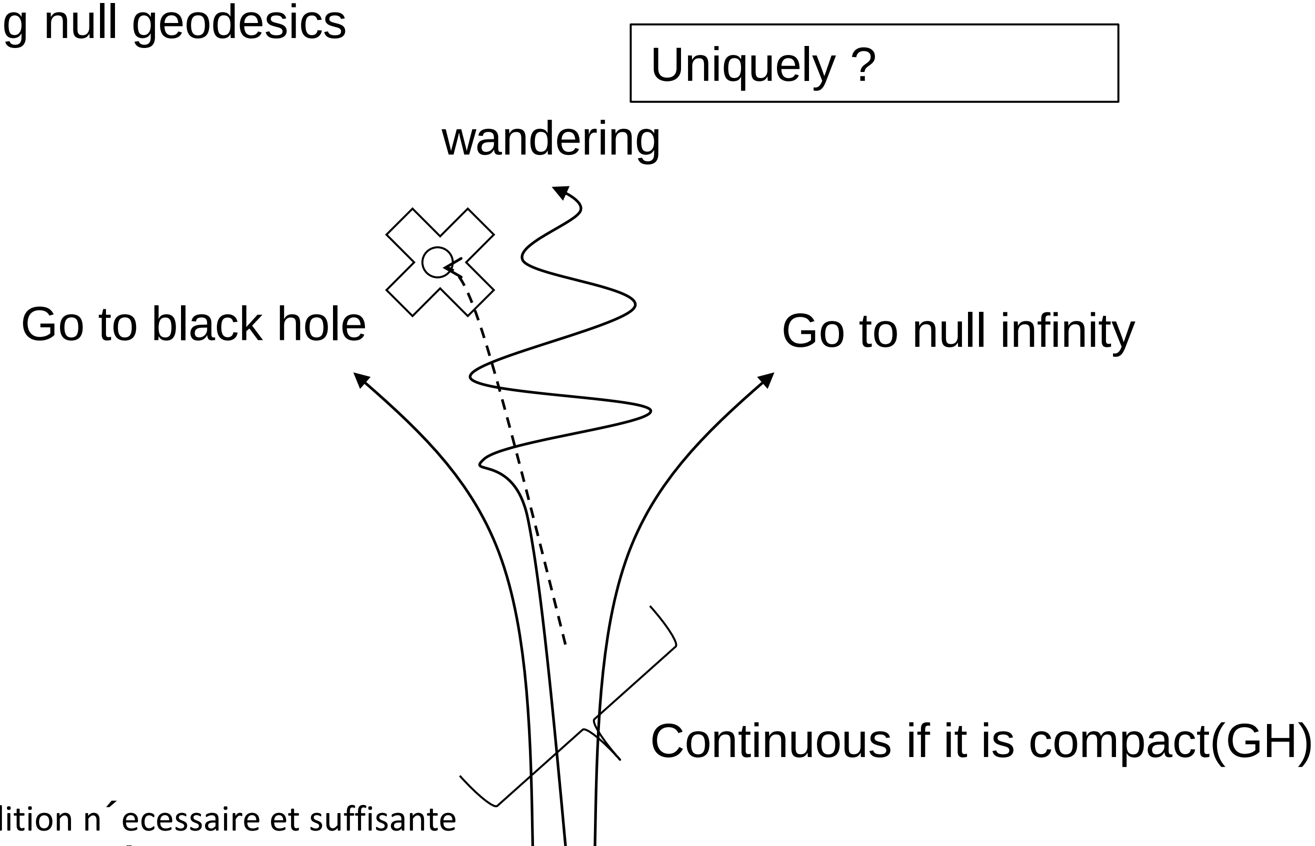
○globally hyperbolicなら
A=S[t]∩J^+(p)はcompactで連結

○smooth GH spacetimeなら
Λ＝S[t]∩ {λ[p:q]}はS_2のAへの連続なimmersion

Wandering null geodesicの汎存性

From the Libsitz continuity (possible on the surface of star and global hyperbolicity

All the black hole (Rabcd≠0、E≥0）spacetime **point** possesses wandering null geodesics



Hiroshi Okamura. Condition n´ecessaire et suffisante remplie par les ´equations diff´erentielles ordinaires sans points de peano. *Mem. Coll. Sci., Kyoto Imperial Univ., A (in French), 24: 21–28*, 1942.

3.1 with small perturbative Weyl curvature

In the theorem 1, we have demonstarated that in the formation of conformally flat formation of black hole is not accompanied by the truncated wandering set. In general, however, the black hole is not conformally flat at the formation of itself. So, in now we consider the black hole is formed with a small Weyl curvature bounded above.

Now we suppose the situation where a small Weyl curvature is included to the system of the null geodesic congruence equations.

deviation equation

conformal trs. で
この項を除去

$$\frac{d\theta}{ds} = -\frac{1}{2}\theta^2 - \hat{\sigma}_{ab}\hat{\sigma}^{ab} - R_{cd}k^c k^d$$

$$k^c\nabla_c\hat{\sigma}_{ab} = -\theta\hat{\sigma}_{ab} + \widehat{C_{abcd}k^c k^d}, \tag{4}$$

where s is the affine parameter.

Firstly we want to distiguish the effect of Ricci curvature $R_{ab}k^a k^b$. Considering a conformal transformation $\tilde{g}_{ab} = \Omega^2 g_{ab}$, while the affine connection changes to $\tilde{\nabla}_b v^a = \nabla v^a + C^a_{bc}v^b$, $C^a_{bc} = 2\delta^a_{(b}\nabla_{c)}\ln\Omega - g_{bc}g^{ad}\nabla_d \ln\Omega$, the null geodesics is maintained as

$$k^b\tilde{\nabla}_b k^a = k^b\nabla_b k^a + 2k^a k^c\nabla_c \ln\Omega - \mathbf{g}(\mathbf{k},\mathbf{k})g^{ad}\nabla_d \ln\Omega \tag{5}$$

$$= 2\frac{d\ln\Omega}{ds}k^a,$$

null測地線はnull測地線のままアフィンで無くなる

where k^a is the affine parametrized tangent null vector $(\frac{\partial}{\partial s})^a$. The affine parametrized tangent null vector $\tilde{k}^a = f(s)k^a = (\frac{\partial}{\partial \tilde{s}})^a$ for $\tilde{g}_{ab}$ geometry, is given by

$$f(s)f'(s) + 2f(s)^2\frac{d\ln\Omega}{ds} = 0 \tag{7}$$

$$f(s) = -\frac{1}{c\Omega}, \quad \frac{d\tilde{s}}{ds} = c\Omega^2, \tag{8}$$

where c is a constant.

Now we assume that $R_{ab}k^a k^b$ in Eq. (2.1) vanishes by this conformal transformation, then the conformal transformation o
: Ricci tensor,

$$\tilde{R}_{ac} = R_{ac} - 2\nabla_a \nabla_c \ln\Omega - g_{ac} g^{de} \nabla_d \nabla_e \ln\Omega + 2\nabla_a \ln\Omega \nabla_c \ln\Omega - 2g_{ac} g^{de} \nabla_d \ln\Omega \nabla_e \ln\Omega \qquad (3.5)$$

(3.6

quires the conformal factor Ω satisfies a relation,

$$\begin{aligned} \tilde{R}_{ac}\tilde{k}^a\tilde{k}^c &= \frac{1}{(c\Omega)^2}\left[k^a k^c R_{ac} - 2k^a k^c \nabla_a \nabla_c \ln\Omega + 2\left(k^a \nabla_a \ln\Omega\right)^2\right] &(3.7\\ &= \frac{1}{c^2\Omega^4}\left[R_{ac}k^a k^c - 2\frac{d^2\ln\Omega}{ds^2} + 2\left(\frac{d\ln\Omega}{ds}\right)^2\right] &(3.8\\ &= 0, &(3.9 \end{aligned}$$

ere we consult $k^a k^c \nabla_a \nabla_c F = k^a \nabla_a (k^c \nabla_c F) - (k^a \nabla_a k^c) \nabla_c F = k^a \nabla_a (\frac{dF}{ds})$. **注意、リッチによる特異点**

Here we introduce $\omega(s)$ which satisfies $\omega' = -\omega^2 - \frac{1}{2}[R_{ab}k^a k^b](s)$ along the null geodesic. Then, there exists Ω such tha
satisfies $\omega(s) = -d\ln\Omega/ds$ on the null geodesic congruence. The smoothness of Ω will be destroyed by the divergence o
If $\omega(s)$ takes a negative value ω_0 at s_0, it diverge within $s - s_0 \le 1/\omega_0$ provided that $R_{ab}k^a k^b$ is always non-negative. Th
vergence directly means a conjugate point appears by the effect of the Ricci curvature. This, however, is not the case of ou
nost vacuum situation. So, we will assume the existence of a regular solution for such $\omega(s)$.

By the conformal transformation, the deviation equation of null congruence reduces to

$$\begin{cases} \dot{\tilde{\theta}} = -\frac{1}{2}\tilde{\theta}^2 - \mathrm{Tr}\tilde{\boldsymbol{\Sigma}}^2 \\ \dot{\tilde{\boldsymbol{\Sigma}}} = -\tilde{\theta}\tilde{\boldsymbol{\Sigma}} + \widehat{\mathbf{C}}, \end{cases} \qquad (3.10$$

ere $\tilde{\theta}$ and $(\tilde{\boldsymbol{\Sigma}})^a_c \equiv \tilde{g}^{ab}\sigma_{bc}$ are for $\tilde{k}^a$ and $\tilde{\nabla}_a$, and $(\widehat{\mathbf{C}})^a_d$ is $\widehat{C^{\tilde{a}}_{\ bcd}\tilde{k}^b\tilde{k}^c}$, where the bold face is used to express matrix value
riables. $\dot{F} = dF/d\tilde{s}$ is the derivative of F by the conformally transformed affine parameter $\tilde{s}$.

Supposing $\widehat{\mathbf{C}}$ is limited by any matrix norm[5] $||.||$ as $||\widehat{\mathbf{C}}|| < C_0 < 1$, we will consider perturbations by small $\widehat{\mathbf{C}}$ expansions.

By the series expansion $F = \sum_i F_i$ of the variables for the order of $\sim C_0^i$, we define

$$\tilde{\theta} = \tilde{\theta}_0 + \tilde{\theta}_1 + O[C_0^2], \quad \tilde{\Sigma} = \tilde{\Sigma}_0 + \tilde{\Sigma}_1 + O[C_0^2]$$

ワイル曲率が小さいとして
線形摂動

he 0-th order background solution is given by

$$\tilde{\theta}_0(\tilde{s}) = \frac{\theta_{ini}}{\tilde{s} - \tilde{s}_0}, \quad \tilde{\Sigma}_0(\tilde{s}) = \frac{\Sigma_{\mathbf{ini}}}{\tilde{s} - \tilde{s}_0},$$

発散しない

here θ_{ini} and $\Sigma_{\mathbf{ini}}$ satisfy algebraic equations deduced from the equatio

直線解

out source term. This solutions co
straight light rays, and of course not divergent with initial conditions xpanding congruence. The equatio
-st order are given by

first order

$$\dot{\tilde{\theta}}_1 = -\tilde{\theta}_0\tilde{\theta}_1 - 2\mathrm{Tr}\tilde{\Sigma}_0\tilde{\Sigma}_1 = -\frac{\theta_{ini}}{\tilde{s} - \tilde{s}_0}\tilde{\theta}_1 - \frac{2}{\tilde{s} - \tilde{s}_0}\mathrm{Tr}(\Sigma_{\mathbf{ini}}\tilde{\Sigma}_1),$$

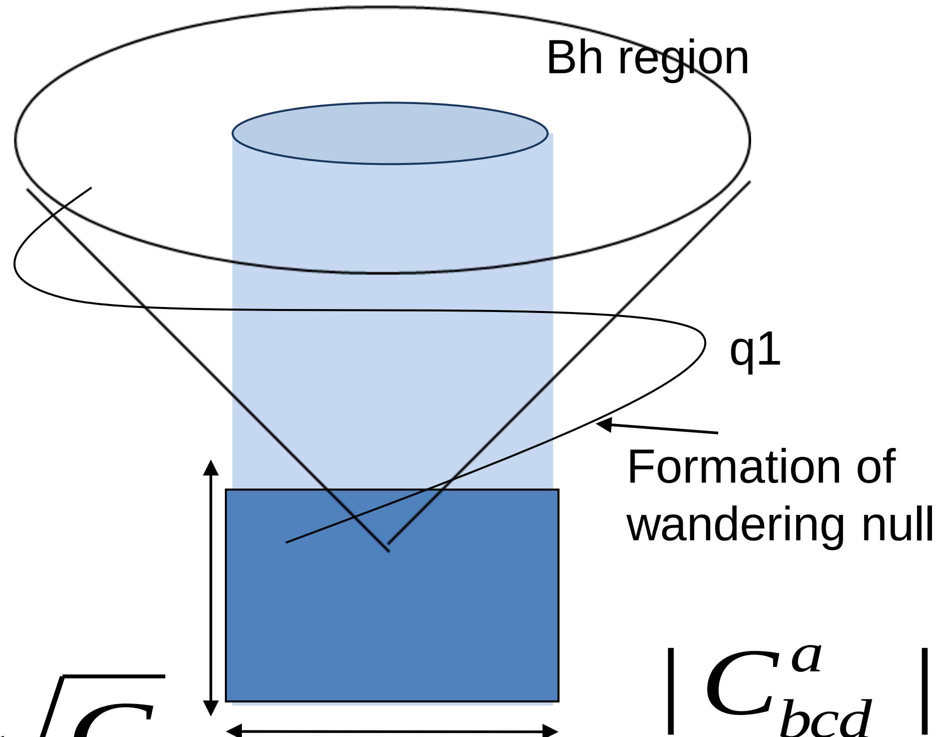
$$\dot{\tilde{\Sigma}}_1 = -\tilde{\Sigma}_0\tilde{\theta}_1 - \tilde{\theta}_0\tilde{\Sigma}_1 + \widehat{\mathbf{C}} = -\frac{\Sigma_{\mathbf{ini}}}{s - s_0}\tilde{\theta}_1 - \frac{\theta_0}{\tilde{s} - \tilde{s}_0}\tilde{\Sigma}_1 + \widehat{\mathbf{C}}.$$

The homogeneous solution of these first order differential equations will be given by $\tilde{\theta}_1 = \tilde{\theta}_0/(\tilde{s} - \tilde{s}_0)$, $\tilde{\Sigma}_1 = \tilde{\Sigma}_0$
hen the order of magnitude of the inhomogeneous solutions are of order $\sim ||\widehat{\mathbf{C}}|||\tilde{s} - \tilde{s}_0|$. From the initial condition
rder, $s - s_0$ is positive and this first order solutions are not divergent. Consequently, as long as the perturbation is c
here appears no conjugate points.

Since the order of magnitude of 0-th order solution is $\sim 1/|\tilde{s} - \tilde{s}_0|$ and that of 1-st order solution is $\sim ||\widehat{\mathbf{C}}|||\tilde{s} - \tilde{s}_0$
$-\tilde{s}_0| \ll 1/\sqrt{C_0}$, the perturbation is consistent. Therefore, there is no conjugate point for this $\tilde{g}_{ab}$ geometry in such a
e original geometry g_{ab}, from Eq. (3.4) and $\tilde{C}^a_{bcd}\tilde{k}^b\tilde{k}^c = \frac{1}{c^2\Omega^4}C^a_{bcd}k^bk^c$ that condition turns to $|s - s_0| \ll 1/\sqrt{C_1}$ when
bounded by C_1 as $|C^a_{bcd}k^bk^c| < C_1 < 1$.

摂動のconsistencyがワイルを抑える

Evaluation for 共形構造



$$1/\sqrt{C}$$

$$|C^{a}_{bcd}| < C$$

<u>By solving Conformally transformed geodesic deviation equation</u>

Skip calc.

系

大域的共形平坦なら

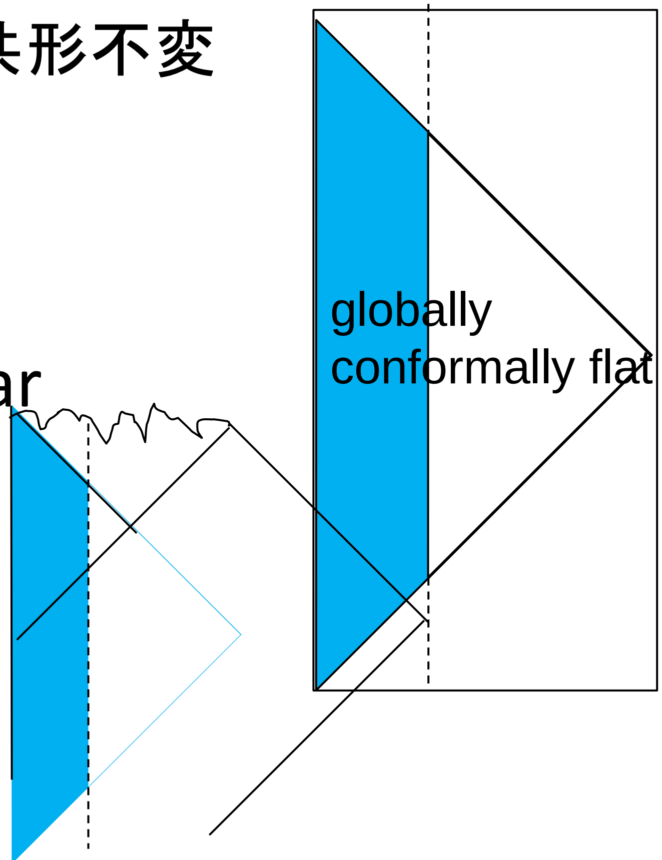
No conjugate point

光測地線は共形不変

〇$C_{abcd}=0$, locally no wandering set

〇Inside of Oppenheimer Snyder star

No truncated wandering null

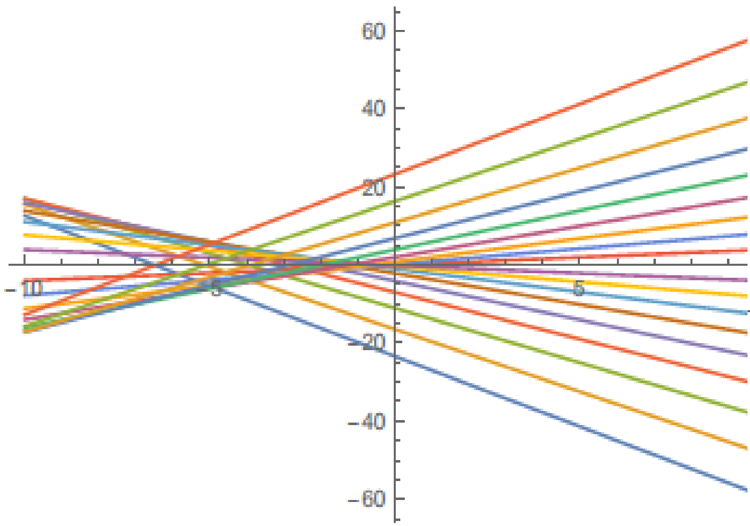


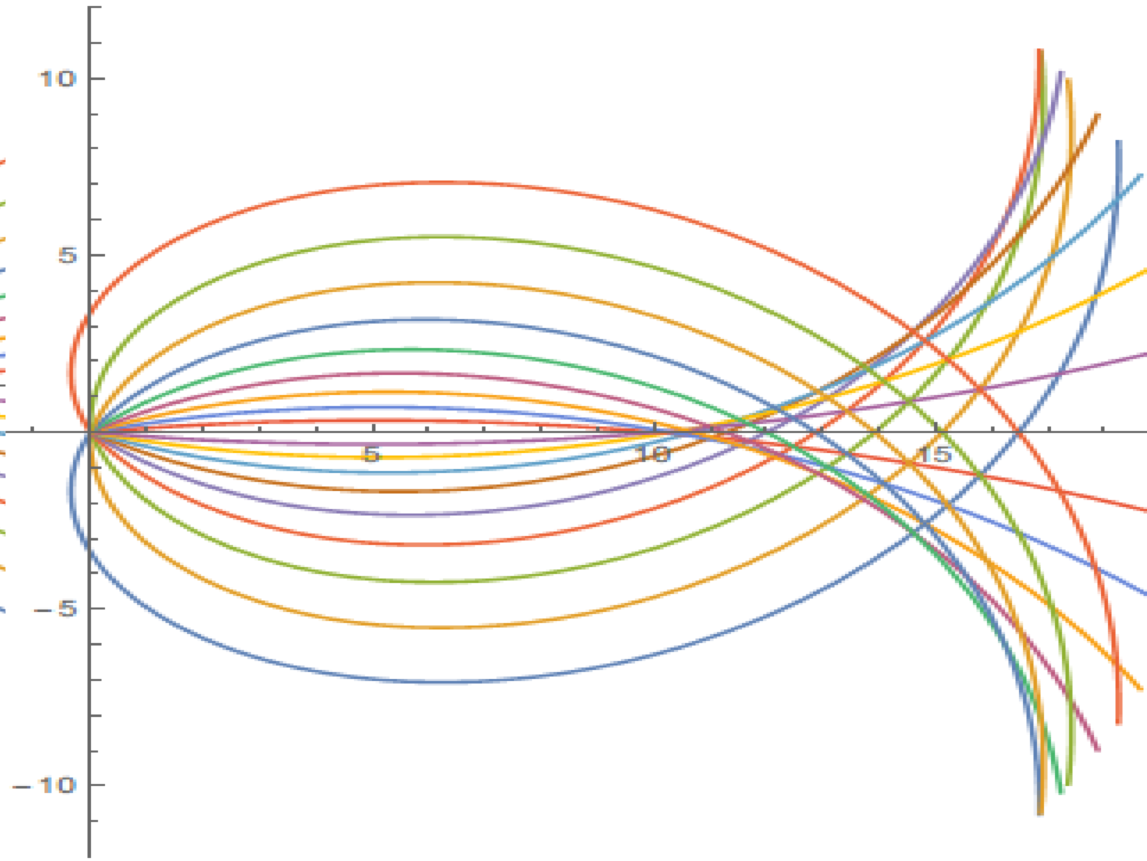
3.1 Thom's theorem and geometrical optics

From the Thom's theorem, in the case of the control parameter is three dimensional only five types of potential funtion is strucrurally stable in the equivalence class at the diffeomorphisms.

3.2 cusp: $\tau = x^4 + ux^2 + vx$

Conjugate pointはくさびのカタストロフィー（構想安定）を伴う事ができる。

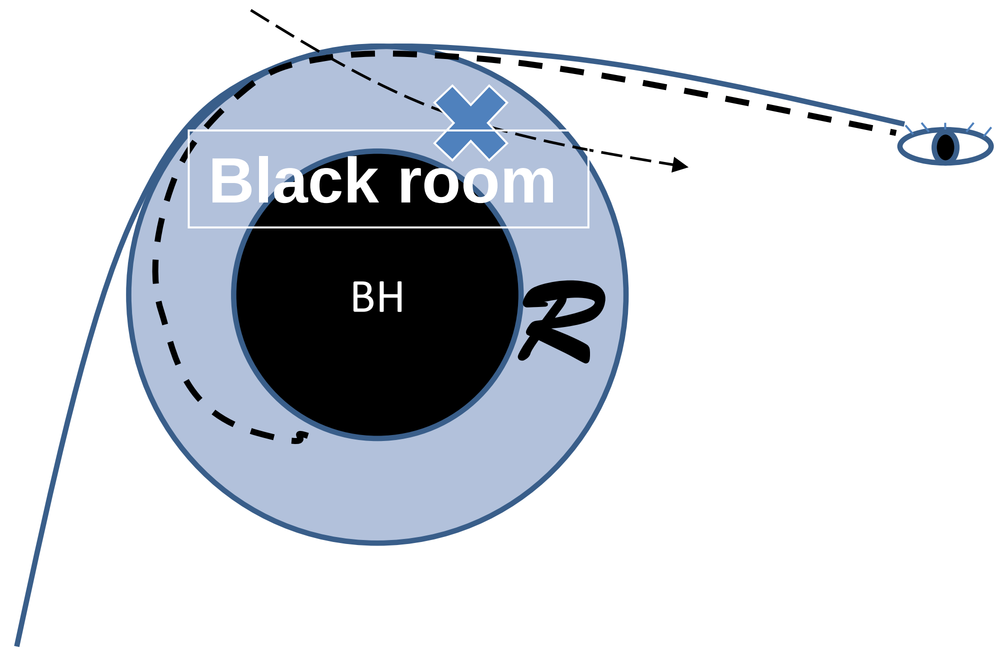




MBR: Maximal Black room $\mathcal{R}$

空間的に有限、null geodesic が入ってきたら出て行かない

Black hole is also black room
Black room≧Black hole



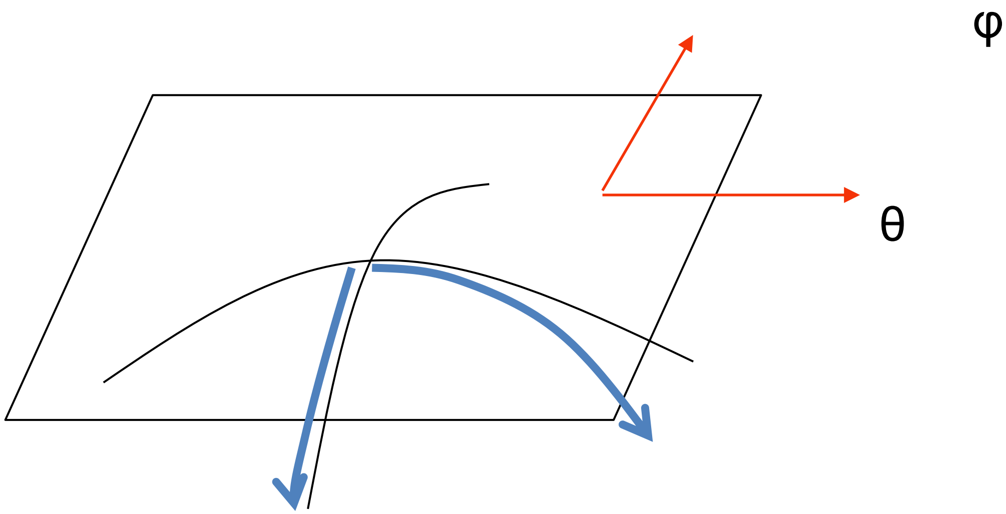
Maximal black room $\mathcal{R}$:
a black room include
all black room

$\partial\mathcal{R}$ Is corresponding to the photon sphere

Not Photon surface but Rays surfacee

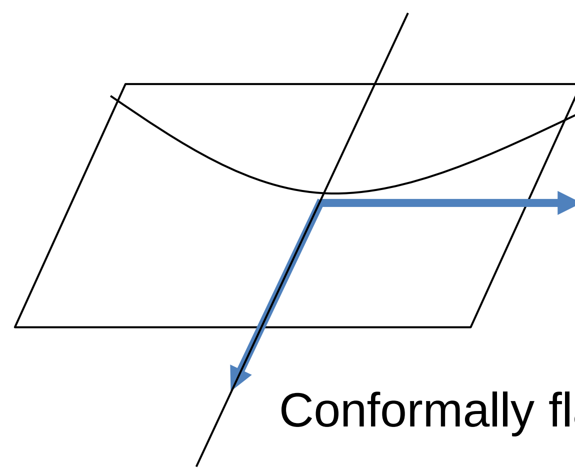
Prop.2 in PRD <u>106</u> 044020 (2022)
MBR is bounded by a non spacelike surface containing at least one null geodesic
i.e., horizon generator, wandering null geo. , or future endpoint [光線曲面≃rays surface]

under the energy condition, generic condition



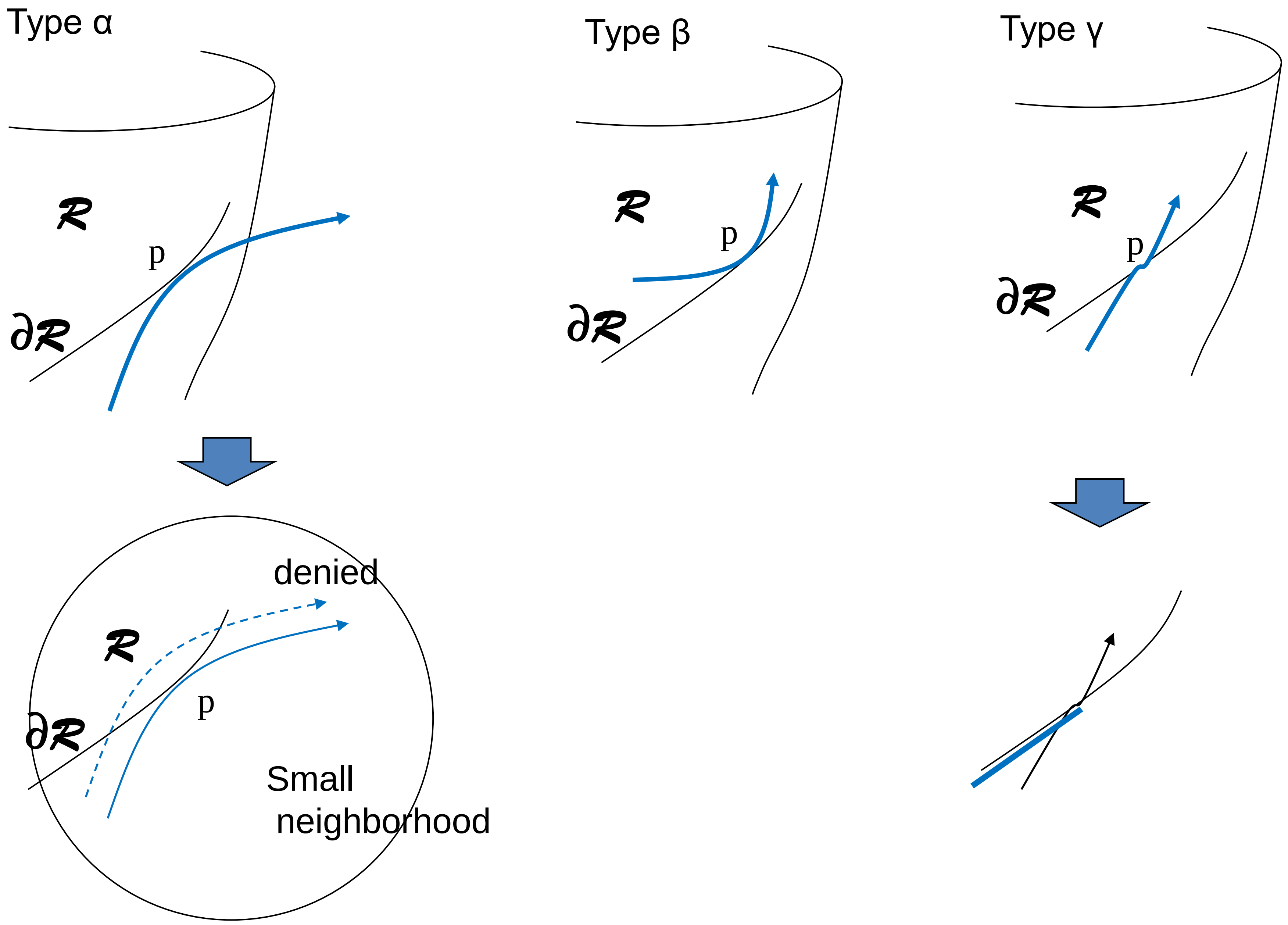
Direction where
apparent Mass is small

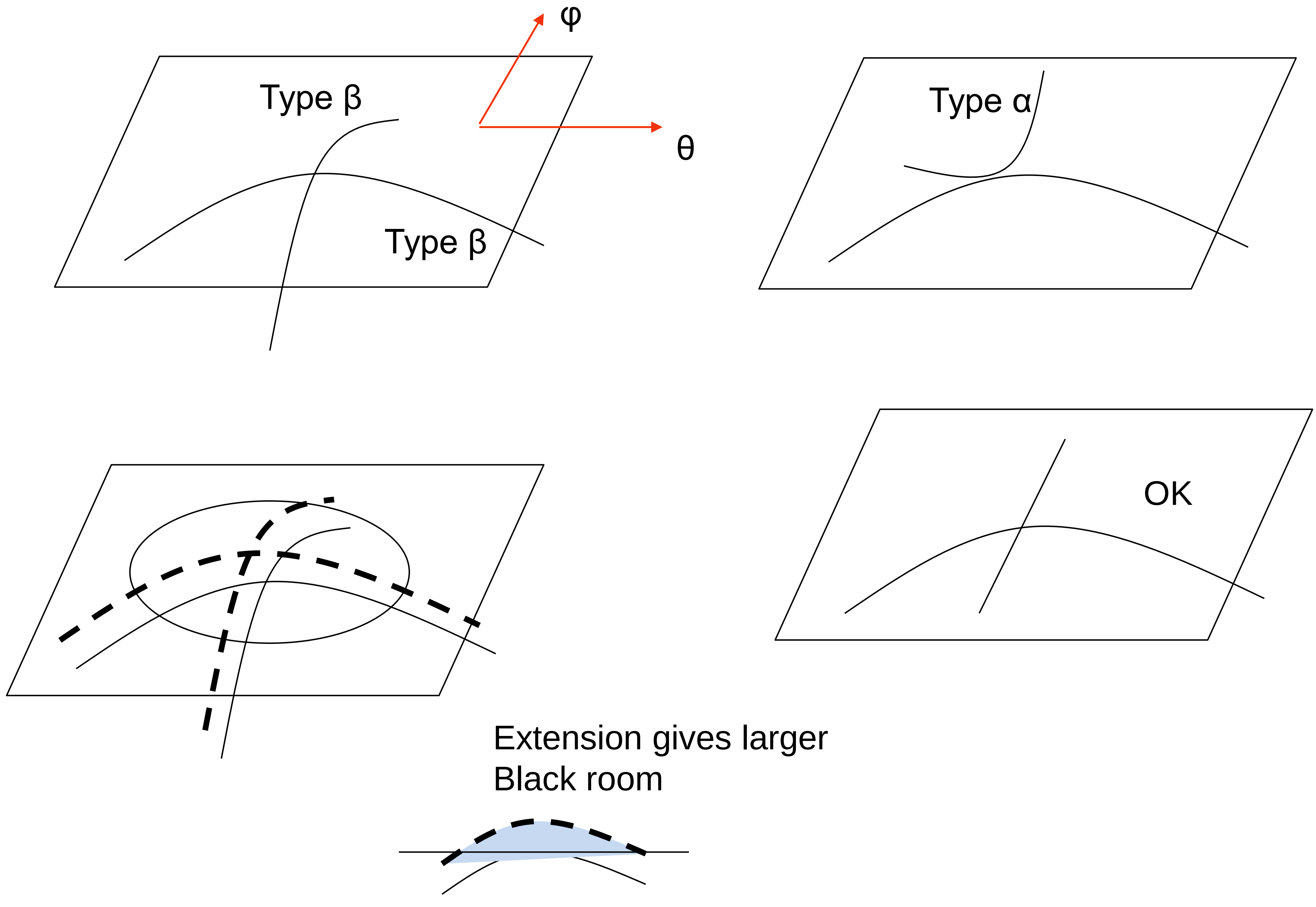
e.g., Kerr



Conformally flat pict.

Supposing null geo. not on $\partial\mathcal{R}$ there is three types of null geo.

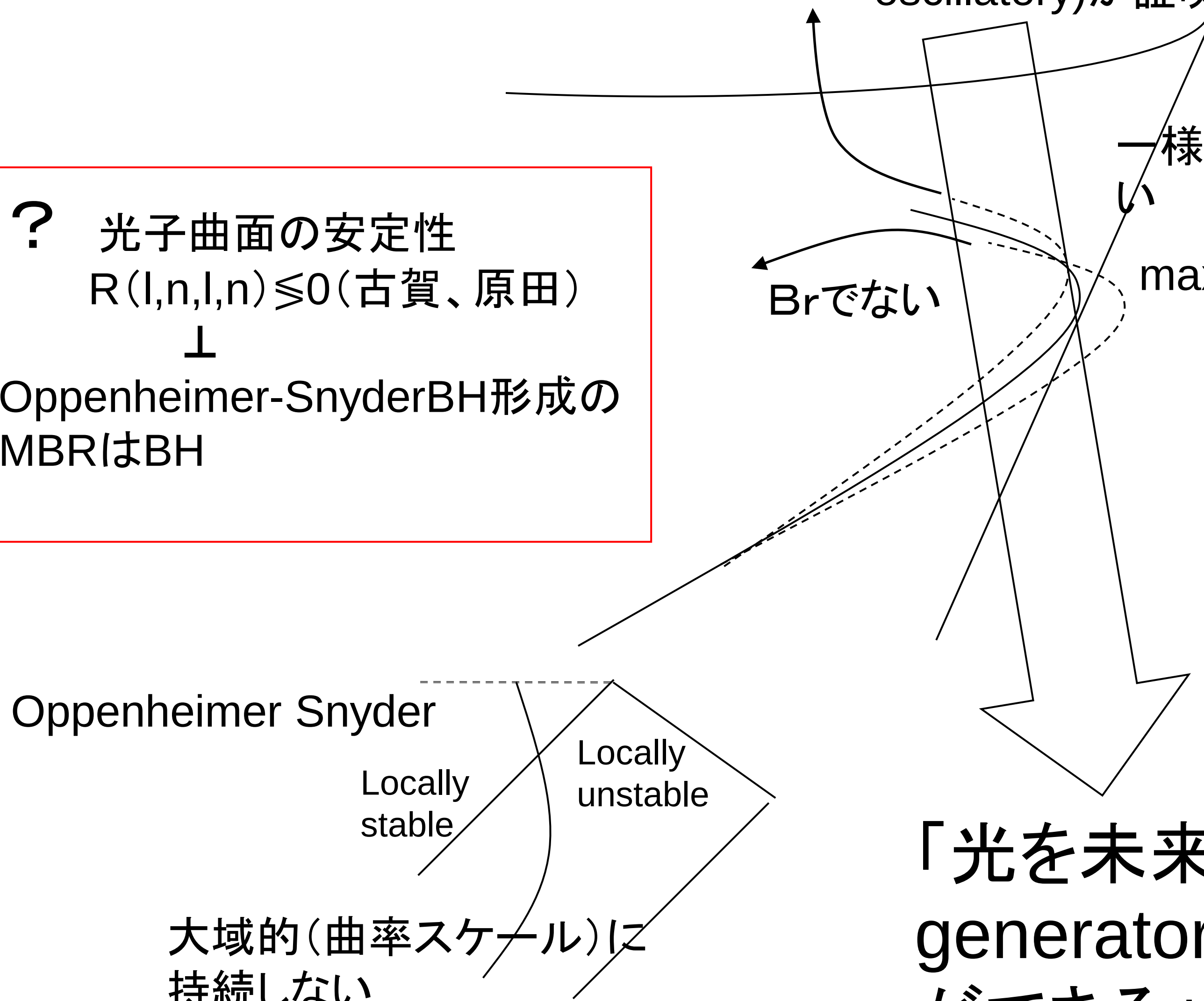



φ
θ
Type β
Type β
Type α
OK
Extension gives larger
Black room

MBRの不安定性

光線曲面と同じような証明で**総体的に**不安定（non oscillatory)が証明できる

？　光子曲面の安定性
R（l,n,l,n）≦0（古賀、原田）
⊥
Oppenheimer-SnyderBH形成のMBRはBH



「光を未来に飛ばしてgeneratorを探すことができる」

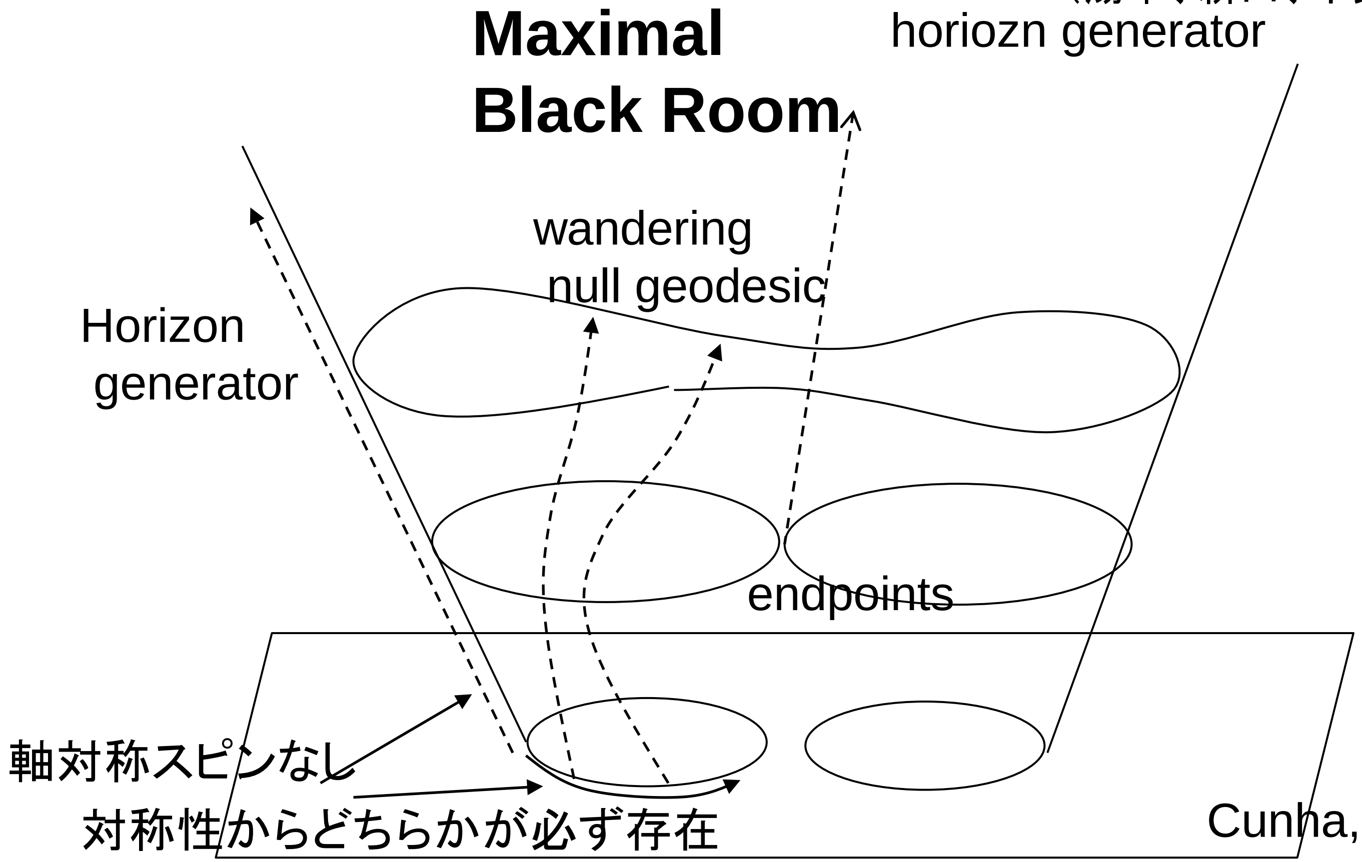
Bounded by null geodesic of:

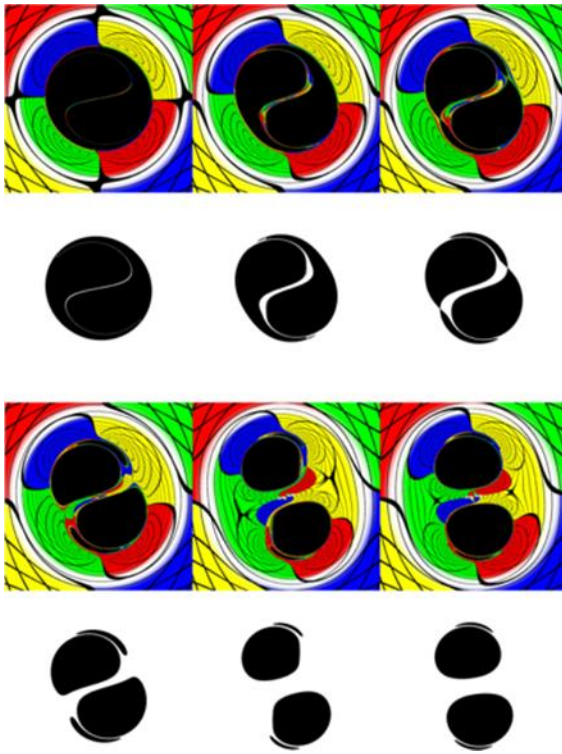
null generator(null sur.) → horizon generator

In time like surface → wandering null geodesic

endpoint→ only for horizon generator

KTBH （岡林、安積、中尾 2020），（湯本、新田、千葉、杉山 2012,）





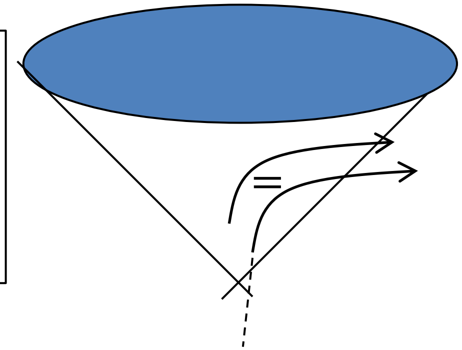
Cunha,Herdeiro,Rodriguez 2018

MBRの存在性について

論理的に言って.: maximal black room formation is corresponding to event horizon

ブラックホール⇒MBR exists（選択公理？）

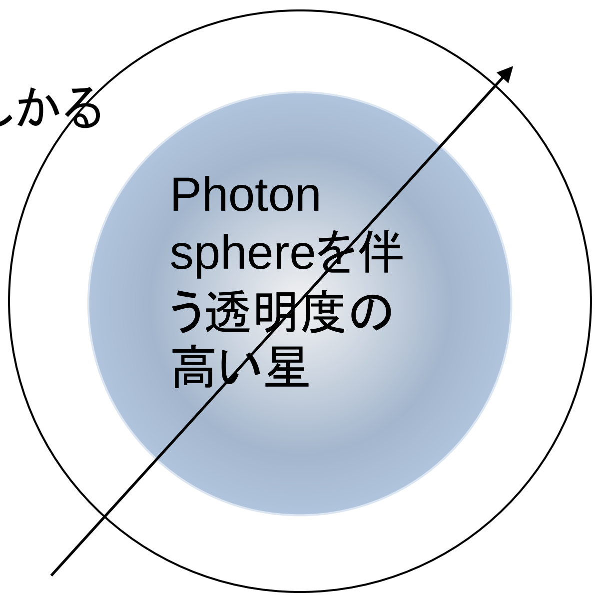
ブラックホールformation⇒ブラックホール≡MBR



MBRはエキゾチック星を区別できる？

エキゾチック星（asymptotically simple）

c.f. 球対称静的なら
photon surfaceはBH(naked sing.)かしかるべく重い物質
Claudel, Virbhadra,Ellis



Dynamicalに光子球面があったとしてもblack roomは無いと言ってよいのでは？

エキゾチック星
c.f. 球対称定常なら　photon surfaceはBHかしかるべく重い物質

光子球面があったとしてもblack roomは無いと言ってよいのでは？ー＞
証明割愛

expansion may reach the origin, and give trapped surface. If not, t
verse inside. For the spherical metric

$$ds^2 = -dt^2 + X^2(r,t)dr^2 + A^2(r,t)(d\theta^2 + sin^2\theta d\phi^2)$$
$$= -dt^2 + \frac{A'^2}{1-k(r)}dr^2 + A^2(r,t)(d\theta^2 + \sin^2\theta d\phi^2)$$

e have second line and The general accelaration equation is given b

$$\frac{2}{3}\frac{\ddot{A}}{A} + \frac{1}{3}\frac{\ddot{A}'}{A'} = -\frac{4\pi G}{3}(\rho + 3P)$$

225%　5ページ中5ページ

Aのr依存性次第である。
但しこの時はペンローズダイアグラムも変わってしまうので、
子宇宙があると言うべきだろう
MBH⇒BH（naked sing.） or 子宇宙（void?）（**Claudel, Virbhadra,Ellis**よりは強い主張？）

軸対称でもまたしかり

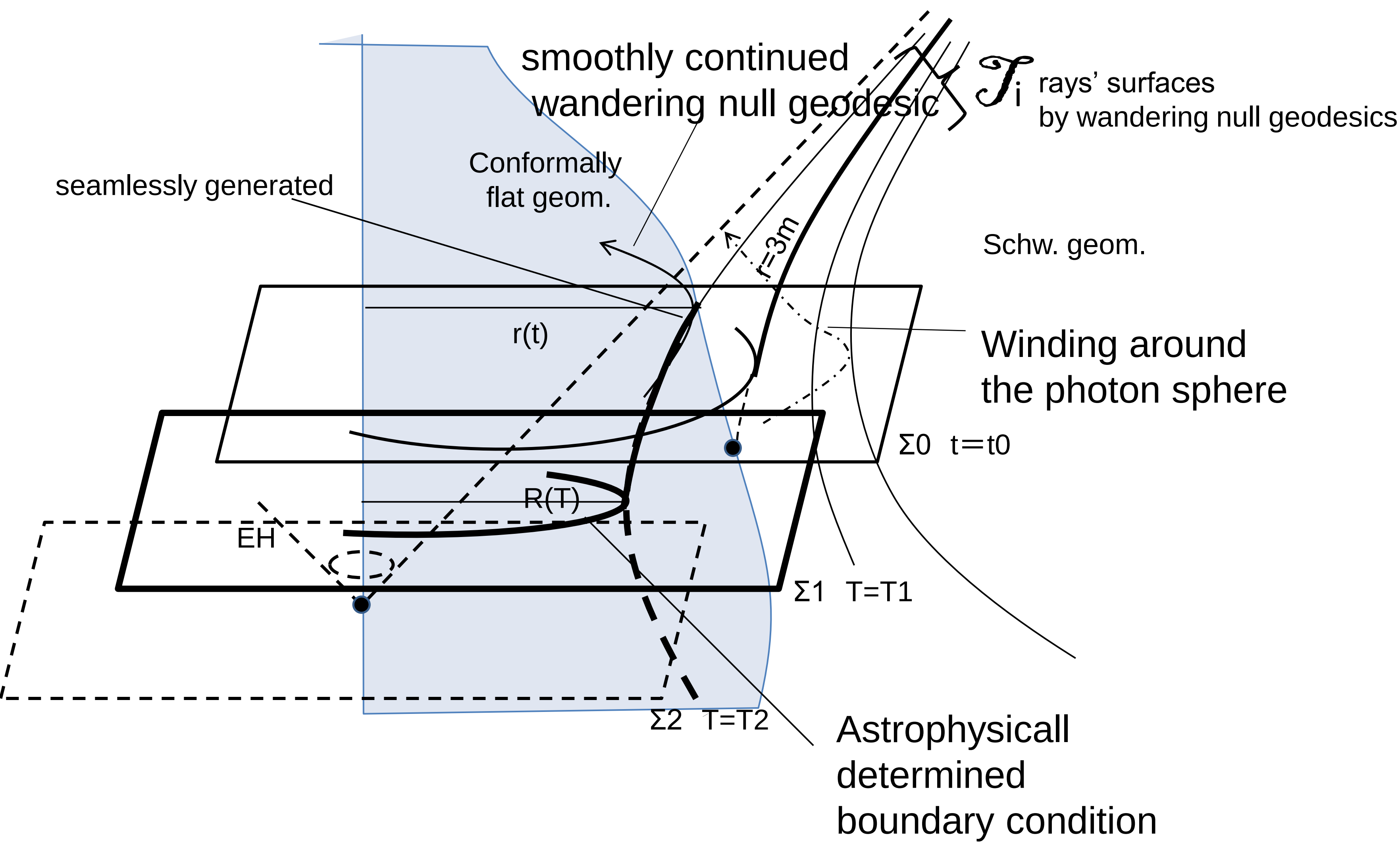
Maximal Black Room を探す

Initial t＝－∞なら不安定性より母線のtotally wandering null geodesicが
ほぼuniqueに決まる

Initial t=t0 noBH なら、MBR=BH
Initial t=t0 with BHなら、境界条件を与える必要がある。

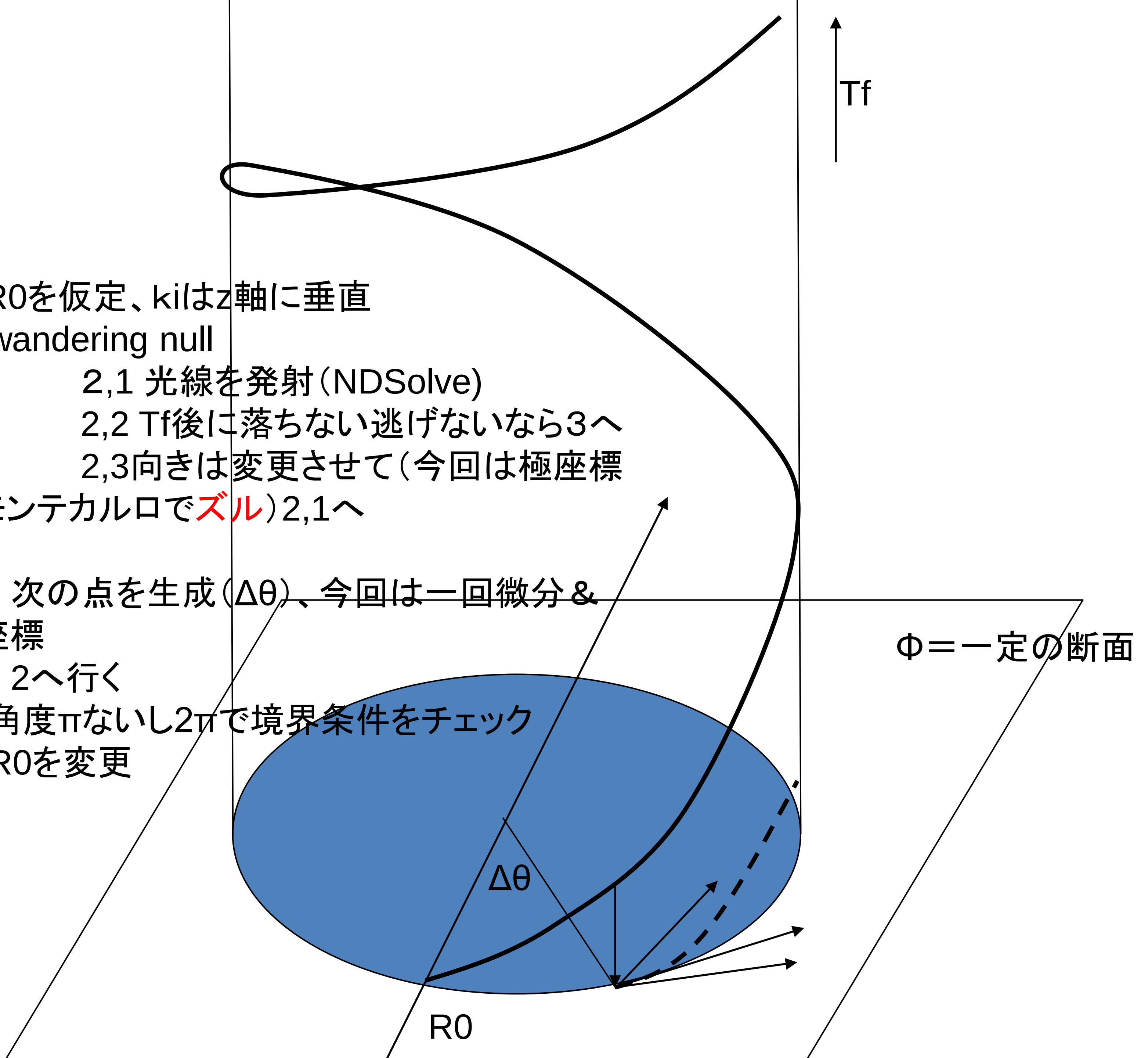
〇初期面は光り始めなどの物理で決める
〇∂MBR　は初期面に垂直（orbit concept）

→球対称はある程度解析的にわかる
→非球対称なら数値解析が必要。
何本あるかも分からないので難しい
が軸対称などは議論が可能


smoothly continued
wandering null geodesic
$\mathcal{T}_i$
rays' surfaces
by wandering null geodesics
Conformally
flat geom.
seamlessly generated
r=3m
Schw. geom.
r(t)
Winding around
the photon sphere
Σ0 t=t0
R(T)
EH
Σ1 T=T1
Σ2 T=T2
Astrophysicall
determined
boundary condition

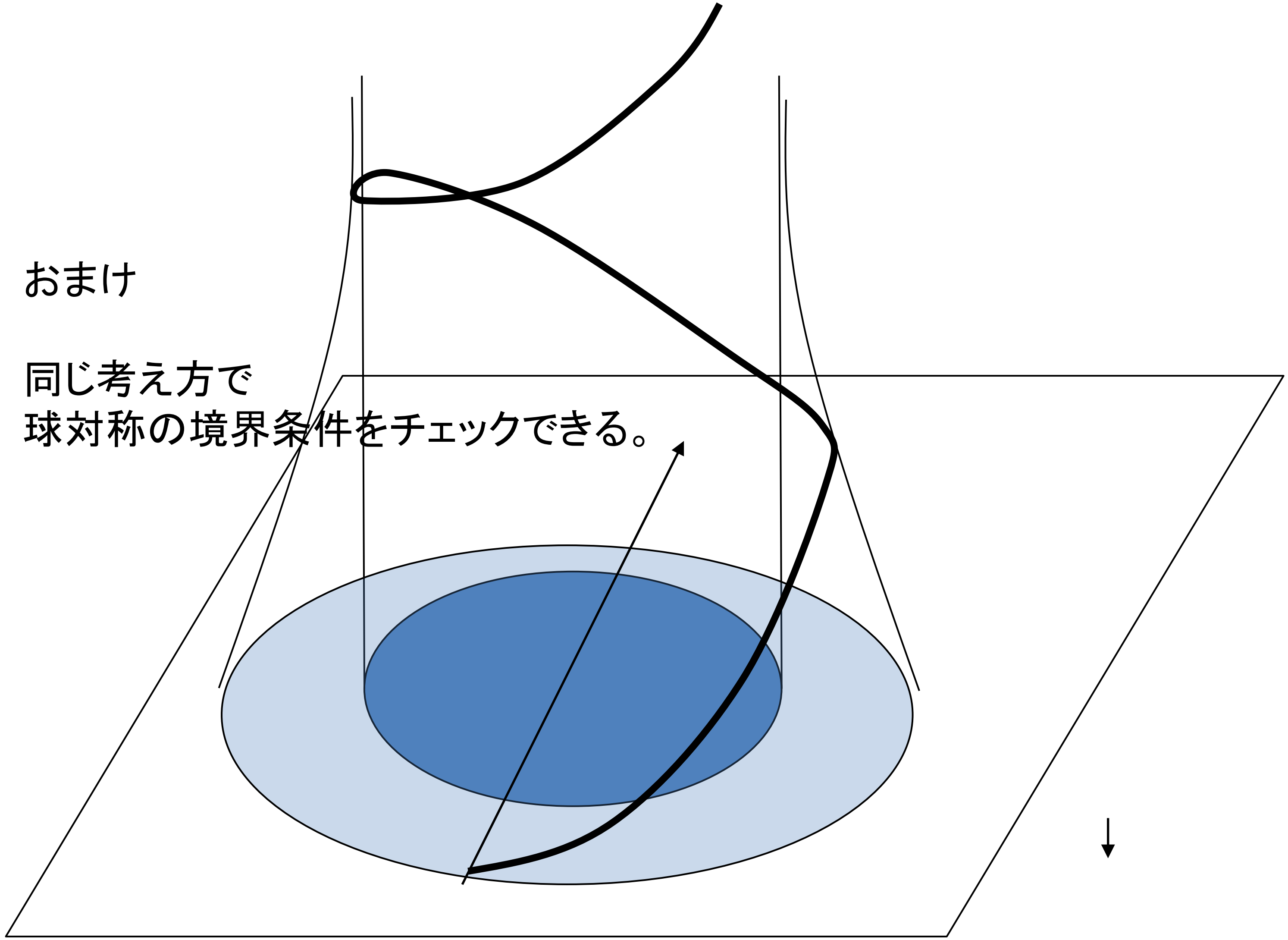
1、R0を仮定、kiはz軸に垂直
２、wandering null
　２,1 光線を発射（NDSolve)
　2,2 Tf後に落ちない逃げないなら3へ
　2,3向きは変更させて（今回は極座標&モンテカルロでズル）2,1へ

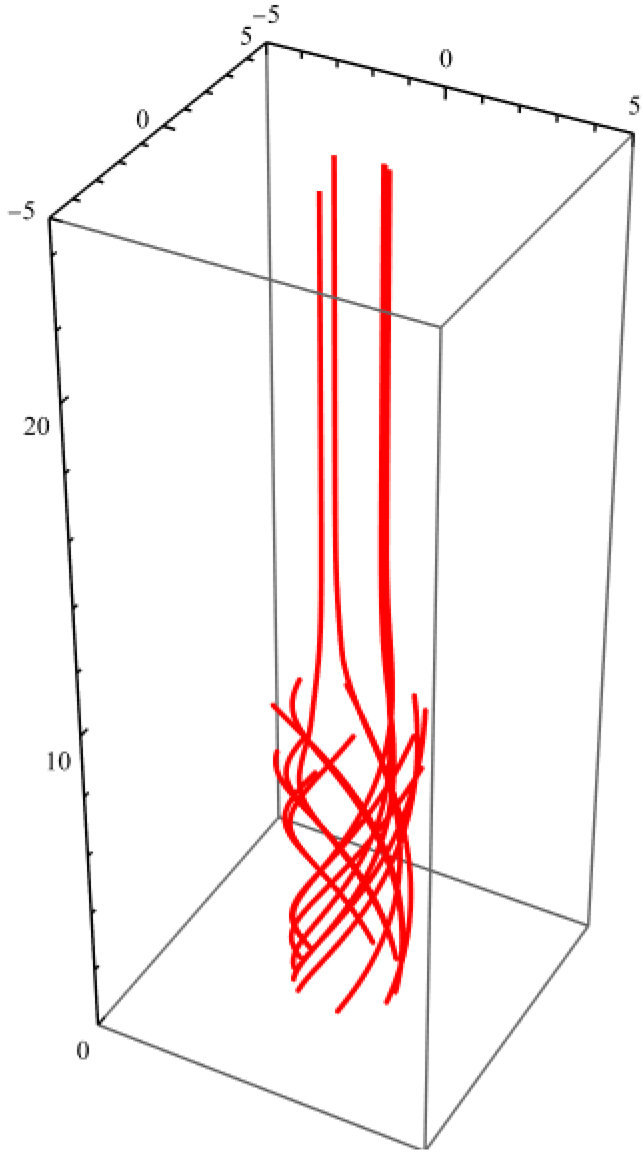
３、 次の点を生成（Δθ）、今回は一回微分&極座標
４、 2へ行く
５、角度πないし2πで境界条件をチェック
６、R0を変更



おまけ

同じ考え方で
球対称の境界条件をチェックできる。





今後は

○中心をずらす

○座標変換で擬似的に
球対称でなくす

10回程度の試行錯誤によるwandering

まとめ

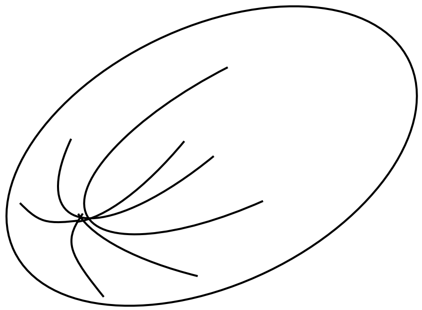
まとめ

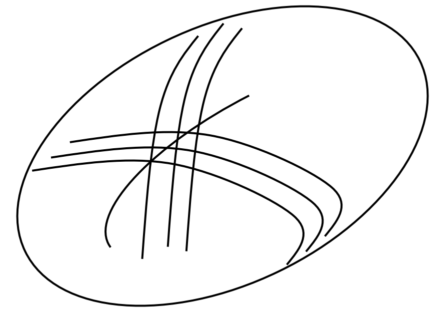
球対称ならかなり簡単に決定

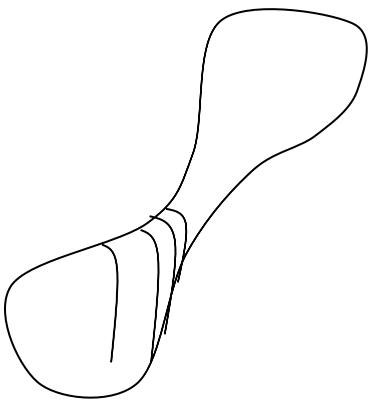
軸対称性が利用できる場合もあり
その数値解析プログラムを開発しつつある

question

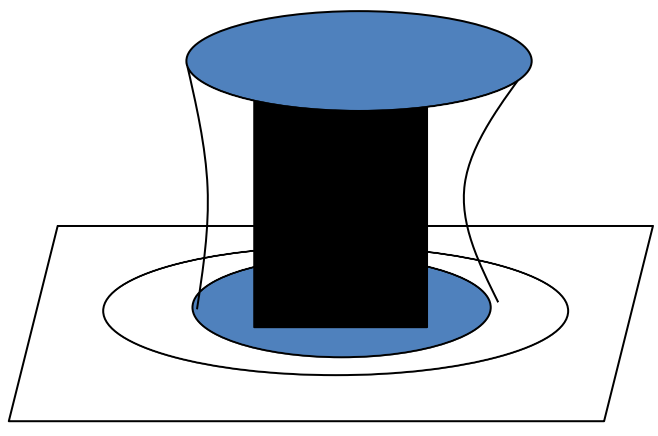
非等方なMBRの様相







アパレントな概念　　EHにとってのAH



例えば（吉野、泉、白水、富川＋天羽）の面

まとめ

- black hole shadow is gegnerally explained by wandering null geodesics

展望

一般的構造（面？、安定性？など）

by catastrophe (=gravitational lens)

- Blackhole identity とMaximal black room

Cheeger理論と測度距離空間

〇Ricci flowによる多様体の崩壊→測度距離空間、距離空間
〇量子論的な時空の病像→離散的？非可換空間？、string？,,Loop表現？
→量子多様体？？？
〇重力は距離さえあれば定義できる。
〇metricityの無い接続も流行っている。

測度距離空間
〇リフシッツ的Cheeger微分による。。。
測度論的可微分構造でもってtangentbundleを
定義。→上限、下限という意味で曲率が考察されている。
→unmetricity???

〇運動方程式～gradient flowの測度距離空間への十分条件で
拡張（upper gradient)
$u'(t)=-\nabla f(u(t)) \quad \Leftrightarrow \quad (f\circ u)'(t)\le -1/2||u'(t)||^2-1/2||\nabla f(u(t))||^2$
$||\nabla f(x)||\le g(x)$で抑える

Newton理論　ポテンシャルは～Ｉ／d（x、y）

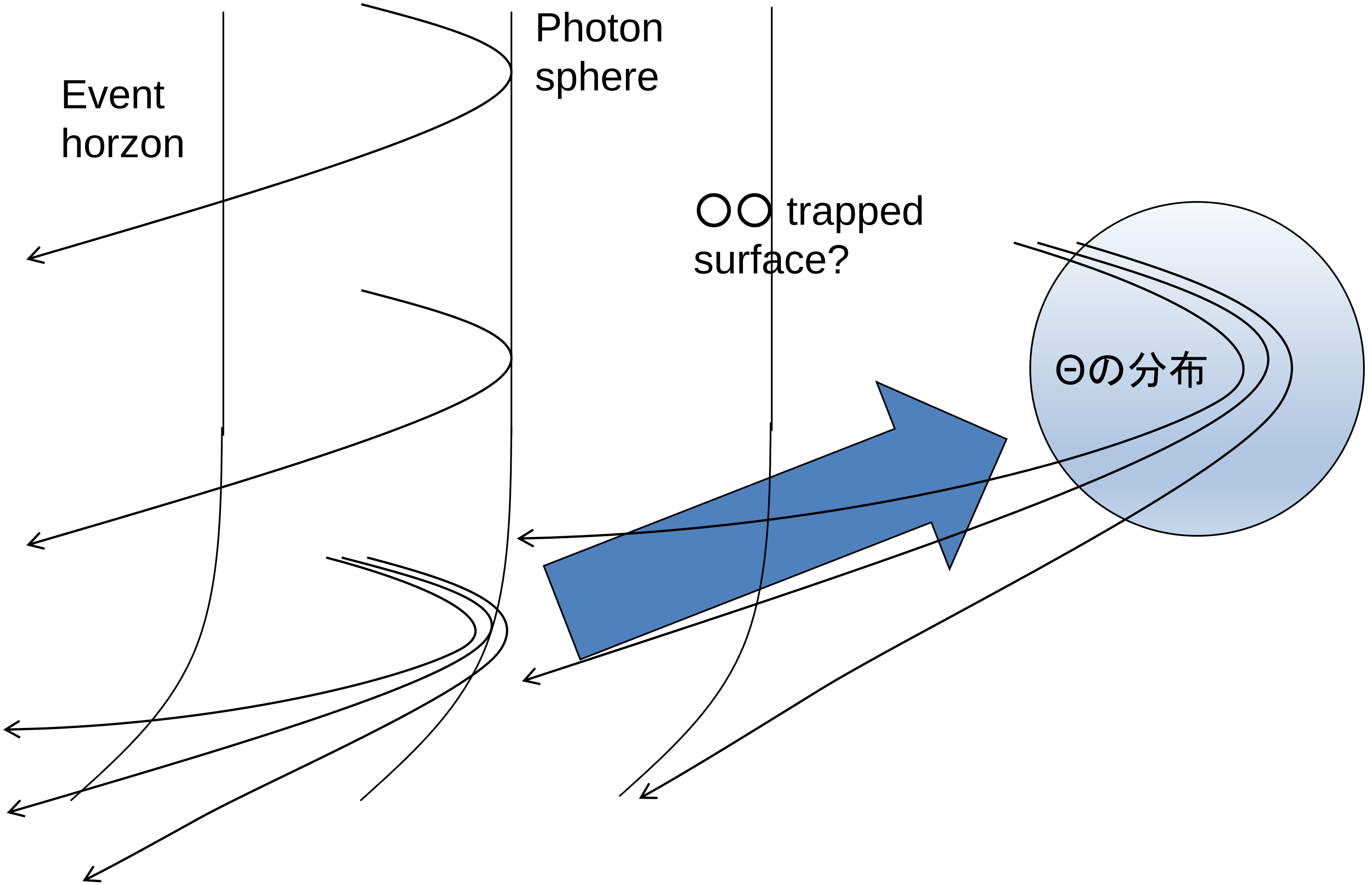
前提

Black hole image
→一般化photon sphere
→{完備無限個
conjugate pointsヌル測地線}
→ぐるぐる集合（wandering set）
arXive:

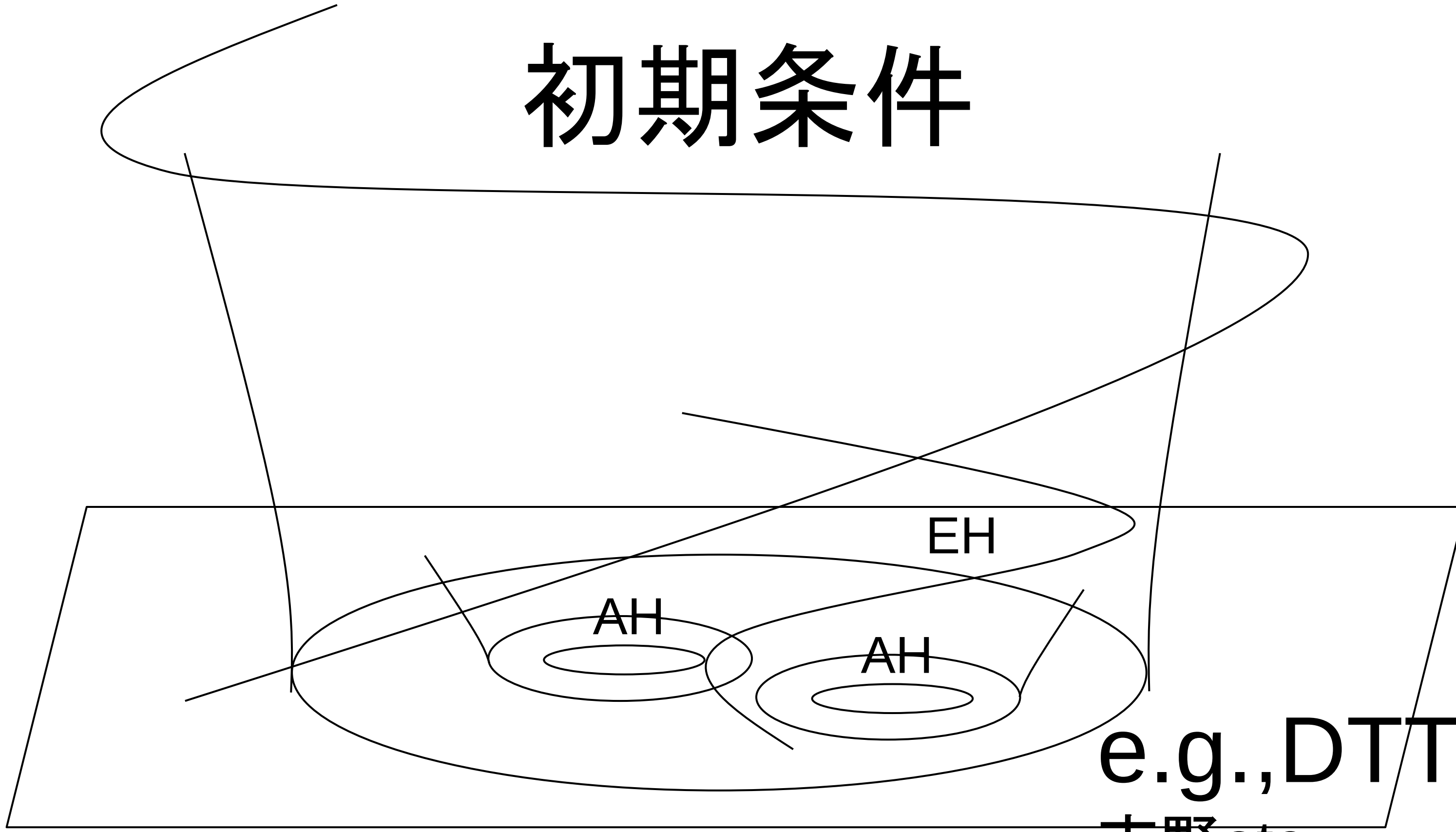
未来向き（過去向き）ぐるぐる集合

摘まんだぐるぐる集合

Wandering null geodesic　を探す



初期条件



e.g.,DTTS
吉野etc,

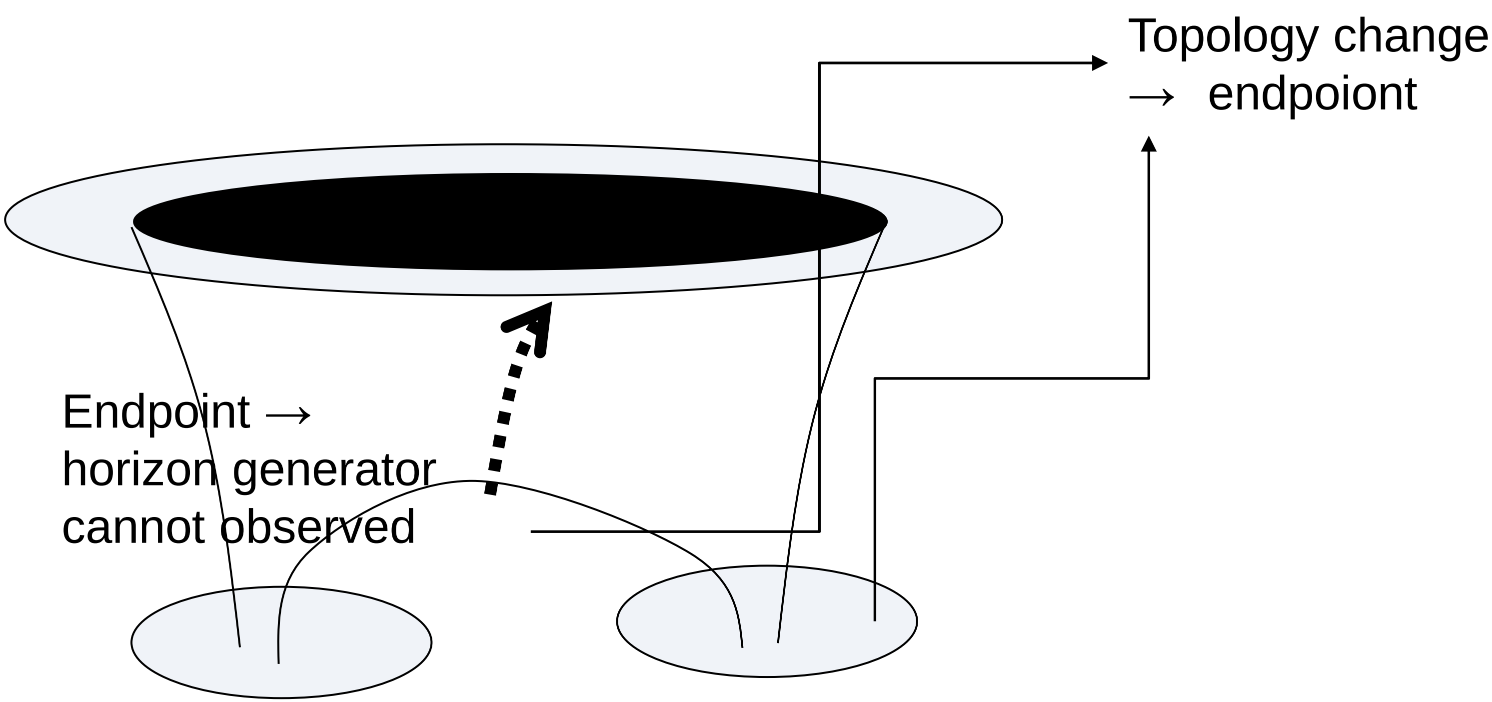
○汎存性定理が多分正しいので、寧ろ
排除率が在れば役に立つ

○AHやEHには排除率があるので、外には無いと言える面があると良い。

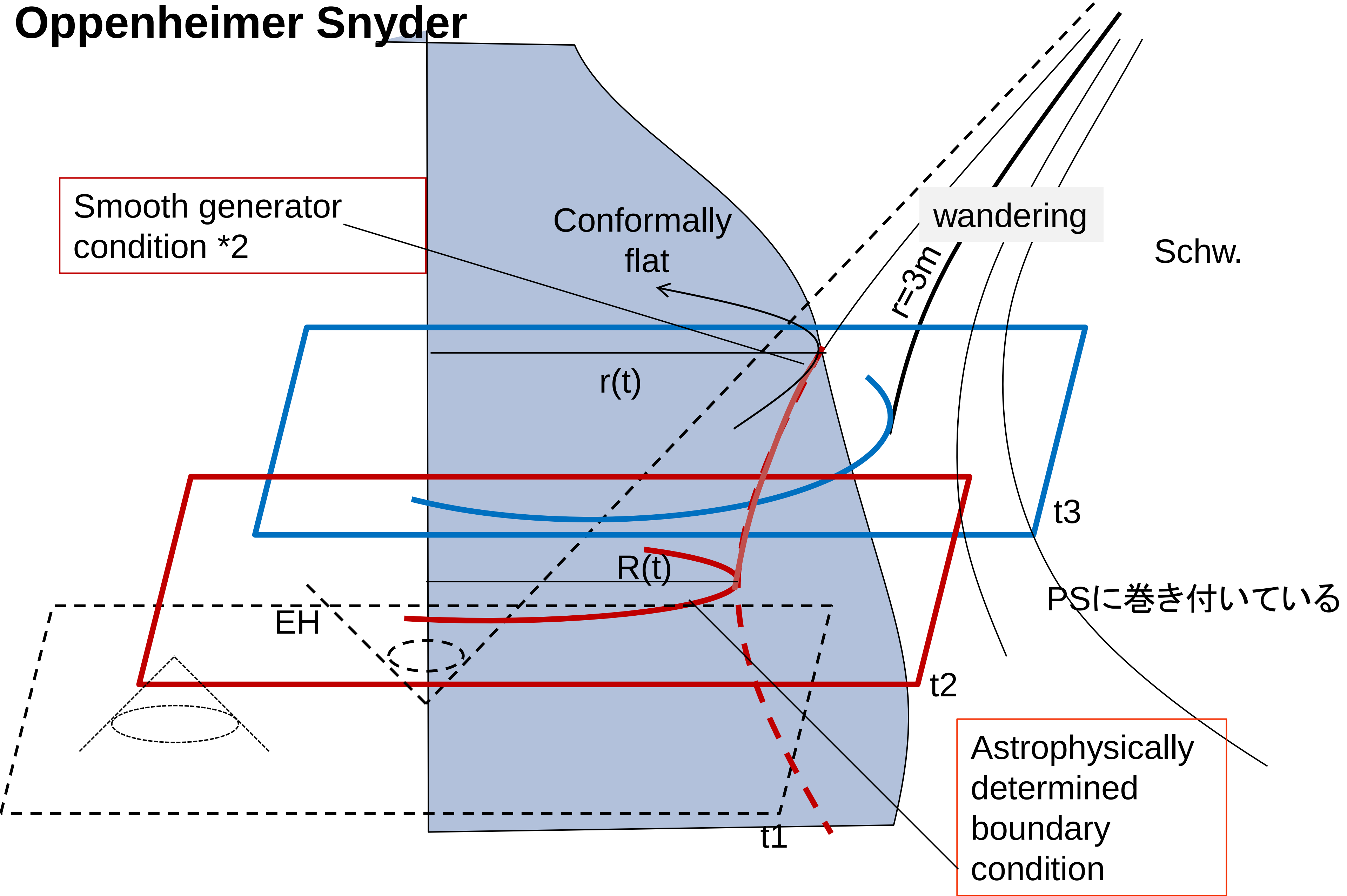
Black room's topology change

○by def.: maximal black room formation is corresponding to event horizon
○endpoint for saddle point is of horizon generator

Ex. Kastor Trascen BH never merge　Okabayashi, Azumi, Nakao（2020）



初期条件をあたえる時刻によるMBR時間遷移の例
Oppenheimer Snyder

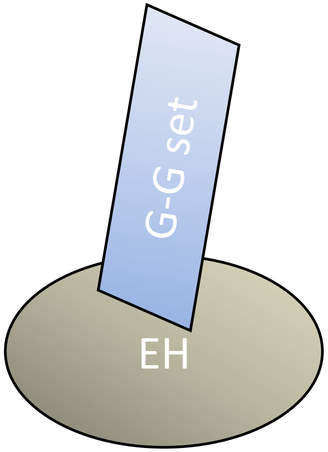


Summary and problem

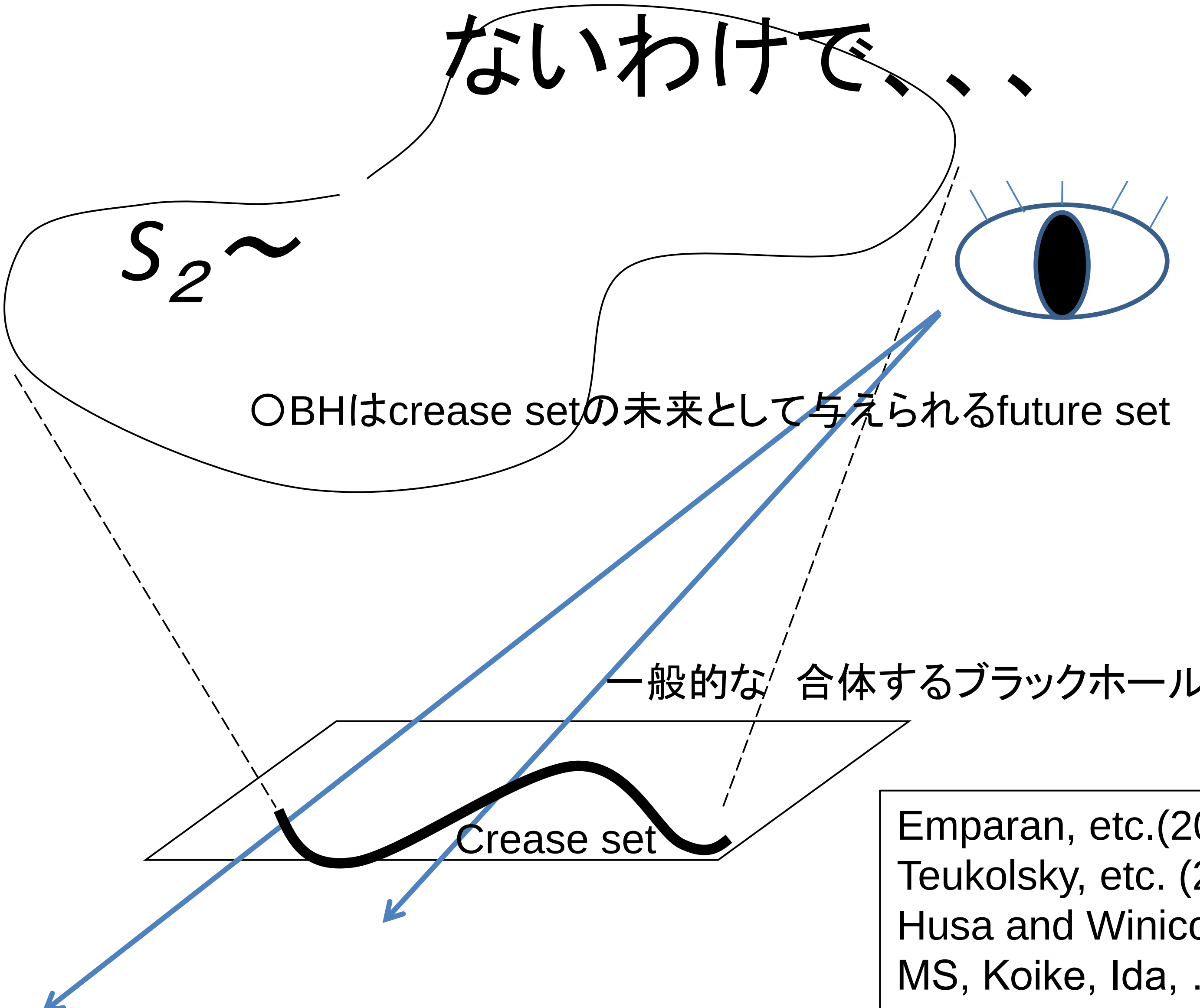
- Interesting but not clear for observation
- Name for Guru-Guru set
  - ○○○ photon surface?
  - Imprison , bounded null, surrounding null,
  - ～horizon,～surface

- Existence theorem is not necessary condition
- Rather integration of the collapsed congruence
- The boundary of Guru-Guru set?
- topology

- Stability for more general situation



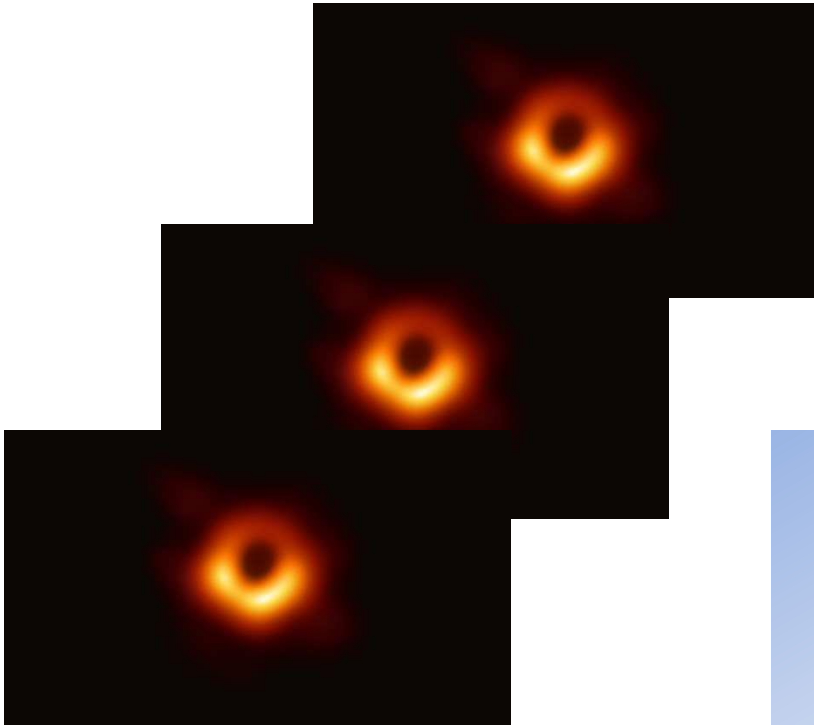
そもそも原理的にはブラックホールは見えないわけで、、、

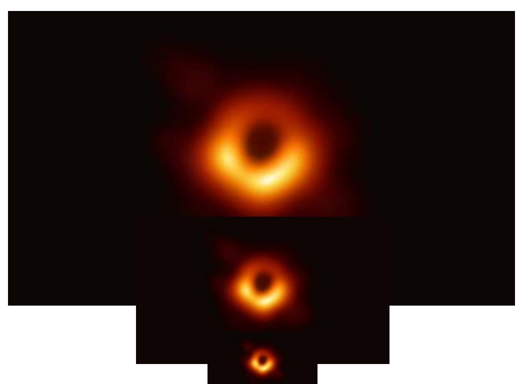


Emparan, etc.(2018)
Teukolsky, etc. (2016)
Husa and Winicour (1999)
MS, Koike, Ida, …(from 1998)

ブラックホールのアイデンティティとは？

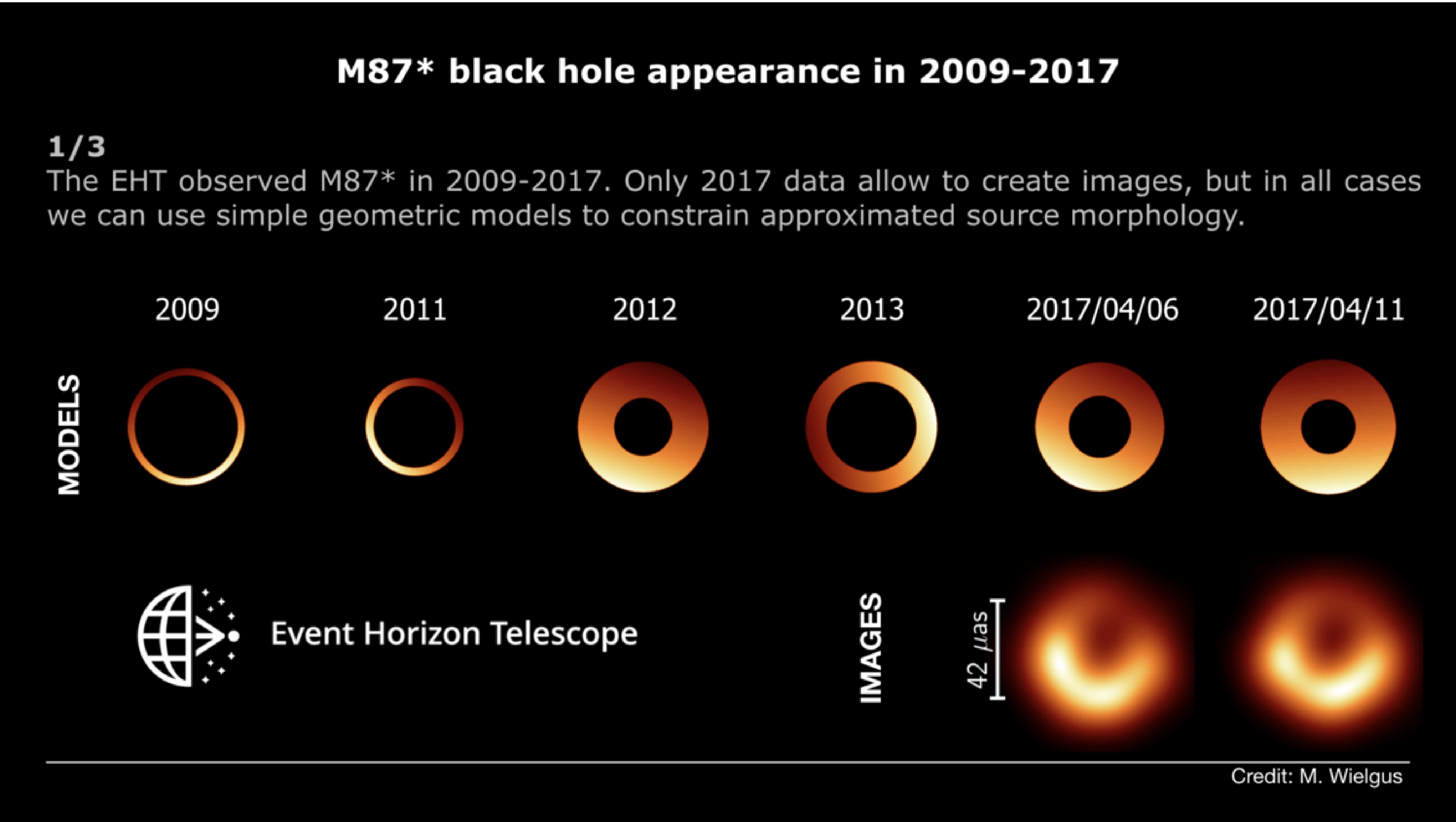
動いているブラックホールはどう見える？





M. Wielgus et al.: “Monitoring the Morphology of M87* in 2009-2017 with the Event Horizon Telescope”, in Astrophysical Journal (2020, September 23)

DOI: 10.3847/1538-4357/abac0d



Totally wandering null geodesics ー＞一般光子球面

- Totally (future past) wandering null geodesic

→generalized photon sphere

- Truncated wandering null geodesic

→formation of photon sphere

Questions:

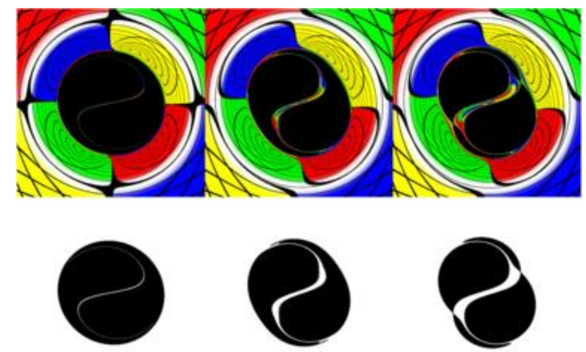
○What is this quasi photon surface

$K_{ab} k^a k^b \geqq 0$ equal at least one k^a

Causal concepts

○Can explain

final state of merger?

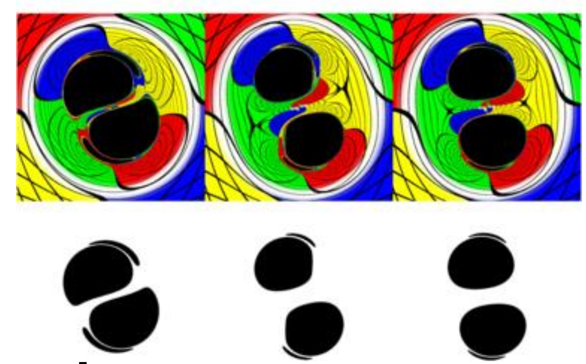


Other problem:

○Catastrophic nature

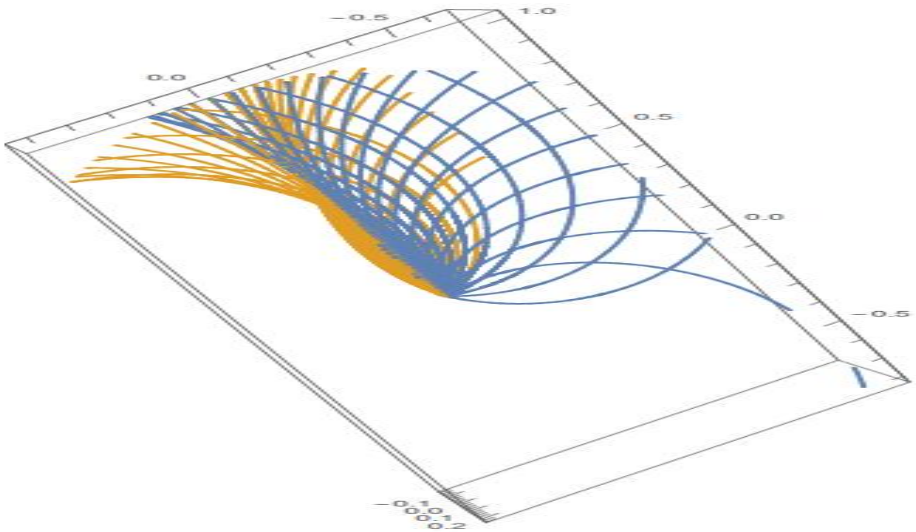
○axial symmetry

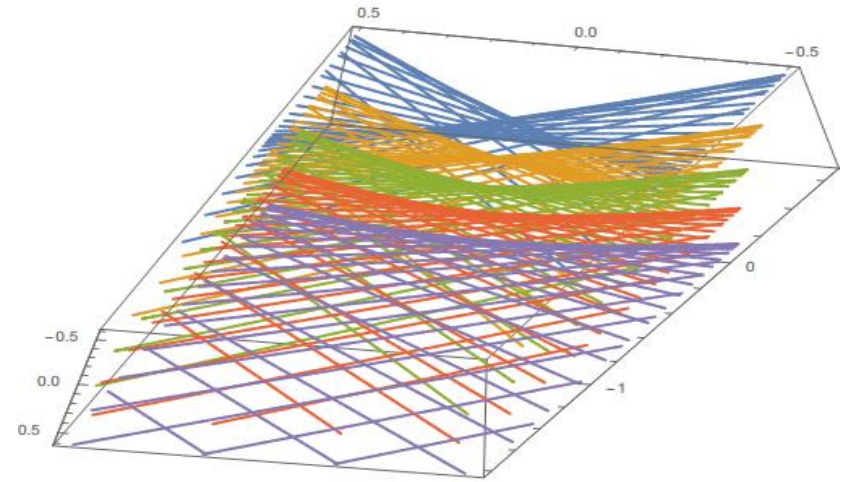
○black room and quantum physics



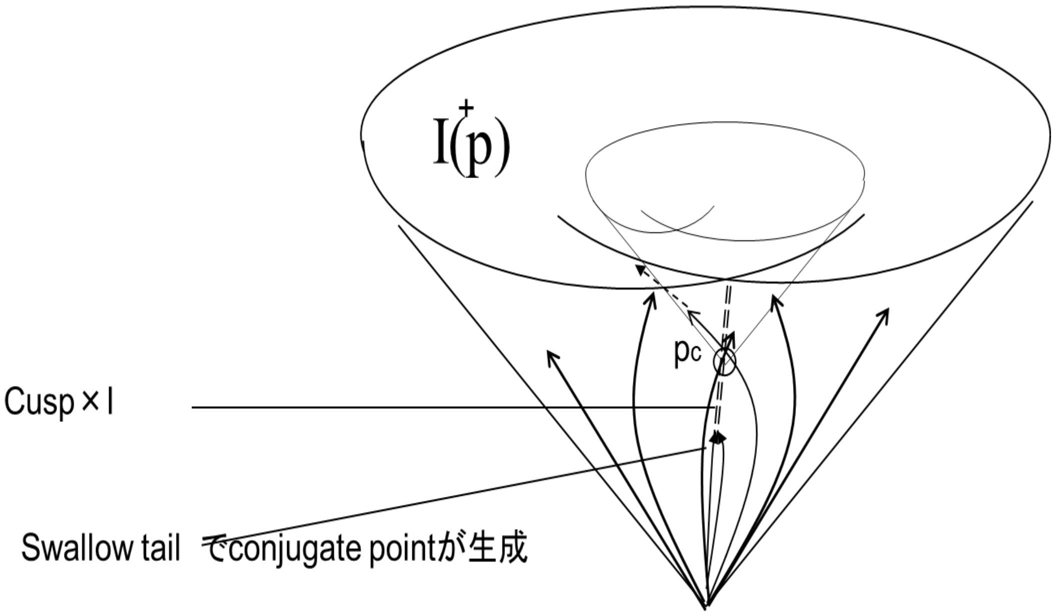
Schwartzschilld	球対称　×折り目（動径座標）
球対称、安定→不安定	くさび（2次元角度座標、時間）×くさび（動径座標）
非球対称、永続的	くさび（2次元角度座標）×折り目（動径座標）
非球対称、形成	燕の尾（2次元角度座標、時間）×折り目（動径座標）
非球対称、安定→不安定	くさび（2次元角度座標、時間）×くさび（動径座標）
その他　局所的に上の性質を実現	直積でないもの　は　蝶　双曲へそ　楕円へそ　放物へそ

3.3 swallow tail: $\tau = x^5 + ux^3 + vx^2 + wx$

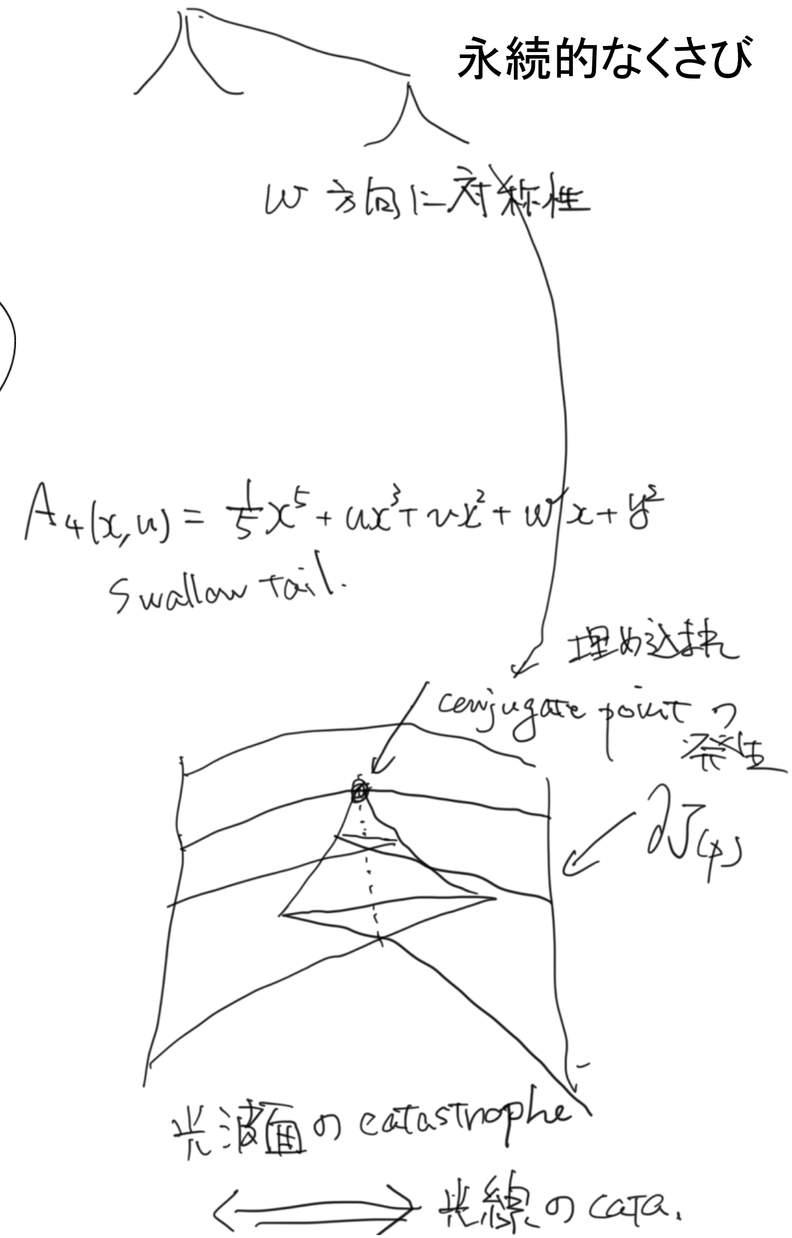




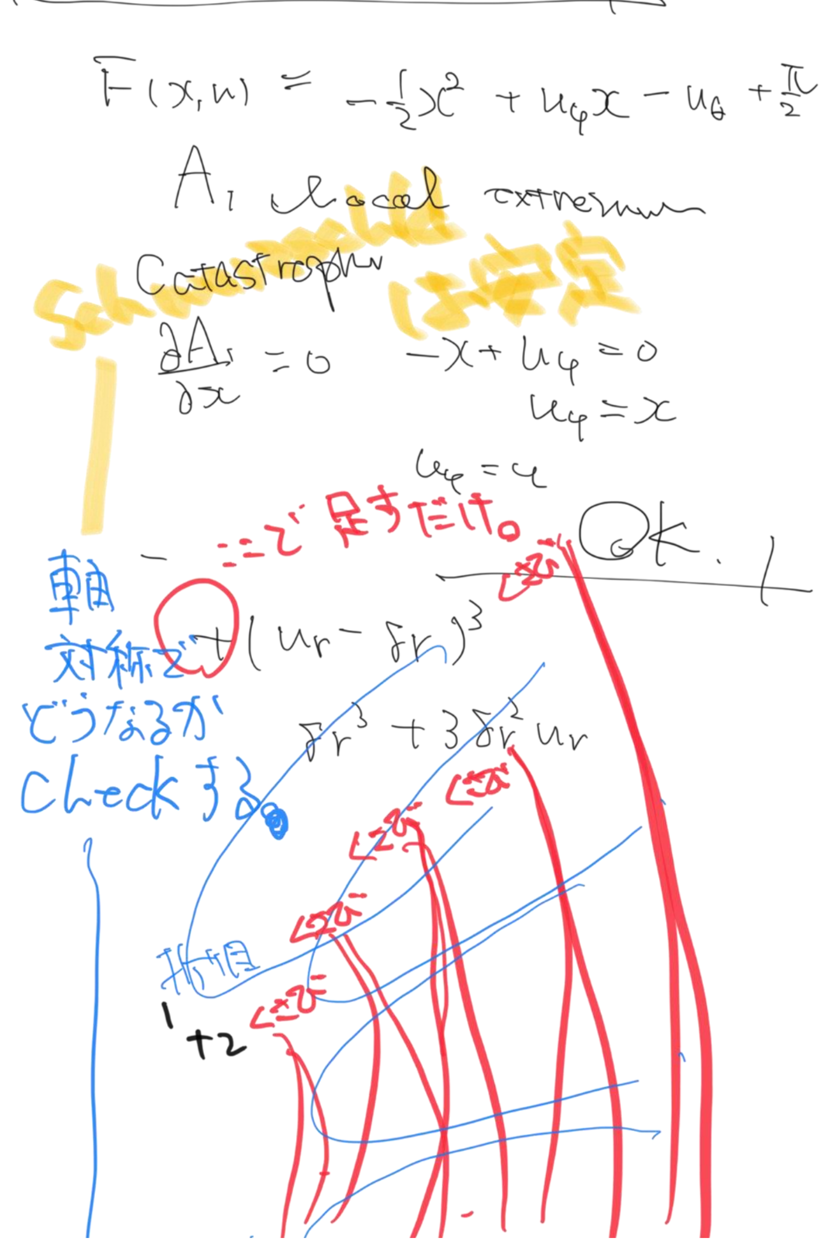
燕の尾によるくさびの生成



永続的なくさび



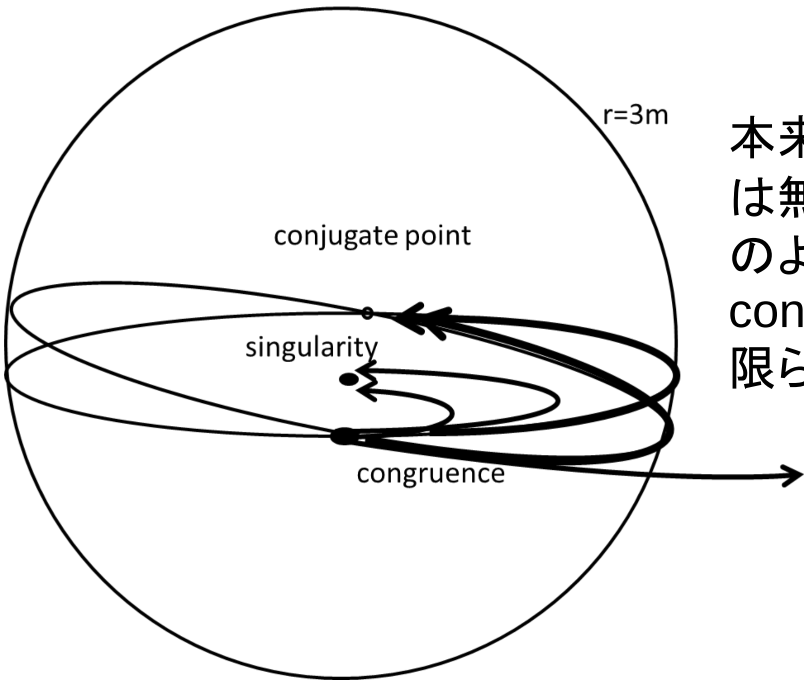
くさび×折り目



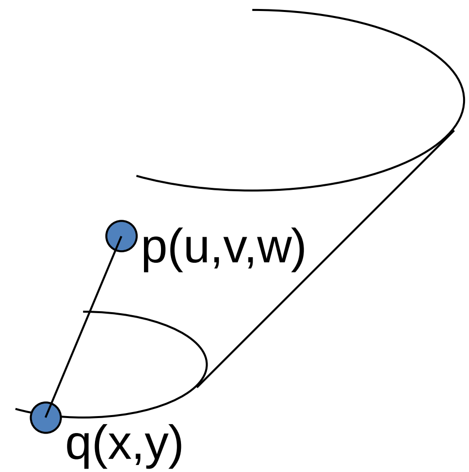
r＝3m

r＞3m

Catastrophyとconjugate point



本来congruenceは無限小なのでこのような有限なcongruenceとは限らない



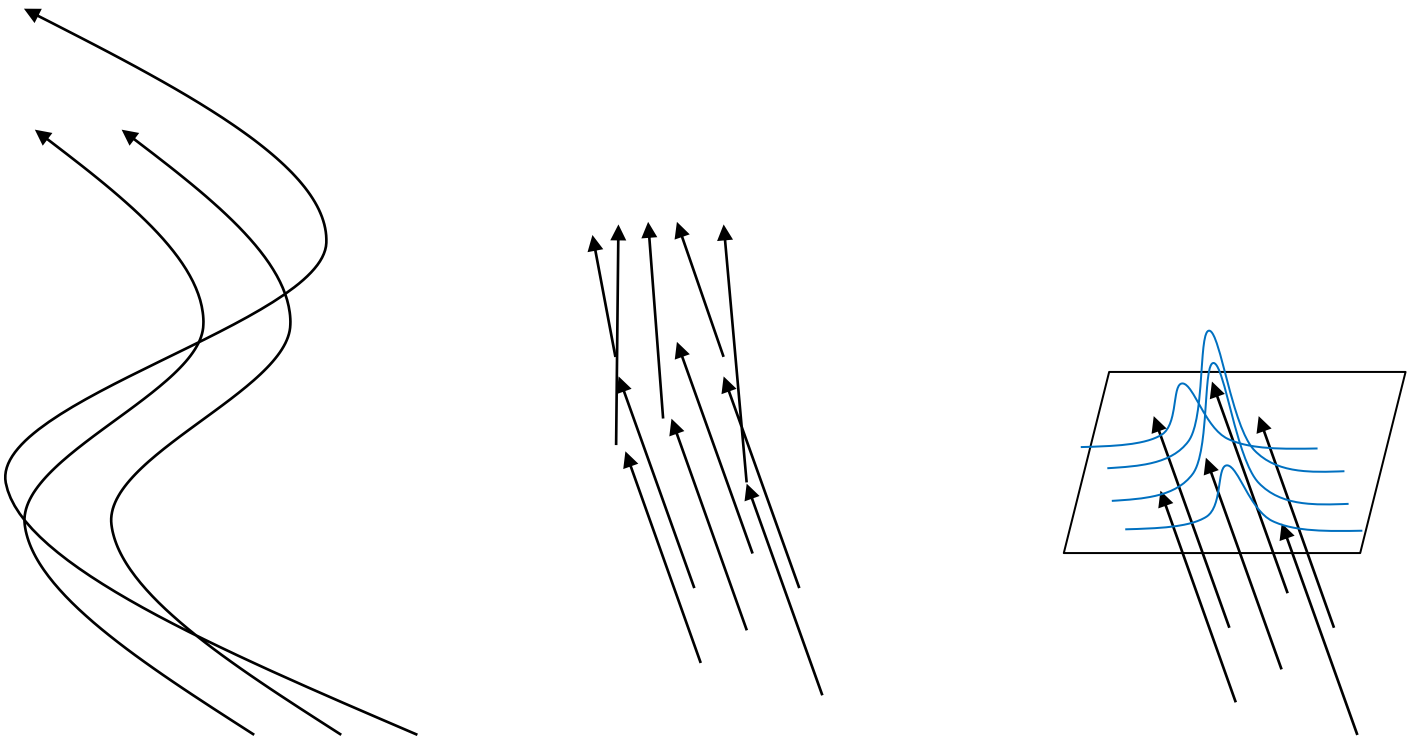
距離の
一階微分が零　測地線　　カタストロフィー集合
二階微分が零　　　　　　分岐集合↓
三階微分が零　共役点
高階微分も零　面になる

構造安定なconjugate pointを調べる

距離L=L(x,y;u,v,w)の変分原理

→Thomの定理より候補は二つ

algorism



Try and error

photon surface

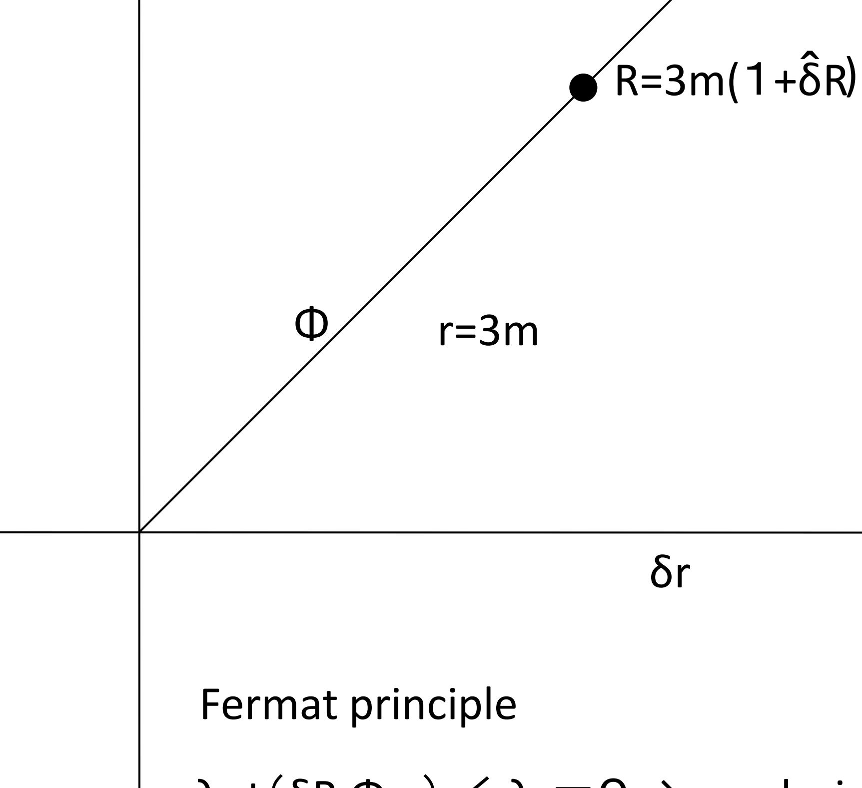
peak of potential

closed orbit

region

Not suitable for dynamical situation

Stability for such,,,



Time function

u=1/r

$$t=\int_0^{\Phi} d\phi \ \ \alpha / [l\, u(\phi)^2(1-2mu(\phi))]$$

Geodesics(elliptic fn.) around r=3m

$\delta(u) \sim C \cosh m(\phi+\phi_0)$

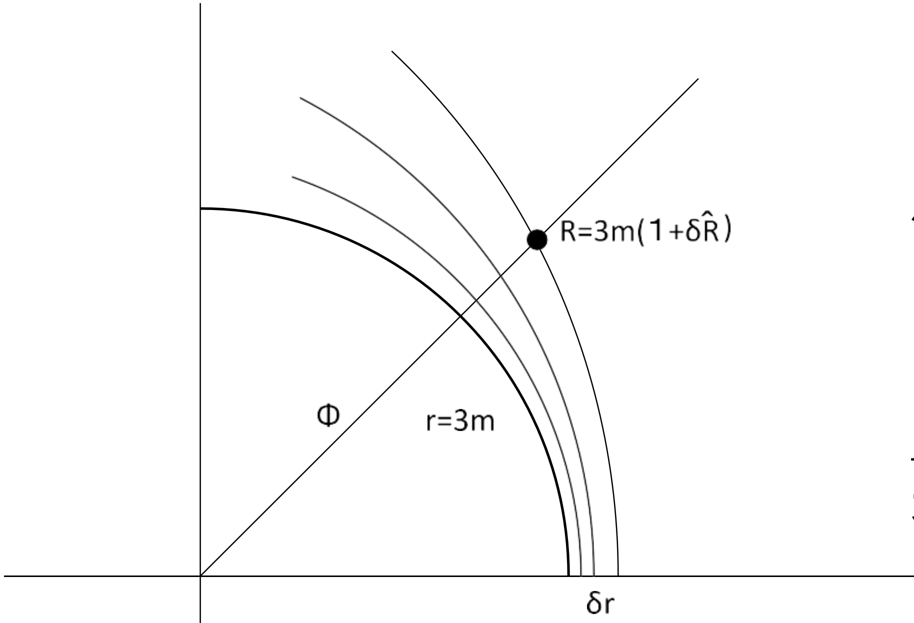
Fermat principle

$\delta r = m\, x$

$\partial\ t(\delta R,\Phi;x) / \partial x = 0 \rightarrow$ geodesic

$$t=\left(\frac{3(6m-\delta R)\Phi}{2l}+O[\Phi]^2\right)+\left(-\frac{3(m-2\delta R)\Phi}{2l}+O[\Phi]^2\right)x+\left(-\frac{5\delta R\Phi}{2l}+O[\Phi]^2\right)x^2$$
$$+\left(\frac{(15m+44\delta R)\Phi}{18l}+O[\Phi]^2\right)x^3+\left(-\frac{(22m+35\delta R)\Phi}{18l}+O[\Phi]^2\right)x^4+O[x]^5$$

Fold(折り目)catastrophe → structually stable



Fermat principle

シュバルツシルドのphoton sphere
は折り目のカタストロフィー
（構造安定）になっている

→
不安定円軌道と言っている。

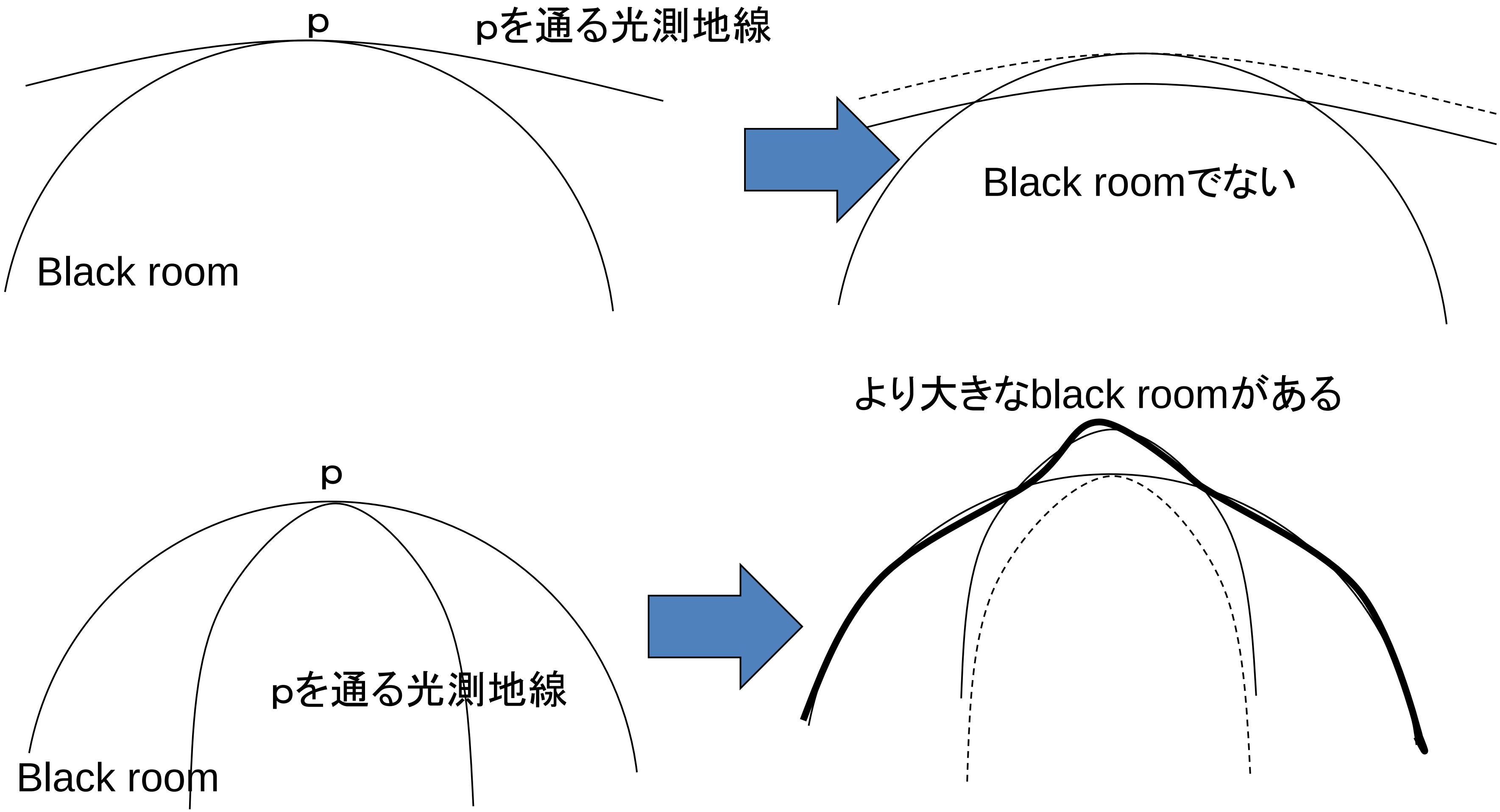
FIG 1. A figure[a] of the null congruence near the region $r = 3m$ on the equatorial plane on the Schwarzschild spacetime plotted in (r, ϕ) plane: the most inner null geodesic in this figure represents the unstable circular orbit on $r = 3m$. Our interest is where the most outer null geodesic starts.

[a] This figure is reproduced from Reference[37].

$R/(3m) - 1$ and $\delta\hat{r} = r(0)/(3m) - 1$, the time function of this light ray can be obtained [1] as

$$\frac{t[\delta\hat{R}, \Phi](\delta\hat{r})}{9\sqrt{3}m/\Phi} = \left(-1 + \frac{3}{\Phi^2}\delta\hat{R}\right)\delta\hat{r}^3 + \left(\frac{1}{2} + \delta\hat{R}\right)\delta\hat{r}^2 - \delta\hat{R}\delta\hat{r} + \frac{\Phi^2}{3} + \mathscr{O}(\delta\hat{R}^2, \Phi^1, \delta\hat{r}^4). \tag{1.1}$$

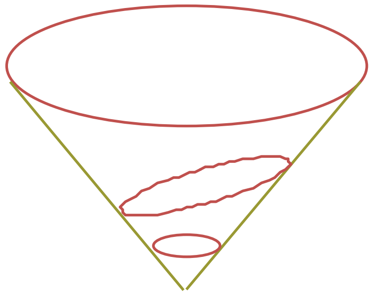
This function is reduced to $t = x^3 + ax + c + \mathscr{O}[x^4]$ by a diffeomorphism in variable space $(\delta\hat{R}, \Phi, \delta\hat{r})$ around the origin. In this case, the Fermat's principle says, considering a photon reaching a point $(r = R > 3m, \phi = \Phi)$, it should be start the point on


p
pを通る光測地線
Black room
Black roomでない
p
pを通る光測地線
Black room
より大きなblack roomがある

TOEH and the crease set Ref. PRD58104016

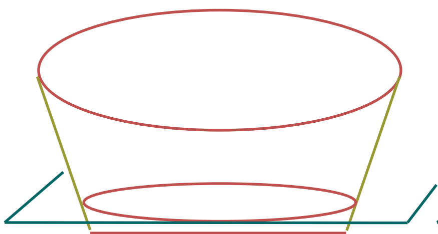
From Poincare=Hopf th. and the fact that the crease set is a connected acausal set, TOEH changes as the following figures by the rearrangement of a timeslicing.

0-d crease set

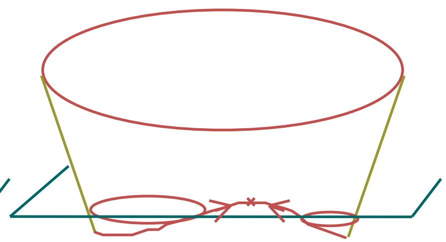


trivial

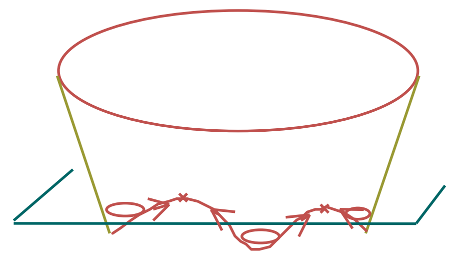
1-d crease set



trivial

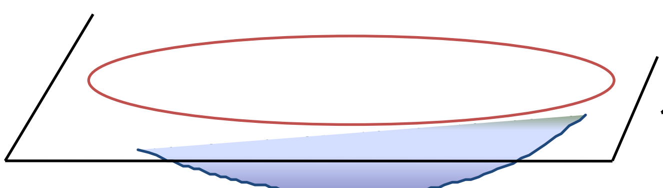


the coalescence of two EHs

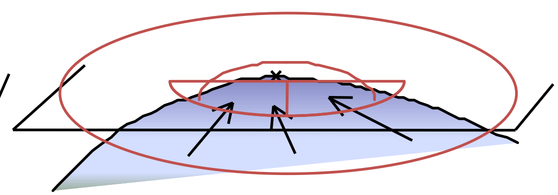


the coalescence of three EHs

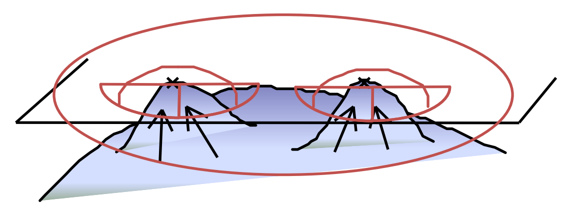
2-d crease set



trivial



torus EH



double torus EH