小玉ベクトルと自由粒子のエネルギー

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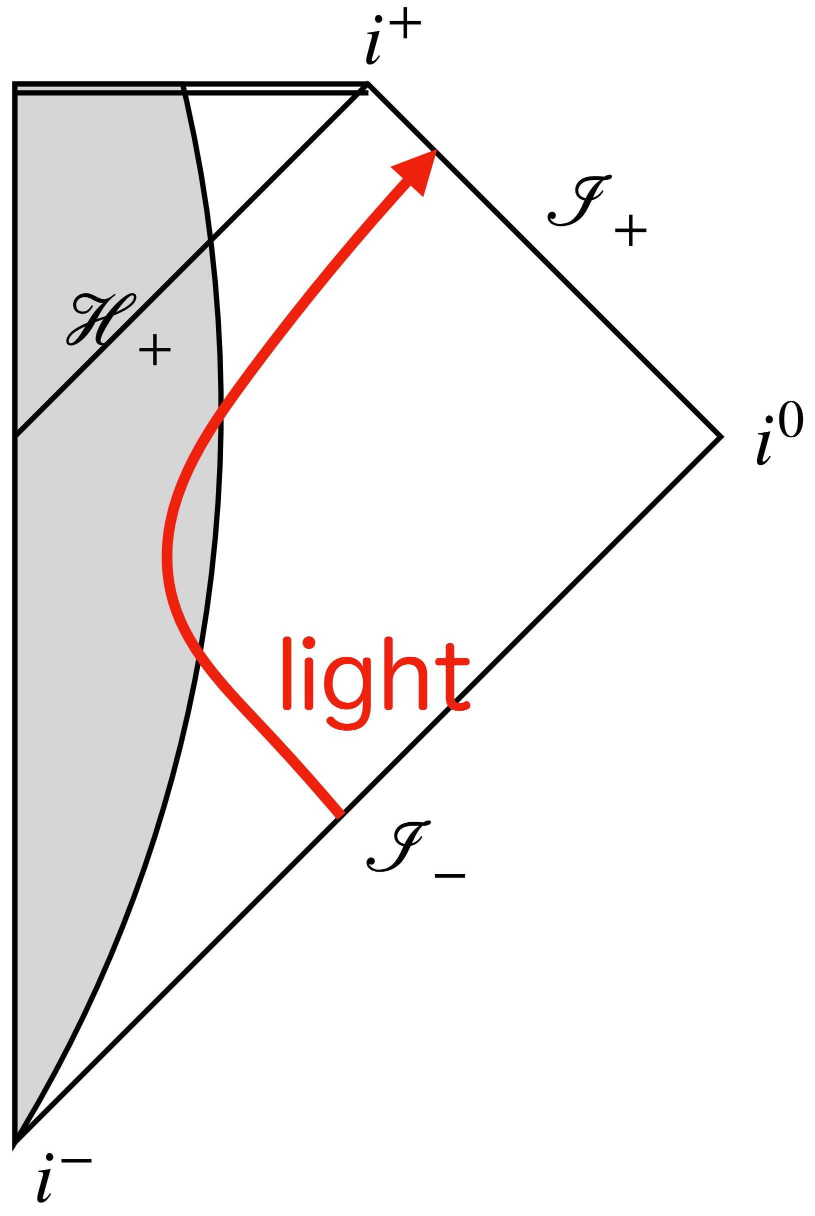
Yasutaka Koga (Nagoya U.)
Work with N. Asaka (TDSE), M. Kimura (DIT), K. Okabayashi (Kyoto U.) (paper in preparation)

1. Introduction

概略:

- 動的な時空における、$\mathscr{I}^-$から$\mathscr{I}^+$への光（null測地線）のエネルギーの赤方偏移 αを議論する。
- 球対称シェルの場合、重力崩壊で赤方偏移$\alpha < 1$, 膨張で青方偏移$\alpha > 1$となる傾向が見える。
- 一般の（漸近平坦）球対称時空の場合、小玉ベクトルをreferenceとしてエネルギーを定義すると、エネルギーの変化が曲率を使って、$\nabla_k E = -4\pi R\widetilde{\mathscr{T}}(k,k)$. また、ニュートン重力の類推が可能で、時空のダイナミクスと赤方偏移の関係を議論する。



Redshift factor

$$\alpha := \frac{E|_{\mathscr{I}^+}}{E|_{\mathscr{I}^-}}$$

$E := -g(k, \partial_t)$, k: null geodesic tangent.

∂_t: time coord. basis w.r.t. the flatness

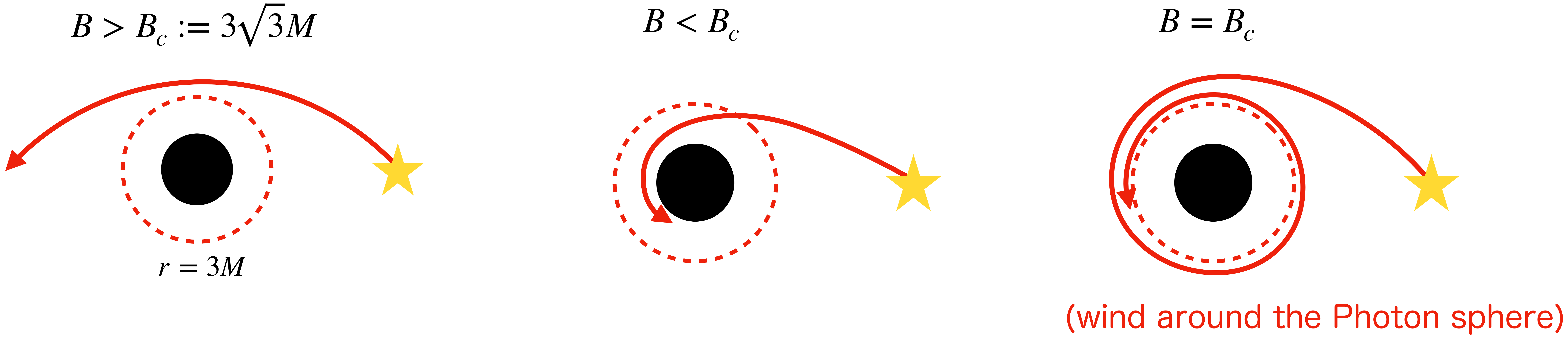
Outline

Outline:

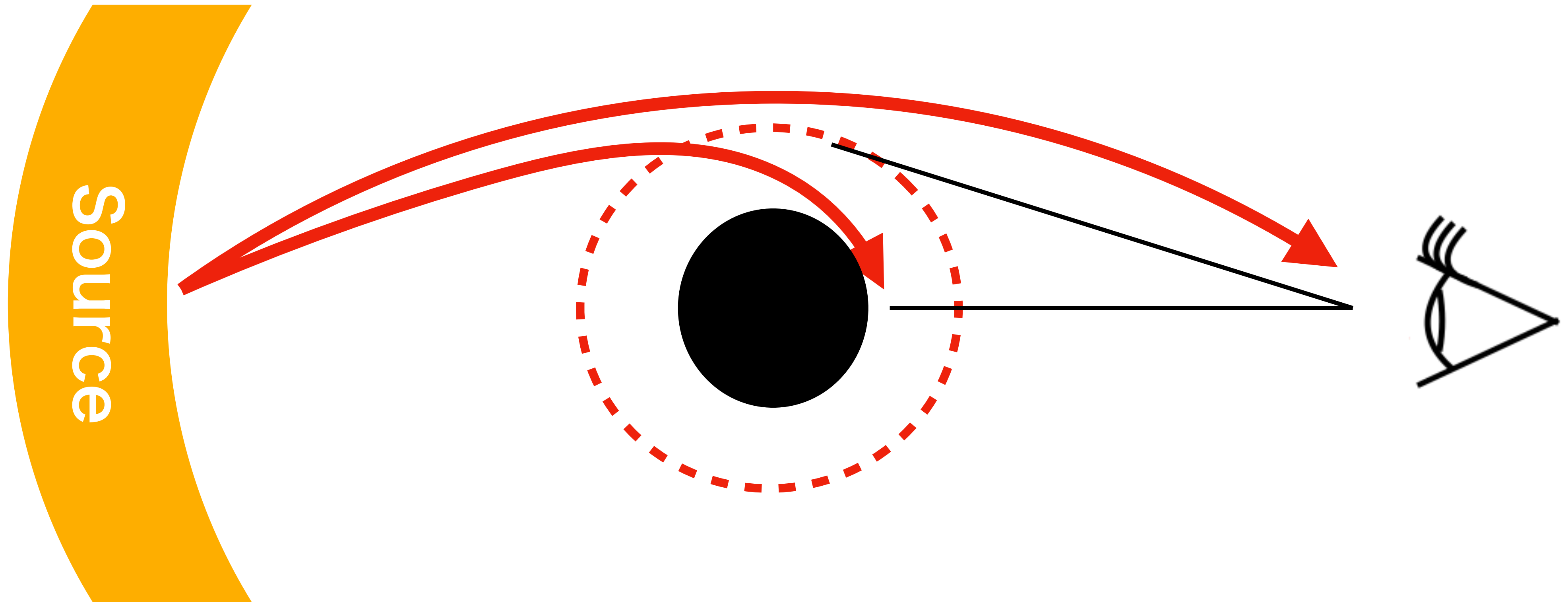
2. Motivation from study of BH shadow

2. Brief review

3 types of light orbit from infinity in Schwarzschild spacetime:



BH shadow with distant light source:



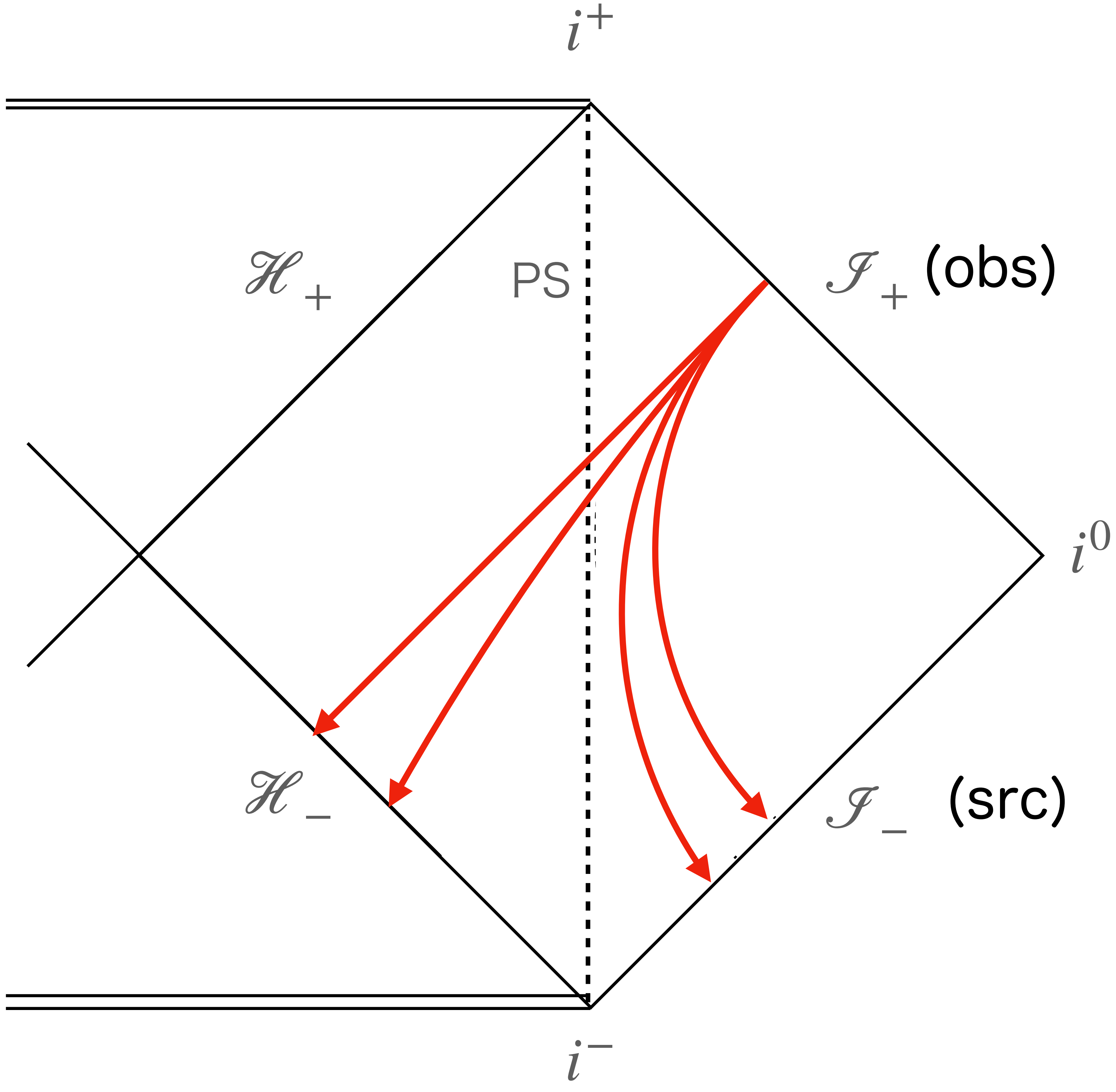
The apparent angular size of the shadow is determined by that of PS (or B_c)

2. Brief review

Causal viewpoint

- Imaging is ray-tracing backward in time.
- Past-directed light from the obs.:
  - For $B > B_c$, from $\mathscr{I}^+$ to $\mathscr{I}^-$ (i.e. light source).
  - For $B < B_c$, from $\mathscr{I}^+$ to $\mathscr{H}^-$ (white hole horizon).

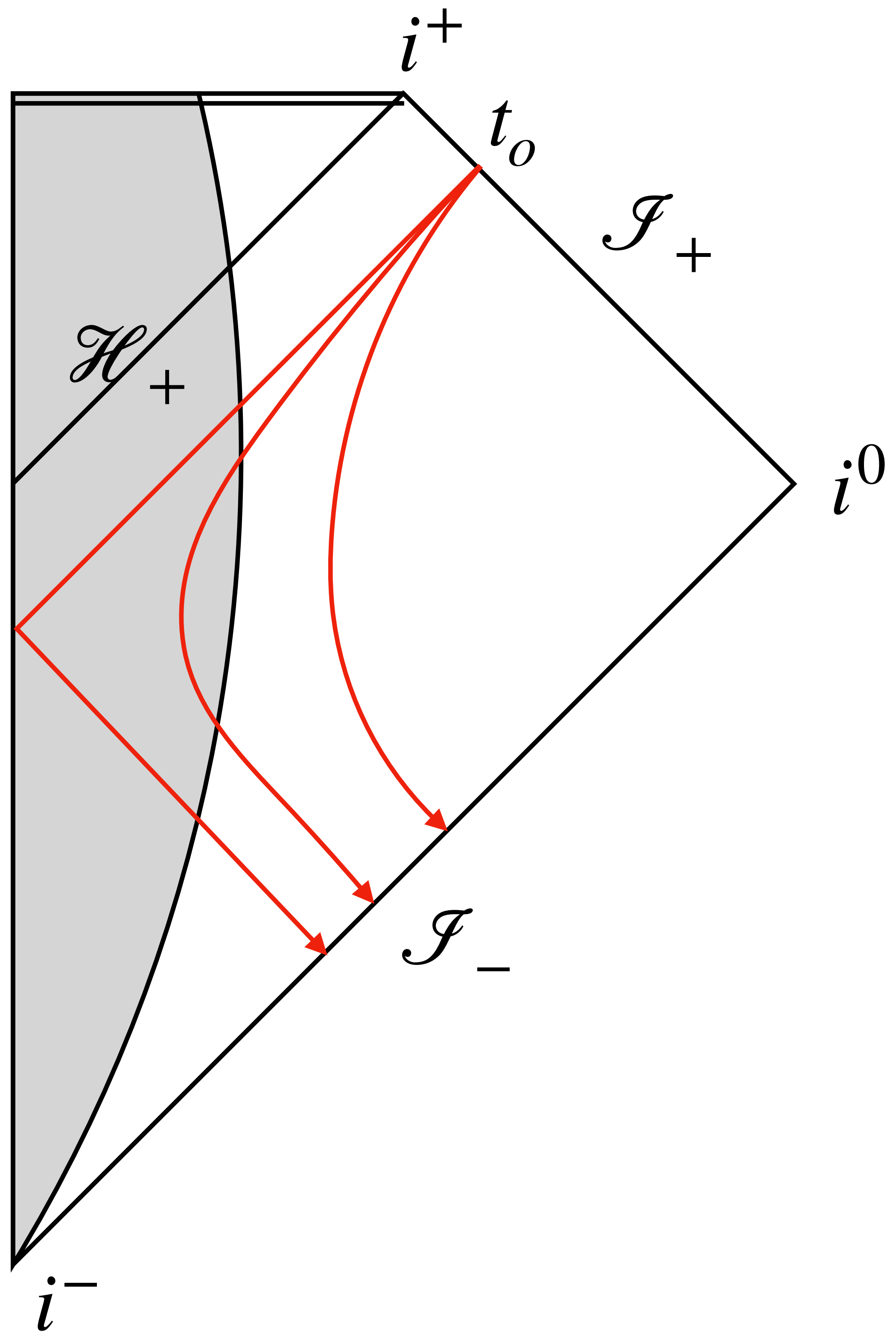
- In an eternal BH spacetime, the shadow is a complemental image of the white hole. The photon sphere works as the threshold. cf. [YK, Asaka, Kimura, Okabayashi (2022)]



2. Brief review

Shadow in gravitational collapse:

- All the past-directed null geodesic from $\mathscr{I}^+$ go to $\mathscr{I}^-$ (no white hole).
- This fact does not mean that the observer never observes shadow.
- The shadow (dark) image should be formed by the effect of redshift of light.



3. Thin shell model

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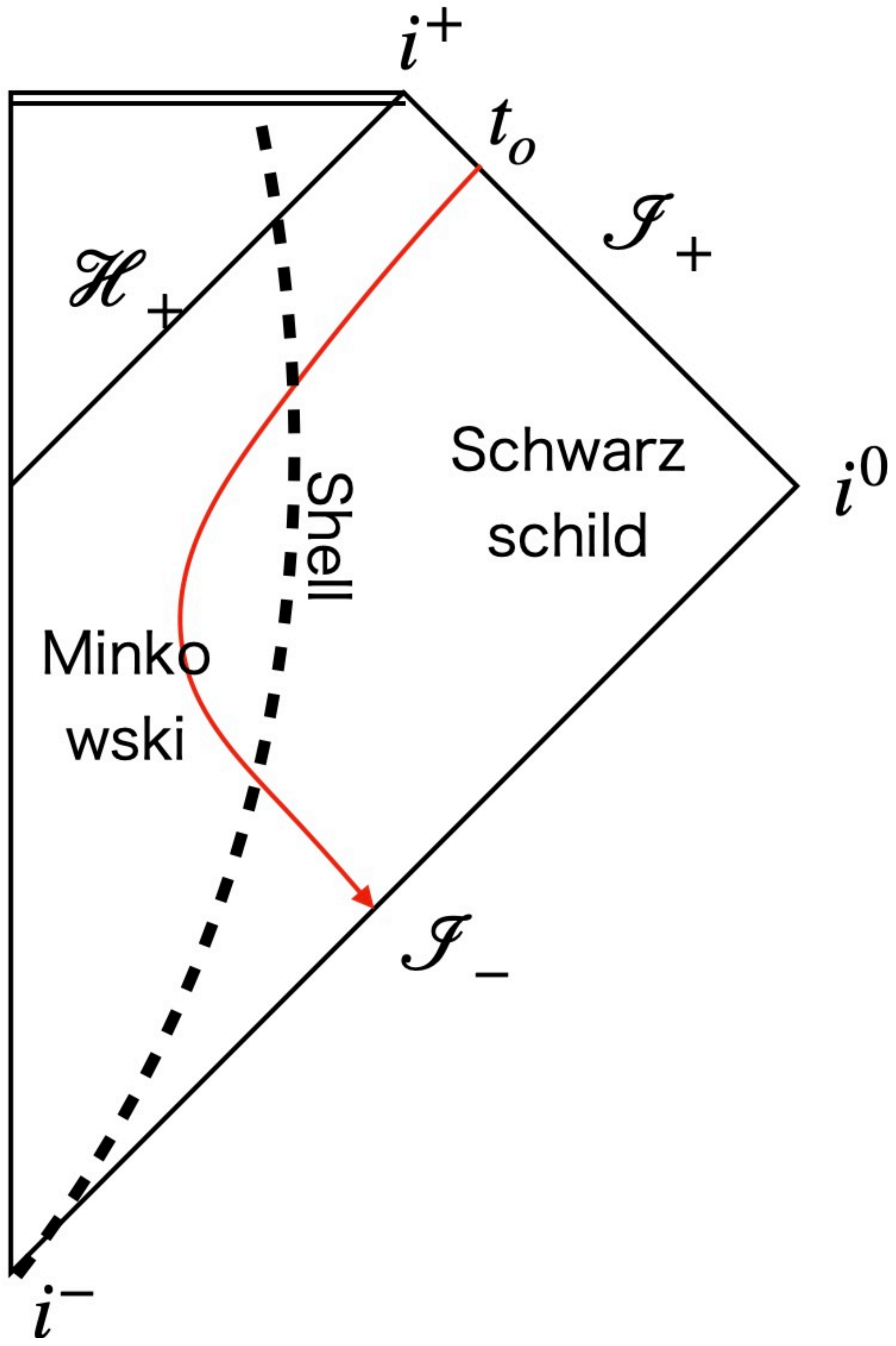
A collapsing thin shell model:



- Exterior: Schwarzschild spacetime

$$ds^2_{\text{ex}} = -f(r)dt^2 + f^{-1}(r)dr^2 + r^2 d\Omega^2, \;\; f(r) = 1 - \frac{2M}{r}$$

- Interior: Minkowski spacetime

$$ds^2_{\text{in}} = -dT^2 + dr^2 + r^2 d\Omega^2$$

- Boundary: a thin shell

$$\Sigma := \partial M_1 \equiv \partial M_2 = \{r = R(\tau)\}$$

τ: proper time of the shell

- Impose 1st junction condition (equivalence of induced metrics on Σ) & specify $R(\tau)$ by hand.

=> spacetime is fixed & coord. trans $t \leftrightarrow T$ is obtained.

Energy of null geodesic

Null geodesic motion:

- In the Schwarzschild region:

$$E := -g(\partial_t, k), \quad L := g(\partial_\phi, k), \quad \dot{r}^2 + V(r) = 0, \quad V(r) := E^2 - f(r) r^{-2} L^2 .$$

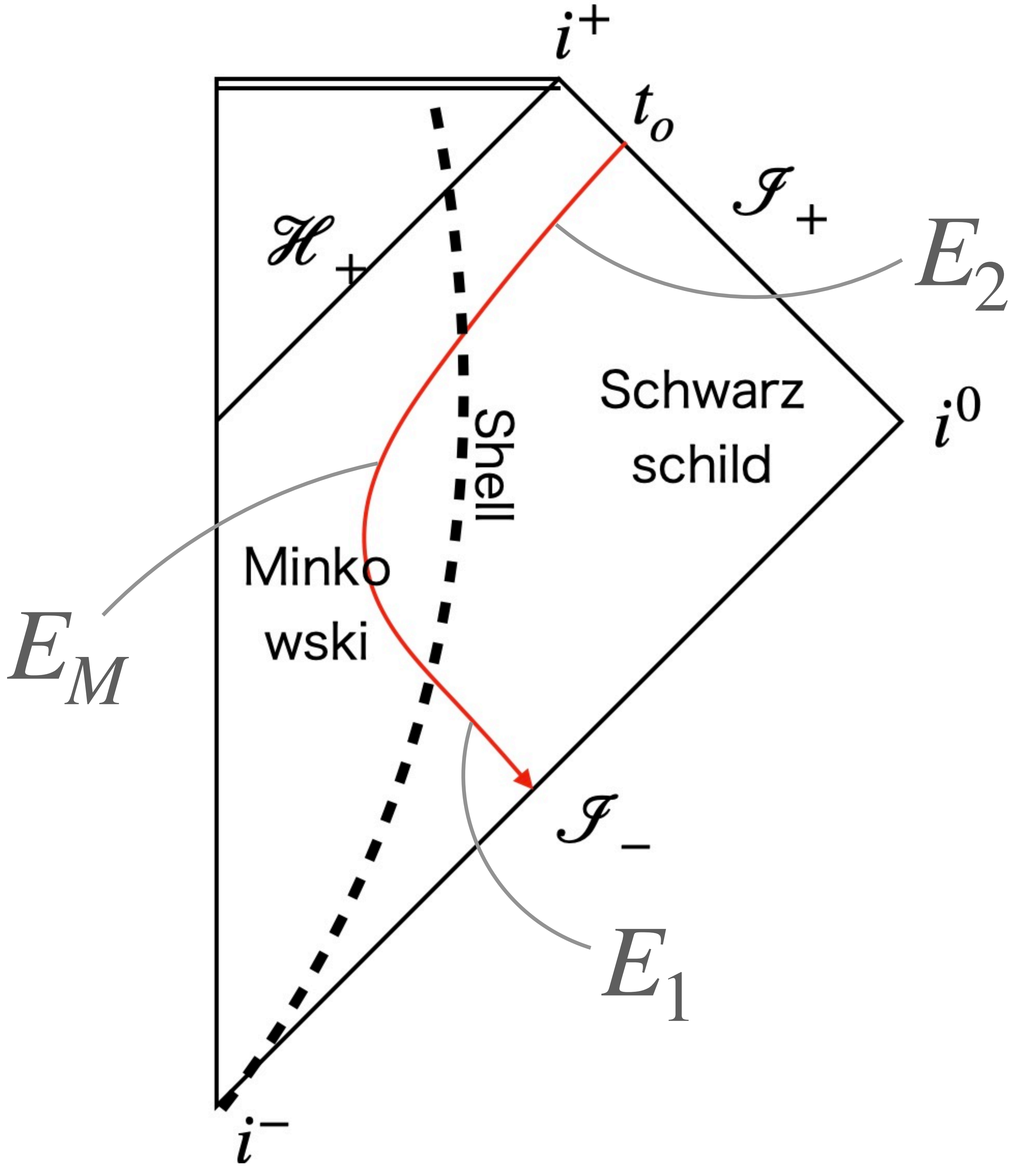
- In the Minkowski region:

$$E_M := -g(\partial_T, k), \quad L := g(\partial_\phi, k), \quad -(T - T_0)^2 + R^2 = L^2 / E_M^2 .$$

- Relation btwn energies, $E \leftrightarrow E_M$:

$$E = -g(\underline{\partial_t}, k) \qquad E_M = -g(\underline{\partial_T}, k)$$

coord. transformation on Σ



Energy of null geodesic

Energy of null geodesic

- Eliminating E_M,

$$\alpha = \frac{E|_{\mathcal{I}^+}}{E|_{\mathcal{I}^-}} = \frac{E_1}{E_2} = \dots$$

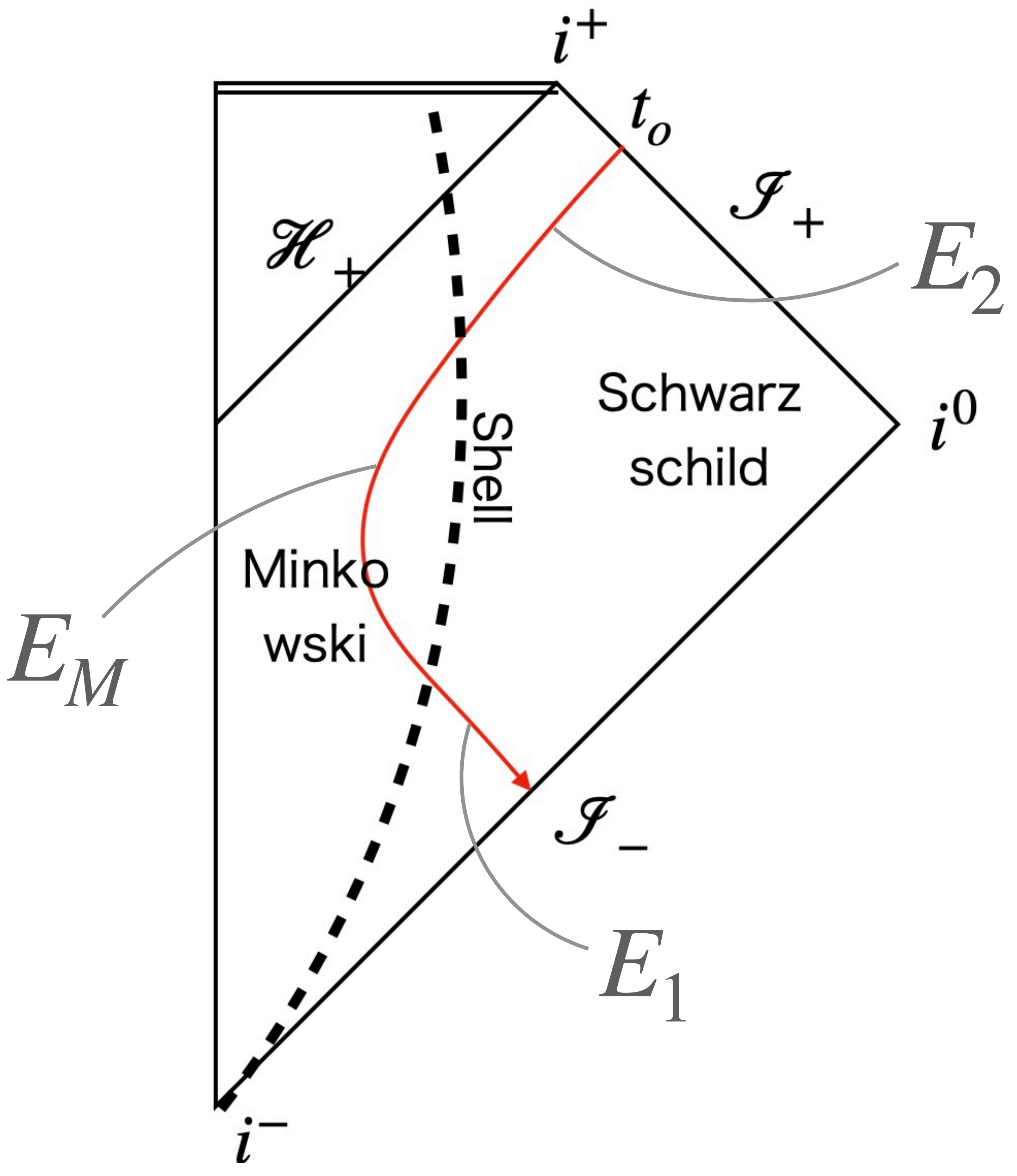
$$= \frac{f_2}{f_1} \frac{A_1\left(A_2 - C_2\sqrt{1 - B_2^2 f_2 r_2^{-2}}\right) + C_1\sqrt{\left(A_2 - C_2\sqrt{1 - B_2^2 f_2 r_2^{-2}}\right)^2 - B_2^2 f_2^2 r_1^{-2}}}{\left(A_2 - C_2\sqrt{1 - B_2^2 f_2 r_2^{-2}}\right)^2 + C_1^2 b B_2^2 f_2^2 f_1^{-1} r_1^{-2}} .$$

$$= \alpha\underline{(r_1, r_2, R_1', R_2', \sigma_1, \sigma_2, B_2)} .$$

7 parameters determine α

$$A = \sqrt{1 + (R')^2}\sqrt{f + (R')^2} - (R')^2,$$

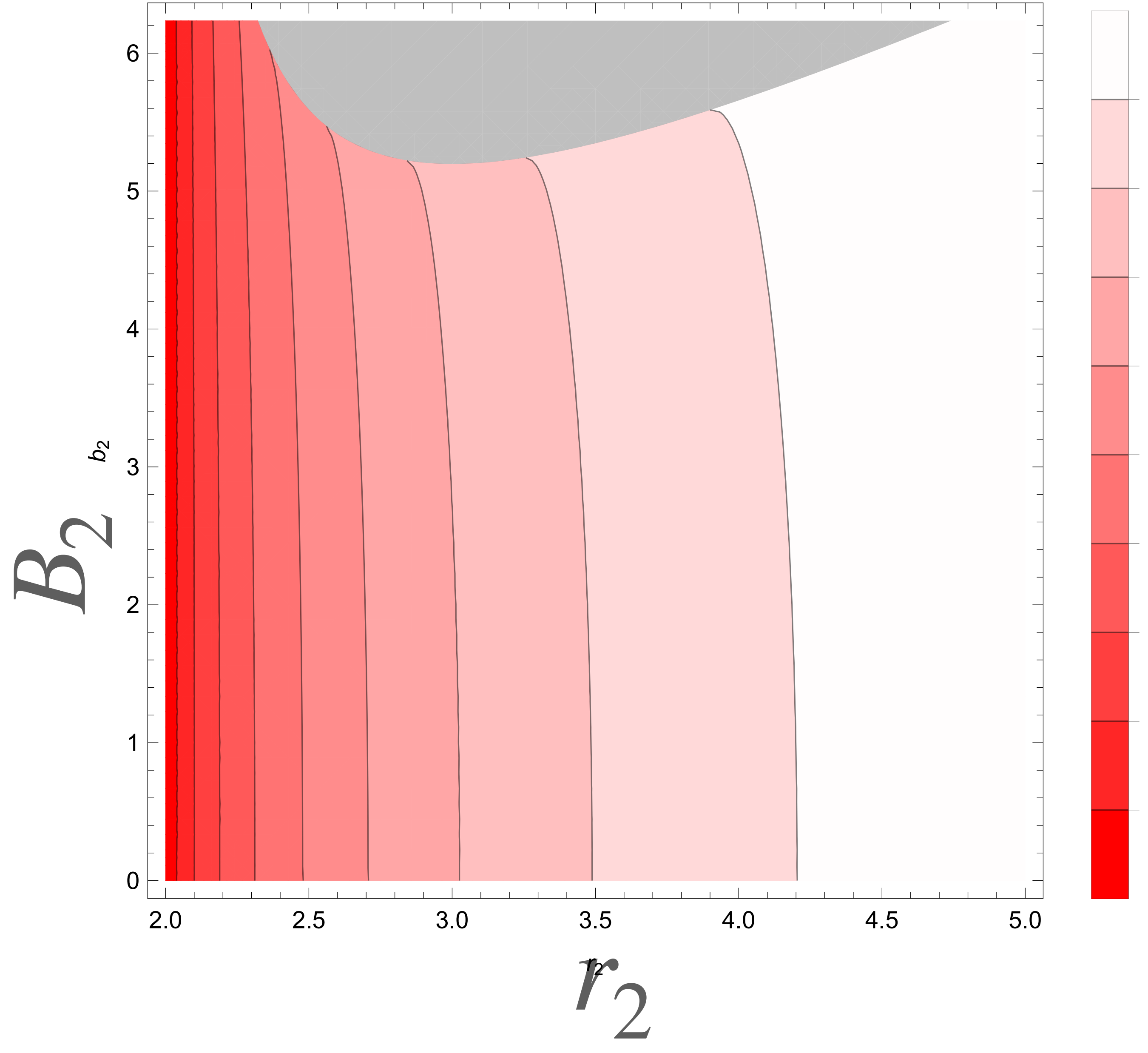
$$C = \sigma(R')\left(\sqrt{1 + (R')^2} - \sqrt{f + (R')^2}\right), \qquad \sigma = \mathrm{Sign}(\dot{r})$$



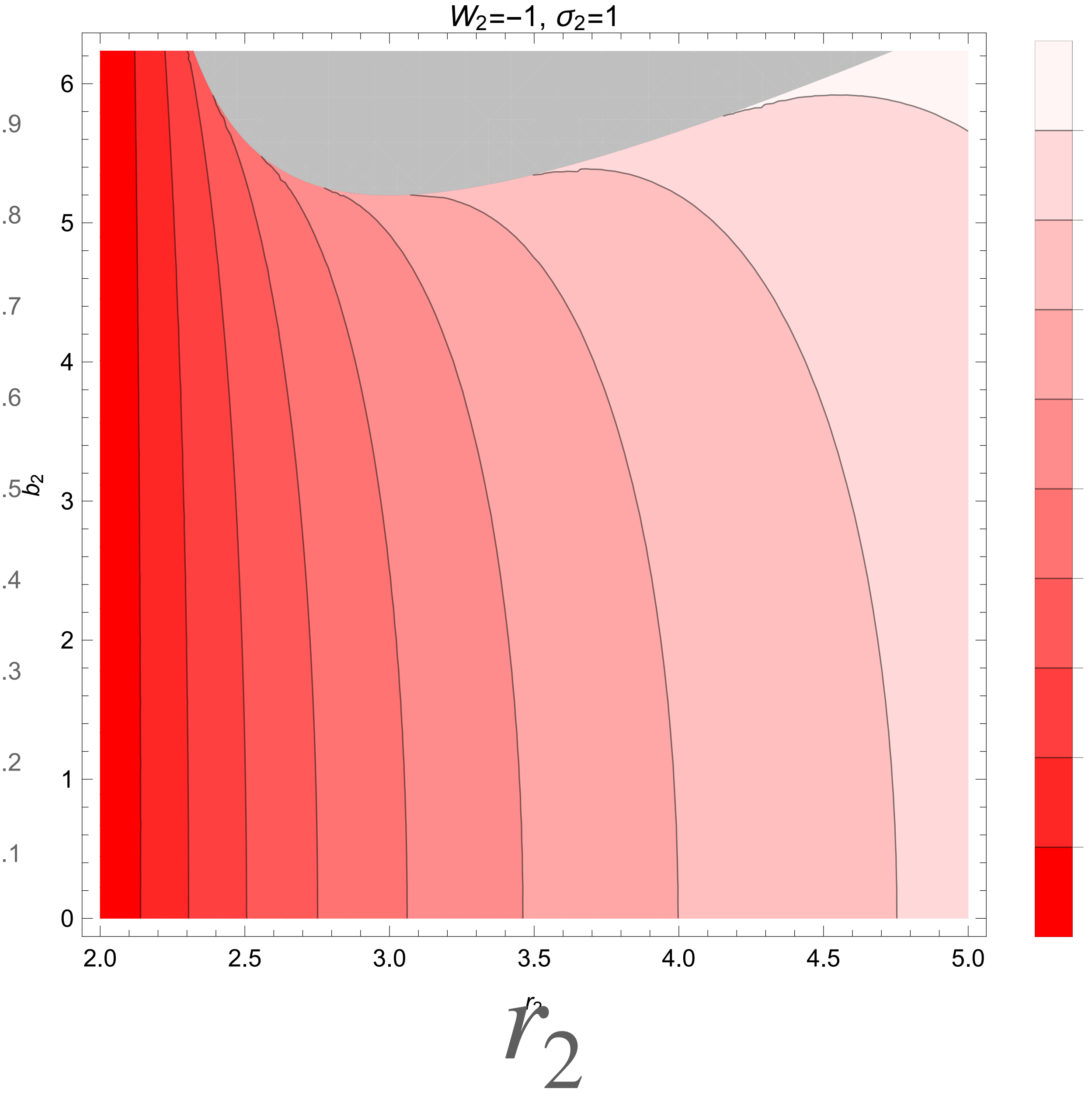
Redshift in monotonic collapse

Redshift Factor for $(r_1, r_2, R'_1, R'_2, \sigma_1, \sigma_2, B_2) = (5M, r_2, 0, R'_2, -1, +1, B_2)$

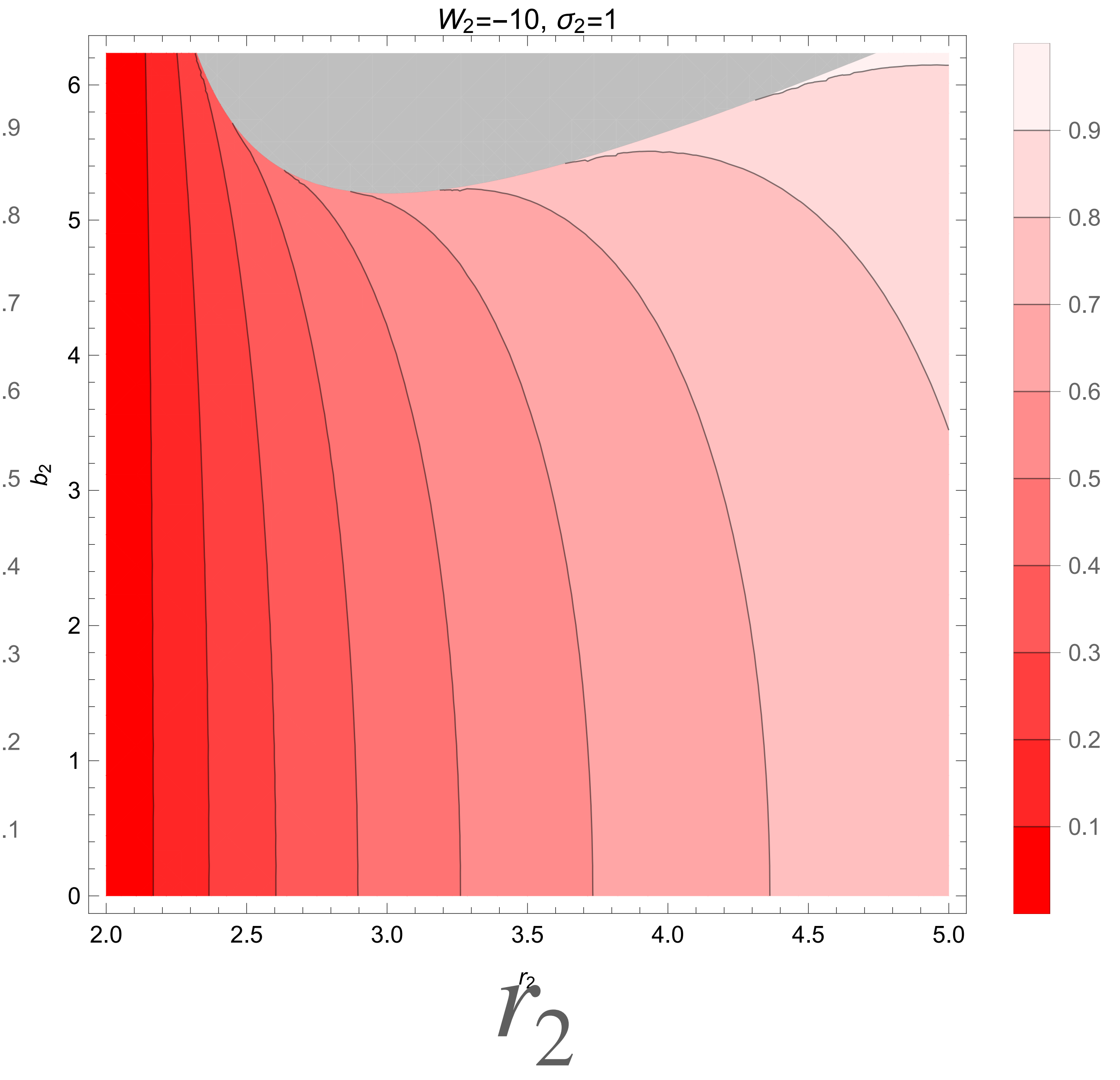
$R'_2 = -0.1$

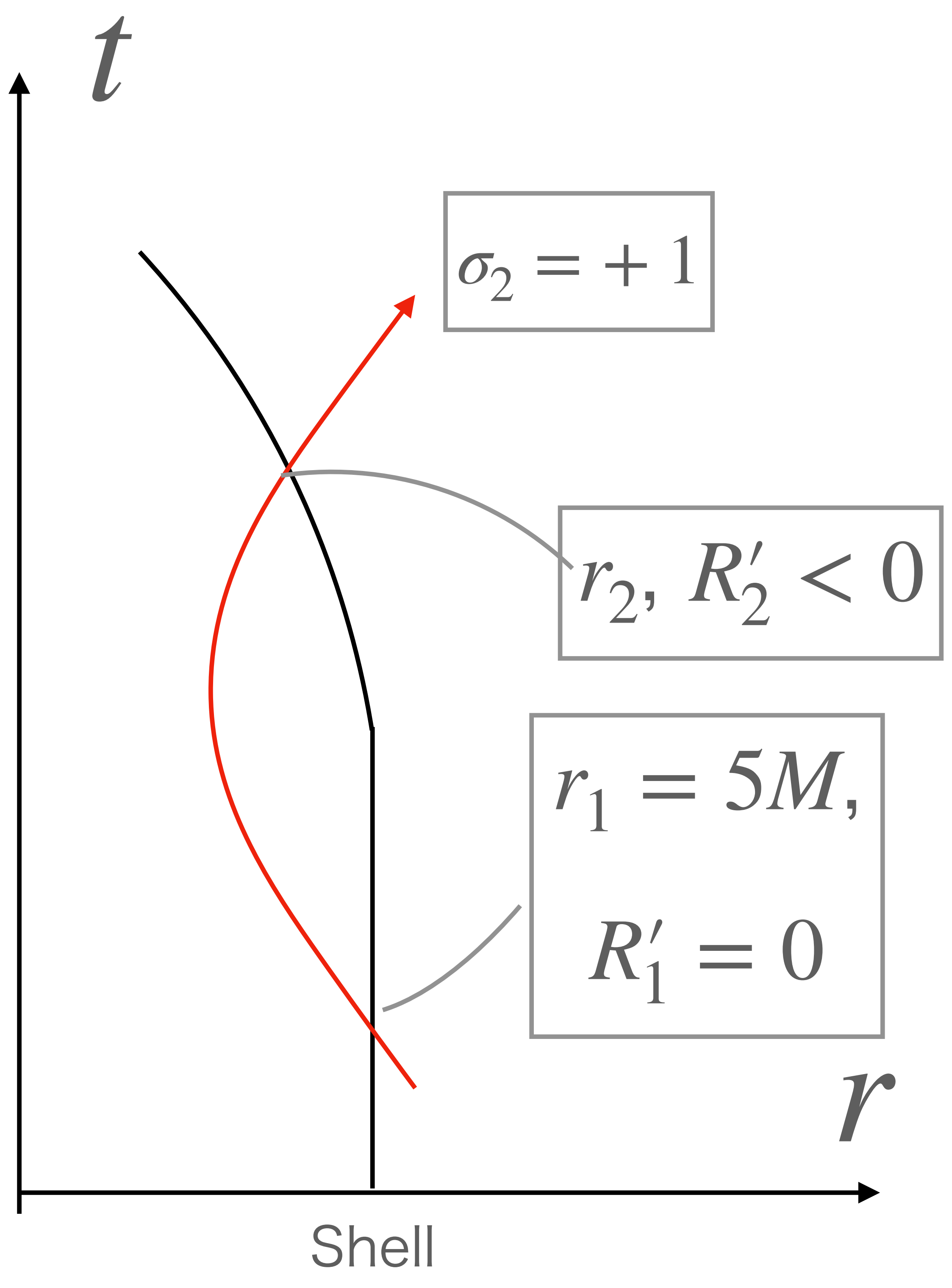


$R'_2 = -1$



$R'_2 = -10$

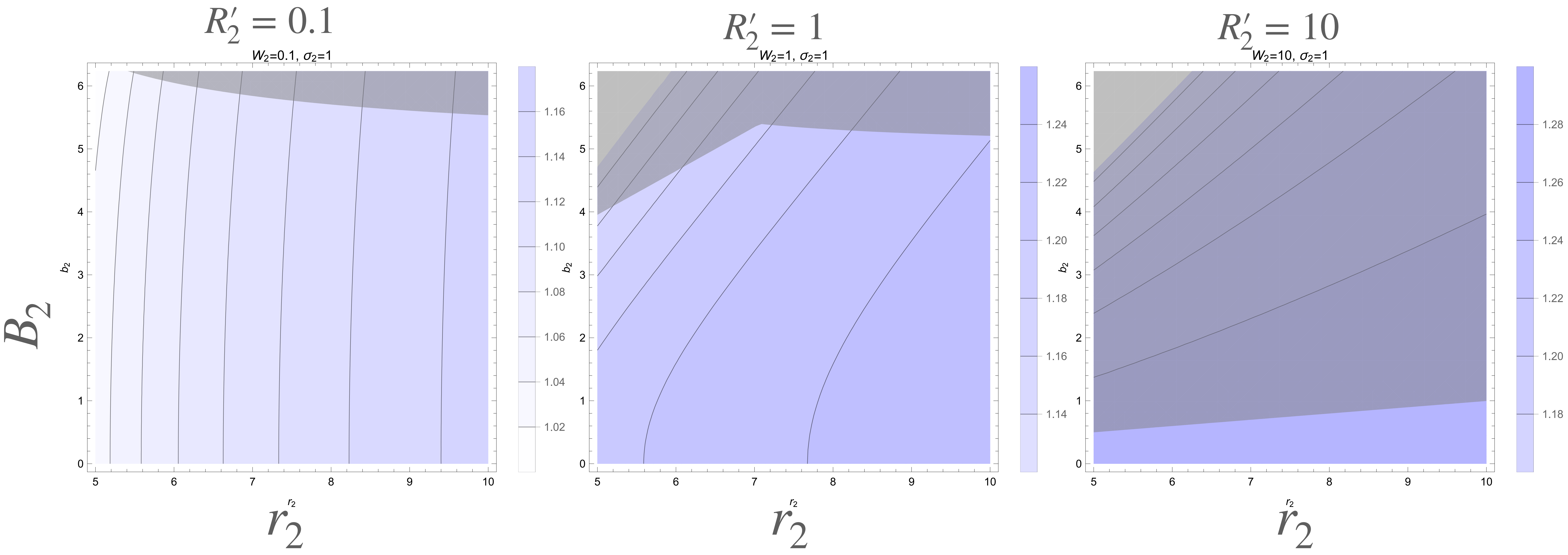


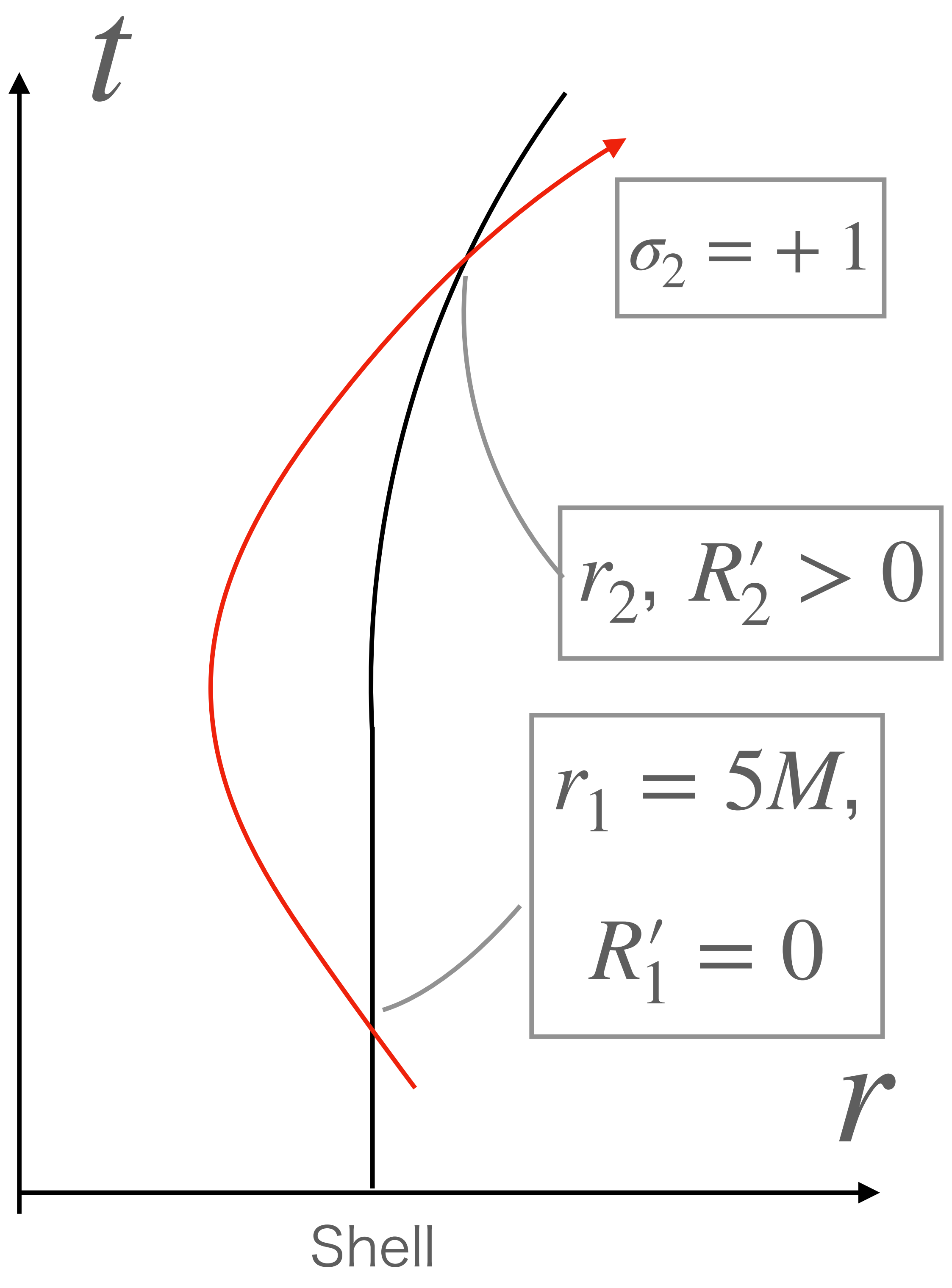


- Every light is redshifted $\alpha < 1$ if the shell is shrinking and $\sigma_2 > 0$. (no blueshift)
- High redshift (small α) for rapid collapse (large $|R'|$) & in the late stage ($r_2 \to 2M$).

Blueshift in monotonic expansion

Redshift Factor for $(r_1, r_2, R'_1, R'_2, \sigma_1, \sigma_2, B_2) = (5M, r_2, 0, R'_2, -1, +1, B_2)$





- **Every light is blueshifted $\alpha > 1$ if the shell is expanding.**
- **High blueshift (large α) for rapid expansion (large $|R'|$).**

4. Generic spacetime and new formula

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Question:

- How is the redshift related to the spacetime dynamics?
- No unique preferred reference of time. -> one of good candidates: Kodama vector.

4. Generic spacetime and new formula

General, asymptotically flat, spherically symmetric spacetime $(\mathscr{M}, g)$:

$$g = \underbrace{h_{AB}(x^C)dx^A dx^B}_{\text{submanifold } (\mathscr{N}, h)} + R^2(x^C)(d\theta^2 + \sin^2\theta d\phi^2), \qquad x^C = t, r\,.$$

Kodama vector:

$$K := \mathrm{curl} R = -(\epsilon^{AB}\nabla_B R)\partial_A$$

ϵ^{AB}: the totally anti-sym tensor of $(\mathscr{N}, h)$ [Kodama 1980]

- Properties:
  - $K \cdot \nabla R = 0$,
  - If $(\mathscr{M}, g)$ is static, K is proportional to the static KV, $K \propto \xi_t$.
  - In vacuum and thus in the asymptotic region, $K = \xi_t$ (can be natural extension of the static KV).
  - $g(K, K) = 0$ on a trapping horizon.
  - Conserved current: $\nabla_a J^a = 0$, where $J^a := T^a{}_b K^b$.
  - Conservation of the quasi-local mass, “Kodama mass” $E(t, r)$, defined by $g(K, K) = 2E/r - 1$.

4. Generic spacetime and new formula

- The energy of light associated with K:

$$E := -g(k, K), \qquad k\text{: a null (/timelike) geodesic tangent}$$

- Redshift factor:

$$\alpha := \frac{E|_{\mathscr{I}^+}}{E|_{\mathscr{I}^-}} = \frac{\int_{-\infty}^{+\infty} \nabla_k E d\lambda + E|_{\mathscr{I}^-}}{E|_{\mathscr{I}^-}}$$

- The derivative:

$$\nabla_k E = -\nabla_k g(k, K) = -\underbrace{\cancel{g(\nabla_k k, K)}}_{\because \text{ geodesic eq.}} - g(k, \nabla_k K) = -\underline{\nabla_{(a}K_{b)}}k^a k^b$$

Symmetric derivative of the Kodama vector
is a geometrical quantity
that characterizes redshift

4. Generic spacetime and new formula

Symmetric derivative of Kodama vector:

Proposition: In a 4-dim spherically symmetric spacetime, the Kodama vector satisfies

$$\nabla_{(a}K_{b)} = 4\pi R \widetilde{\mathscr{T}}_{ab},$$

where

$$\widetilde{\mathscr{T}}_{ab} := \epsilon_a{}^c \mathscr{T}_{cb}$$

is the dual of $\mathscr{T}$,

$$\mathscr{T}_{ab} = T^{\mathscr{N}}_{ab} - \frac{1}{2} h_{ab} h^{cd} T^{\mathscr{N}}_{cd}$$

is the trace-free part of $T^{\mathscr{N}}_{ab}$, and $T^{\mathscr{N}}_{ab}$ is the restriction of the energy momentum tensor T_{ab} onto $(\mathscr{N}, h)$.

Proof: Einstein equation. ∎

4. Generic spacetime and new formula

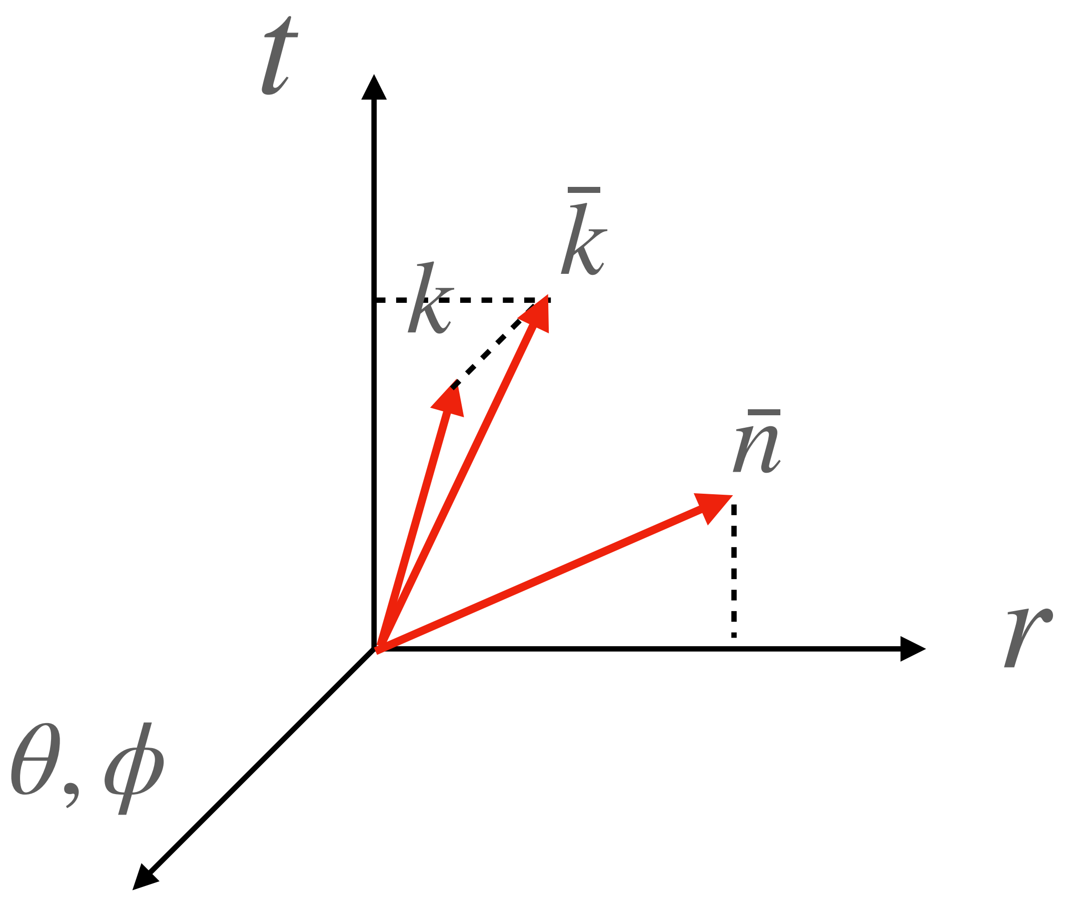
Derivative of the energy:

Theorem: The local redshift of energy of a light associated with Kodama vector is given by

$$\nabla_k E = -4\pi R \widetilde{\mathscr{T}}(k, k)\,.$$

Note this is valid for a timelike geodesic tangent u instead of null k.

4. Generic spacetime and new formula



Meaning of the theorem:

- Rewrite the energy momentum as

$$\widetilde{\mathcal{T}}(k,k) = \widetilde{\mathcal{T}}(\bar{k},\bar{k}) = \mathcal{T}(\bar{k},\tilde{\bar{k}}) = T_{\mathcal{N}}(\bar{k},\tilde{\bar{k}}) = T(\bar{k},-\bar{n})$$

$\bar{k}$: projection of k onto $(\mathcal{N},h)$, $\tilde{\ }$: contraction with $\epsilon_a{}^b$,

$\bar{n} := -\tilde{\bar{k}} = -\epsilon_b{}^a\bar{k}^b\partial_a$: radial outward vector orthogonal to $\bar{k}$.

The energy current in the direction of $-\bar{n}$.

- Newtonian analogy:

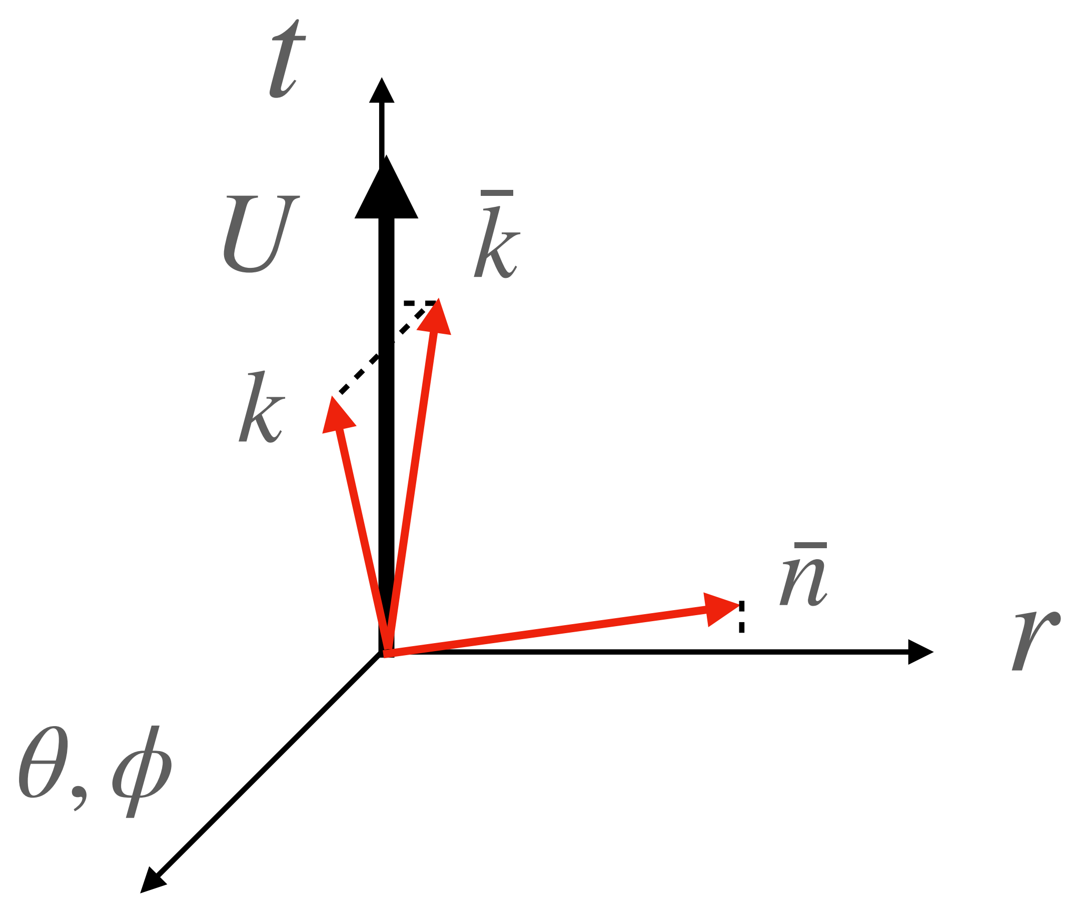
$$\nabla_k E = -4\pi R\widetilde{\mathcal{T}}(k,k) = -\frac{4\pi R^2 T(\bar{k},-\bar{n})}{R} =: -\frac{\delta M}{R}$$

Loss of the potential energy due to the increase of the mass inside the sphere of R at the moment.

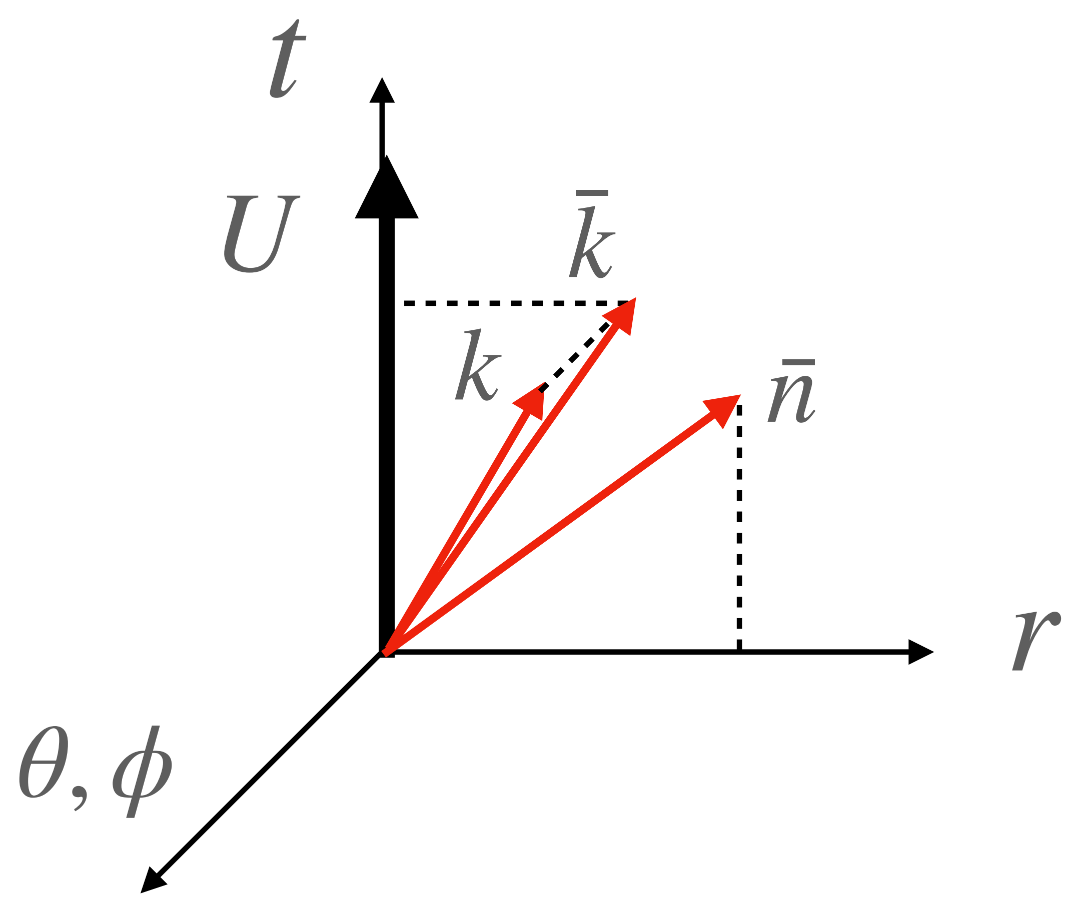
4. Generic spacetime and new formula

- Effective change of δM:

δM: large



δM: small



U: fluid 4-velocity

4. Generic spacetime and new formula

The thin shell case:

- The dual energy momentum tensor:

$$T^{\mathscr{N}}_{ab} = \delta(l)\rho U_a U_b,$$

$$\widetilde{T}_{ab} = \frac{1}{2}\delta(l)\rho\left(U_a U_b + N_a N_b\right),$$

$$\widetilde{\mathscr{T}}_{ab} = -\delta(l)\rho U_{(a} N_{b)}\,.$$

U: shell's 4-velocity, N: orthonomal vector to U,

$\rho = S(U, U)$: shell's rest mass energy,

l: a radial coordinate s.t. $l = 0$ on Σ and $g(dN, dN) = 1$.

- Collapsing shell:

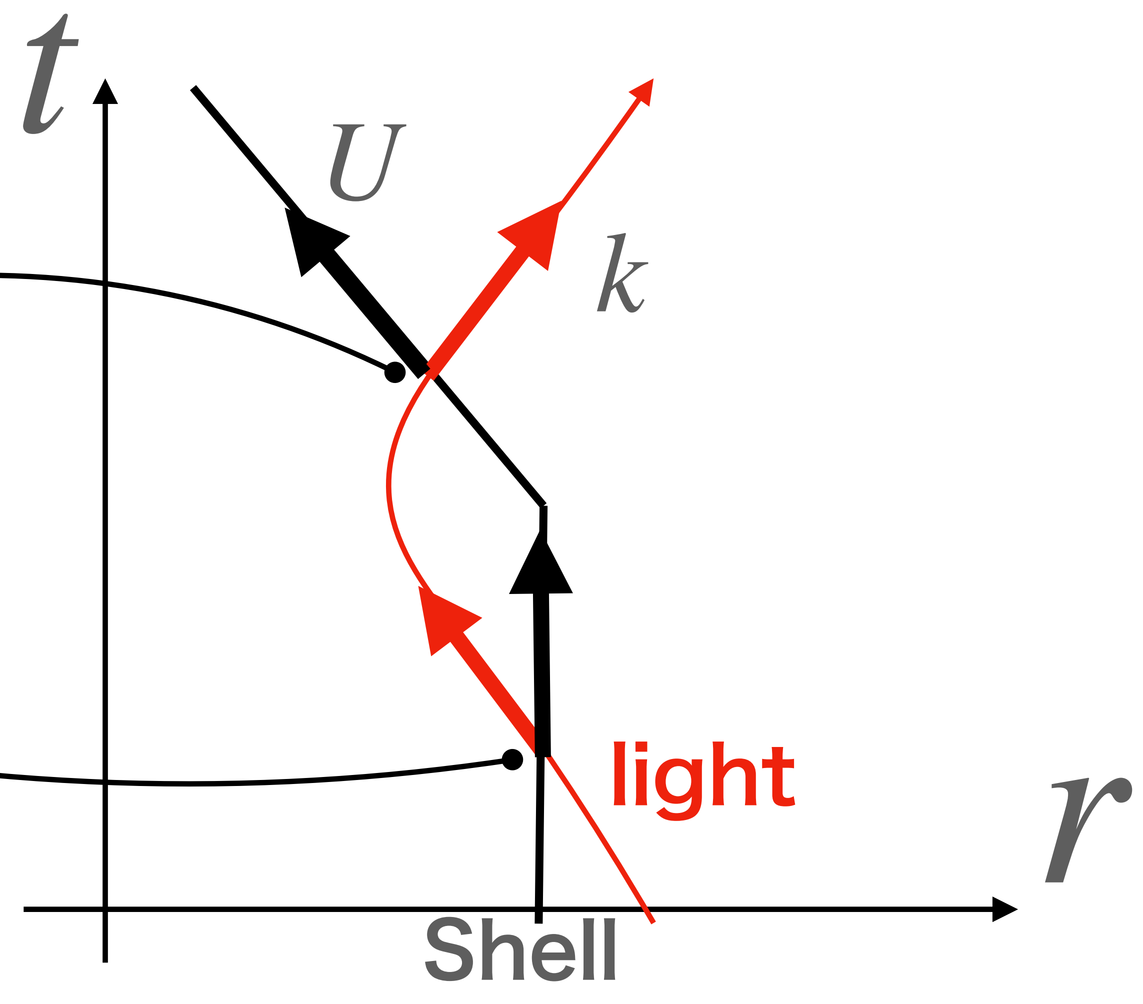
Redshift at exit

$$\Delta E|_{\mathrm{exit}} = -4\pi R\rho\,|\,g(k, U)\,|$$

Blueshift at entry

$$\Delta E|_{\mathrm{entry}} = +4\pi R\rho\,|\,g(k, U)\,|$$

$\Rightarrow\ |\Delta E_{\mathrm{exit}}| > |\Delta E_{\mathrm{entry}}|$ (redshift)



5. Summary

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Summary:

- Tendency of redshift of lights is confirmed in gravitational collapse of a thin shell model.
- Proposed the general covariant formula $\nabla_k E = -4\pi R \widetilde{\mathcal{T}}(k,k)$ of redshift by taking the Kodama vector as a reference.
- The formula gives a very clear interpretation of redshift due to the spacetime dynamics.

Discussion:

- Can we prove the generality of redshift in gravitational collapse?
- Beyond spherical symmetry?
- Application to other gravitational or astrophysical phenomena?