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概アーベルリー群を用いたアインシュタイン方程式の真空解の構成

小研究会「一般相対論と幾何」(GRGeo)

佐藤 雄一郎 (工学院大学 学習支援センター)

j.w. w/ 露木 孝尚 (工学院大学 学習支援センター)

Abstract

Today, we deal with spatially homogeneous spacetimes

$$(M, g_M) = \left(I \times G, -N(t)^2 dt^2 + g_G(t)\right),$$

where I, G denote an open interval and a Lie group, resp.
Given G, we find solutions $N(t)$, $g_G(t)$ which give Ricci-flat metrics.

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Contents

1. Preparation

(M, g_M) : an n-dim. Lorentzian mfd. ($\leftarrow$ we call it a spacetime)

The *Einstein equations* are

$$\mathrm{Ric} - \frac{\mathrm{Scal}}{2} g_M + \Lambda g_M = T.$$

In this talk, the *vacuum* Einstein equations are

$$\mathrm{Ric} - \frac{\mathrm{Scal}}{2} g_M = 0.$$

Namely, in case $T = 0,\ \Lambda = 0$.

If (M, g_M) satisfies the vacuum Einstein equations,
then we call it a *vacuum solution*.

(M, g_M) : vacuum solution $\Longleftrightarrow$ (M, g_M) : Ricci-flat ($\mathrm{Ric} = 0$).

(M, g_M) : *globally hyperbolic*
$:\Longleftrightarrow$ if $\exists$ $(n-1)$-dim. mfd. Σ & $\exists$ a function $N > 0$ on M s.t.
the spacetime (M, g_M) is isometric to

$$\left(\mathbb{R} \times \Sigma, -N^2 dt^2 + h_t\right),$$

where t : the coordinate of $\mathbb{R}$,
h_t : a Riemannian metric on Σ for $\forall t \in \mathbb{R}$.

Moreover, for each $t \in \mathbb{R}$,
$\Sigma_t := \{t\} \times \Sigma$: a *Cauchy hypersurface*

In other words,
a globally hyperbolic spacetime admits a trivial foliation structure

$$\mathcal{F} = \{\Sigma_t\}_{t \in \mathbb{R}}$$

whose leaves consist of Cauchy hypersurfaces.

$I \subset \mathbb{R}$: an open interval.
A globally hyperbolic spacetime

$$(M, g_M) = \left(I \times \Sigma, -N^2 dt^2 + h_t\right), \quad \mathcal{F} = \{\Sigma_t\}_{t \in I}$$

is *spatially homogeneous* $:\Longleftrightarrow$

$\forall t \in I$ & $\forall x, y \in \Sigma_t$, $\exists f \in \mathrm{Isom}(M, g_M) \cap \mathrm{Aut}(\mathcal{F})$ s.t. $f(x) = y$.

Also, it is *spatially isotropic* $:\Longleftrightarrow$

$\forall (t, x) \in I \times \Sigma$ & $\forall v, w \in T_x\Sigma_t$ w/ $g_M(v, v) = g_M(w, w) > 0$,
$\exists f \in \mathrm{Isom}(M, g_M) \cap \mathrm{Aut}(\mathcal{F})$ s.t. $f_*(v) = w$.

By definition,
spatially homogeneous or spatially isotropic,
$\rightsquigarrow N > 0$ is a function on I,
and we call it the *lapse function*.

$I \subset \mathbb{R}$: an open interval,
G : a 1-connected Lie group.
From now on, we consider the case $\Sigma = G$, that is,

$$(M, g_M) = \left(I \times G, -N^2 dt^2 + g_G(t)\right),$$

where $N > 0$: a function on I,
$g_G(t)$: a left-invariant Riemannian metric on G for $\forall t \in I$.

By definition, the globally hyperbolic spacetime is spatially homogeneous, but not necessarily spatially isotropic.

Remark

When $\dim G = 3$,
the globally hyperbolic spacetime is called the *Bianchi spacetime*.

Type I, Type II, Type III, Type IV, Type V, Type VI$_h$, Type VII$_h$, Type VIII, Type IX.

2. Vacuum Einstein equations

G : a 1-connected n-dim. Lie group,
$\{X_1, \ldots, X_n\}$: left-invariant vector fields on G,
$\{\omega^1, \ldots, \omega^n\}$: left-invariant 1-forms on G,
$\{C_{ij}^k\}_{1 \leq i,j,k \leq n}$: the structure constants w.r.t. $\{X_1 \ldots, X_n\}$.

Then a left-invariant Riemannian metric g_G on G can be expressed as

$$g_G = \sum_{1 \leq i,j \leq n} s_{ij}\, \omega^i \otimes \omega^j .$$

In addition, we identify the metric g_G with a matrix as follows:

$$g_G \mapsto [s_{ij}] \in \mathrm{Sym}_n^+ \mathbb{R},$$

where $\mathrm{Sym}_n^+ \mathbb{R}$: the set of $n \times n$ positive definite symmetric matrices.

A spatially homogeneous spacetime can be expressed as

$$(M, g_M) = \left(I \times G, -N(t)^2 dt^2 + \sum_{1 \le i,j \le n} s_{ij}(t)\, \omega^i \otimes \omega^j \right)$$

in general.

$N, a_1, \ldots, a_n$: positive functions on an open interval $I \subset \mathbb{R}$.

Consider a spacetime with the following diagonal metric

$$(M, g_M) = \left(I \times G, -N^2 dt^2 + a_1^2(\omega^1)^2 + \cdots + a_n^2(\omega^n)^2\right),$$

where $a_1, \ldots, a_n$ are called *scale factors* of the space.

By the way, **the universe is expanding.**

We consider higher-dimensional spacetimes, and would like to observe the behavior of the expansion and contraction of space by time evolution.

Proposition 1 [S.–Tsuyuki '23]

Let $1 \leq i, j \leq n$. Then

$$\mathrm{Ric}(X_0, X_0) = \sum_{k=1}^{n} \left(\frac{\dot{N}}{N} \frac{\dot{a}_k}{a_k} - \frac{\ddot{a}_k}{a_k} \right), \quad \mathrm{Ric}(X_0, X_i) = \frac{d}{dt} \log \left[\prod_{k=1}^{n} \left(\frac{a_k}{a_i} \right)^{C_{ik}^k} \right]$$

$$\mathrm{Ric}(X_i, X_j) = \frac{a_i^2}{N^2} \left[\left(\sum_{\substack{k=1 \\ k \neq i}}^{n} \frac{\dot{a}_k}{a_k} - \frac{\dot{N}}{N} \right) \frac{\dot{a}_i}{a_i} + \frac{\ddot{a}_i}{a_i} \right] \delta_{ij}$$
$$+ \frac{1}{4} \sum_{k,l=1}^{n} \left[2 C_{il}^k C_{kj}^l + 2 \frac{C_{lk}^k (C_{jl}^i a_i^2 + C_{il}^j a_j^2)}{a_l^2} \right.$$
$$\left. + \frac{(C_{kl}^i a_i^2 + C_{il}^k a_k^2)(C_{kl}^j a_j^2 + C_{jl}^k a_k^2)}{a_k^2 a_l^2} + \frac{C_{kl}^i C_{jk}^l a_i^2 + C_{kl}^j C_{ik}^l a_j^2 - 3 C_{ki}^l C_{kj}^l a_l^2}{a_k^2} \right],$$

where for a function F on I

$$X_0 := \frac{d}{dt}, \quad \dot{F} := \frac{dF}{dt}, \quad \ddot{F} := \frac{d^2F}{dt^2}.$$

G : a 1-connected n-dim. Lie group.

We define the *moduli space of left-invariant Riemannian metric* on G as

$$\mathfrak{PM} := \mathbb{R}_{>0}\mathrm{Aut}(G)\backslash \mathrm{GL}_n\mathbb{R}/\mathrm{O}(n) \cong \mathbb{R}_{>0}\mathrm{Aut}(G)\backslash \mathrm{Sym}_n^+\mathbb{R},$$

where $\mathrm{Aut}(G)$: the automorphism group of the Lie group G.

- $\mathfrak{PM}$ is connected and Hausdorff.
- The points corresponding to principal orbits form a smooth mfd.
- $\dim \mathfrak{PM} = 0 \iff \mathfrak{PM} = \{*\}$ (singleton)

Theorem 1 [Lauret '03, Kodama–Takahara–Tamaru '11]

$\dim \mathfrak{PM} = 0$ if and only if G is isomorphic to one of

$$\mathbb{R}^n \ (n \geq 1), \quad \mathbb{R}\mathrm{H}^n \ (n \geq 2), \quad H_3 \times \mathbb{R}^{n-3} \ (n \geq 3),$$

where $\mathbb{R}\mathrm{H}^n$: n-dim. real hyperbolic space w/ a Lie group structure,

H_3 : 3-dim. Heisenberg group.

G : a 1-connected n-dim. Lie group.

G : *almost abelian* $:\Longleftrightarrow \exists A \lhd G$: codim. 1 abelian normal subgroup.

By definition, we have
G : almost abelian Lie group/iso. $\stackrel{1:1}{\longleftrightarrow}$ A : real square matrix/equiv.
Note that A is an $(n-1) \times (n-1)$ real square matrix.

Here A is equivalent to B if A is similar to B up to scaling.
We call the equivalence class of A the *associated matrix* of G.

Remark

- G is isomorphic to $\mathbb{R}^n$ w/ a suitable Lie group structure.
- almost abelian $\Longrightarrow$ two-step solvable.
- $n = 2 \Longrightarrow$ almost abelian.
- $n = 3 \Longrightarrow$ almost abelian, or semisimple.
- $\dim \mathfrak{PM} = 0 \Longrightarrow$ almost abelian.

Corollary 1 [S.–Tsuyuki '23]

When G is almost abelian, the Ricci tensor is reduced as follows:

$$\mathrm{Ric}(X_0, X_0) = \sum_{k=1}^{n} \left(\frac{\dot{N}}{N} \frac{\dot{a}_k}{a_k} - \frac{\ddot{a}_k}{a_k} \right),$$

$$\mathrm{Ric}(X_0, X_n) = \frac{d}{dt} \log \left[\prod_{k=1}^{n-1} \left(\frac{a_k}{a_n} \right)^{A_{kk}} \right],$$

$$\mathrm{Ric}(X_n, X_n) = \frac{a_n^2}{N^2} \left[\left(\sum_{k=1}^{n-1} \frac{\dot{a}_k}{a_k} - \frac{\dot{N}}{N} \right) \frac{\dot{a}_n}{a_n} + \frac{\ddot{a}_n}{a_n} \right] - \frac{1}{2} \sum_{k,l=1}^{n-1} \left\{ A_{lk} A_{kl} + (A_{kl})^2 \frac{a_k^2}{a_l^2} \right\},$$

$$\mathrm{Ric}(X_i, X_j) = \frac{a_i^2}{N^2} \left[\left(\sum_{\substack{k=1 \\ k \neq i}}^{n} \frac{\dot{a}_k}{a_k} - \frac{\dot{N}}{N} \right) \frac{\dot{a}_i}{a_i} + \frac{\ddot{a}_i}{a_i} \right] \delta_{ij}$$

$$- \frac{\mathrm{tr} A}{2} \frac{A_{ij} a_i^2 + A_{ji} a_j^2}{a_n^2} + \frac{1}{2} \sum_{k=1}^{n-1} \frac{A_{ki} A_{kj} a_i^2 a_j^2 - A_{ik} A_{jk} a_k^4}{a_n^2 a_k^2},$$

where $A = [A_{ij}]_{1 \leq i,j \leq n-1}$: the associated matrix.
The remaining components identically vanish.

3. Spatially homogeneous solutions

We generalize the Taub solutions in the following sense:

$$G^3 = H_3 \quad \longleftrightarrow \quad A = J(0,2)$$
$$G^n = H_3 \times \mathbb{R}^{n-3} \quad \longleftrightarrow \quad A = J(0,2) \oplus O_{n-3}$$

Namely, we consider all nilpotent matrices of rank one.

$G^n =$	$\mathbb{R}^n$	$\mathbb{R}\mathrm{H}^n$	$H_3 \times \mathbb{R}^{n-3}$
$A =$	O_{n-1}	I_{n-1}	$J(0,2) \oplus O_{n-3}$
$n = 3$ Bianchi spacetime	Kasner, Type I (1921)	Joseph, Type V (1969)	Taub, Type II (1951)
$n \geq 4$	Chodos–Detweiler (1980)	Demaret–Hanquin (1985)	S.–Tsuyuki (2023)

Table1: Vacuum solutions of spatially homogeneous spacetimes

Theorem 2 [S.–Tsuyuki '23]

In case $G = H_3 \times \mathbb{R}^{n-3}$, the spatially homogeneous spacetime

$$(M, g_M) = \left(I \times G, -N^2 dt^2 + g_G(t)\right),$$

is Ricci-flat if and only if

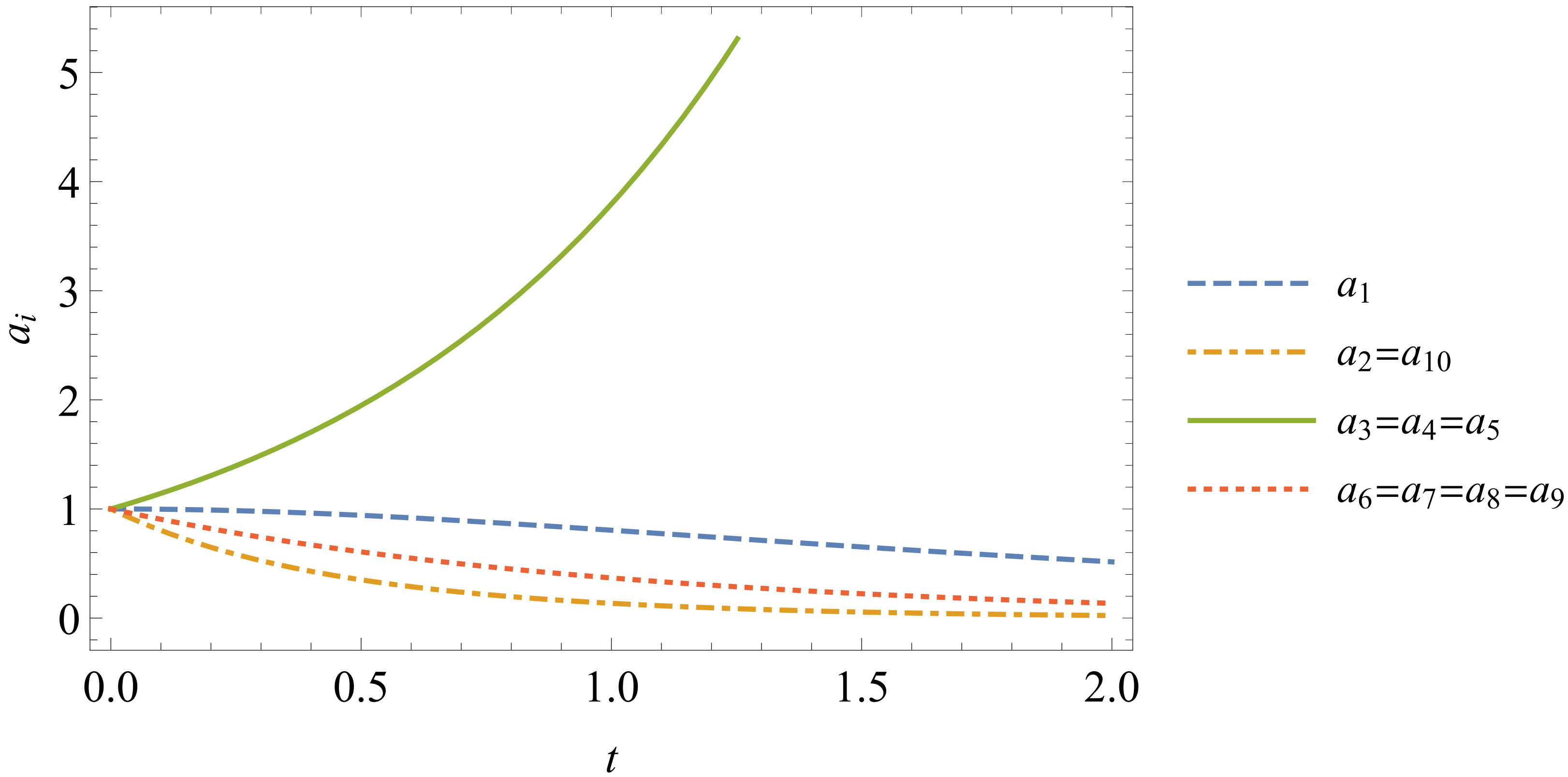
$$\begin{aligned} g_M = &\ \cosh(kt)(-c_0^2 e^{2(\mathrm{tr}\,h)t} dt^2 + c_2^2 e^{2h_2 t} dx_2^2 + c_n^2 e^{2h_n t} dx_n^2) \\ &+ \frac{c_1^2 e^{2h_1 t}}{\cosh(kt)} (dx_1 - x_n dx_2)^2 + \sum_{i=3}^{n-1} c_i^2 e^{2h_i t} dx_i^2, \\ k = &\frac{c_0 c_1}{c_2 c_n},\ 2h_1 = -\sum_{i=3}^{n-1} h_i,\ (\mathrm{tr}\,h)^2 - (\mathrm{tr}\,h^2) = \frac{k^2}{2}, \end{aligned}$$

where $h = \mathrm{diag}(h_1, h_2, \dots, h_n)$.

When $n = 3$, these solutions induce the Taub solutions.

Corollary 2 [S.–Tsuyuki '23]

The exact vacuum solutions for $G = H_3 \times \mathbb{R}^{n-3}$ are spatially homogeneous but not spatially isotropic.
Moreover, each of the spatial dimensions cannot expand or contract simultaneously in the late-time limit.



Proposition 2 [S.–Tsuyuki '23]

$A = J(0,2) \oplus I_{n-2}\,(n > 2)$.
G : the almost abelian Lie group corresponding to A.

Then there is no vacuum solution for $I \times G$.

Remark [Taketomi–Tamaru '18]

The above moduli space is one-dimensional, i.e. $\dim \mathfrak{PM} = 1$.
$\mathfrak{PM} \approx \{\mathrm{diag}(\lambda, 1, \ldots, 1) \mid \lambda > 0\}$.

Proposition 3 [Igawa '24]

A : an arbitrary nilpotent matrix.
G : the almost abelian Lie group corresponding to A.

Then there exist vacuum solutions for $I \times G$.
Igawa found particular solutions of the vacuum Einstein equations in his master's thesis.

4. Discussions

- Can one **construct** a Ricci-flat Lorentzian mfd. using other Lie group?

  $\rightsquigarrow$ For example, compact Lie groups.

  $\rightsquigarrow$ $\mathrm{SO}(n) \subset \mathrm{Aut}(G) \Longrightarrow \mathfrak{PM} \approx \{\mathrm{diag}(k_1, \ldots, k_n)\}$.

- Can one **consider** the case of homogeneous spaces in the same way?

  $\rightsquigarrow$ For example, symmetric spaces.

  $\rightsquigarrow$ Structure of the moduli space of invariant Riemannian metric.

- Can one **classify** Ricci-flat Lorentzian homogeneous spaces?

  $\rightsquigarrow$ Symmetric case is completed (Cahen–Wallach spaces).

  $\rightsquigarrow$ Almost abelian case is in preparation by S.–Tsuyuki.