全スカラー曲率の極限定理

小研究会「一般相対論と幾何」

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Table of contents

Weak notions of $R \geq K$

Gromov (and Bamler) proved the following.

Gromov (2014 [G1]), Bamler (2016 [B])

Let M be a smooth manifold and g a C^2-Riemannian metric on M. Suppose that a sequence of C^2-Riemannian metrics g_i on M that converges to g in the local C^0-sense. Assume that for all $i = 1, 2, \cdots$ $R(g_i) \geq \kappa$ on M for some $\kappa \in C^0(M)$. Then $R(g) \geq \kappa$ on M.

Definition [G]

For a C^0-met. g and a conti. fct. κ, we say "$R(g) \geq \kappa$ *in the Gromov sense*" if $\exists C^2$-Riem. met's (g_i) s.t. $R(g_i) \geq \kappa$ and $g_i \xrightarrow{C^0} g$.

Gromov type theorem in even weaker topology

Recently, Lee and Topping proved that non-negativity of scalar curvature is **NOT** preserved on the sphere in dimension at least four in the sense of uniform convergence of Riemannian distance. More precisely,

Lee–Topping (2022 [LTo])

Let $n \geq 4, f \in C^0(S^n)$. Then $\exists (g_i) \subset \mathcal{M}(S^n)$ s.t. $R(g_i) > 0$ and $d_{g_i} \to d_f$ uniformly on $S^n \times S^n$, where d_f is the Riemannian distance of the met. $e^f g_{std}$.
In paticular, $(S^n, d_{g_i}) \to (S^n, d_f)$ in the Gromov-Hausdorff sense as $i \to \infty$.
Moreover, $\exists C > \infty$ s.t.

$$C^{-1} g_{std} \leq g_i \leq C g_{std}$$

on S^n for all i.

Note: We can always choose the function $f \in C^2(S^n)$ s.t. $e^f g_{std}$ has negative scalar curvature at some point on S^n. Q.: $n = 3$?

Weak notions of $R \geq K$

- **[Bamler 2016 [B]]** An alternative proof using Ricci(-DeTurck) flow.
- **[Burkhardt-Guim 2019 [BG]]** Gave a definition of scalar curvature lower bounds of metric tensors with only C^0-regularity on a closed mfd using Ricci(-DeTurck) flow.

  Note: [BG] $\Leftrightarrow$ [G].
- **[D.Lee and P. G. LeFloch 2015 [LL]]** Defined "scalar curvature lower bounds in the distributional sense" for $g \in L^\infty_{\mathrm{loc}} \cap W^{1,2}_{\mathrm{loc}}$ with $g^{-1} \in L^\infty_{\mathrm{loc}}$.
- **[W.Jiang, W.Sheng and H.Zhang 2021 [JSZ]]** For $g \in W^{1,p}(M)$ $(\dim(M) < p \leq \infty)$, ($g$ can be flowed by a RF $((g(t))_{t\in(0,\exists T)}$ a smooth RF and $(M, d_{g(t)}) \xrightarrow{GH} (M, d_g)$) and) "$R(g) \geq \kappa$ $(\kappa \in \mathbb{R})$ in the distributional sense" is preserved under the RF.

  Note: As a corollary,

  "$R(g) \geq \kappa$ $(\kappa \in \mathbb{R})$ in the distributional sense" $\Rightarrow R(g) \geq \kappa$ in the sense of [BG] ($\Leftrightarrow$ [G]).

- **[T.Lamm and M.Simon 2021 [LS]]** For $g \in L^\infty \cap W^{2,2}(M^4)$ (M^4 : closed 4-mfd) with $a^{-1}h \leq G \leq ah$ for some $a > 0$, the following are equivalent.
    - $R(g) \geq \kappa$ ($\kappa \in \mathbb{R}$) in the distributional sense
    - $\exists (g_{i,0}) \in \mathcal{M}(M^4)$ with $b^{-1}h \leq g_{i,0} \leq bh$ for some $1 < b < \infty$, s.t. $R(g_{i,0}) \geq \kappa$ and $g_{i,0} \to g \in W^{2,2}(M^4)$
    - the RDF $(g(t))_{t \in (0,T)}$ of g constructed in [LS] has $R(g(t)) \geq \kappa$ for all $t \in (0, T)$.
- **[Tian and Wang 2023 [TW]]**
  A precompactness theorem for warped product metrics on $S^2 \times S^1$.
- **[Gromov 2014 [G1]]**
  A characterization of $R \geq 0$ on cube-type polyhedrons. (Rigidity $+\alpha \cdots$ C. Li [L1, L2])

Main Theorem 1 (H. 2022 [H1] arXiv:2208.01865)

M^n : a closed (i.e., cpt without boundary) n-mfd ($n \geq 3$) and g : a C^2 Riem. met. on M.
(g_i) : a sequence of Ricci solitons on M (i.e., $-2\,\mathrm{Ric}(g_i) = \mathcal{L}_{Y_i}\, g_i - 2\lambda_i\, g_i$ for some constant $\lambda_i \in \mathbb{R}$ and a vector field $Y_i \in \Gamma(TM)$) with s.t. $g_i \xrightarrow{C^0} g$ on M as $i \to \infty$. Assume

$$(*) \qquad \int_M R(g_i)\, d\mathrm{vol}_{g_i} \geq \kappa \quad \text{for some constant } \kappa \in \mathbb{R}.$$

Moreover, assume $\lambda_i \leq C_+$ $(i = 1, 2, \cdots)$ for some constant $C_+ \in \mathbb{R}$ if $\kappa \geq 0$ (resp. $\lambda_i \geq C_-$, $C_- \in \mathbb{R}$ if $\kappa < 0$).
Then $\int_M R(g)\, d\mathrm{vol}_g \geq \kappa$.

Main Theorem 2 (H. 2022 [H1])

Let $p > n$. Let M^n $(n \geq 3)$ be a closed mfd. Suppose that a sequence of C^2-Riem. met's g_i converges to g in the $W^{1,p}$-sense. Assume that for all i, $R(g_i) \geq 0$ and $(*)$ as above.
Then $\int_M R(g)\, d\mathrm{vol}_g \geq \kappa$.
Moreover, in dimension 3, the assumption "$R(g_i) \geq 0$" is not needed.

Q.: Is "$R(g_i) \geq 0$" necessary (in dim. ≥ 4)?

Direct Corollary (H. 2022 [H1])

Let $p > n$. Let $\mathcal{M}$ be the space of all C^2-Riem. met.s on a closed mfd M.
For any nonnegative conti. fct. $\sigma : M \to [0, \infty)$ and $\kappa \in \mathbb{R}$, the space

$$\left\{ g \in \mathcal{M} \;\middle|\; \int_M R(g)\, d\mathrm{vol}_g \geq \kappa,\ R(g) \geq \sigma \text{ on } M \right\}$$

is $W^{1,p}$-closed in $\mathcal{M}$.

Rem: In Main thm 2, "$R_g \geq 0$" can be replaced with "$R_g \geq \sigma$ and $\mathrm{Vol}(M, g_i) \geq \mathrm{Vol}(M, g)$".

Corollary

- $p > n$.
- M^n : closed n-mfd.
- g : C^2-Riem. metric on M.

(g_i) : a sequence of C^2-Riem. metrics on M s.t. $g_i \xrightarrow{W^{1,p}} g$ on M and $\mathrm{Vol}(M, g_i) = 1$.
Assume that g is a Yamabe metric of $[g]$ $\left(\text{i.e., } Y(M,g) = \inf_{h \in [g]_1} \int_M R(h)\, \mathrm{dvol}_h = R(g)\right)$,
and $\exists \kappa \in \mathbb{R}$, $\exists \sigma \in C^0(M)$ s.t. $\forall i$,

$$Y(M, g_i) \geq \kappa \text{ and } R(g_i) \geq \sigma \text{ on } M.$$

Then $Y(M, g) \geq \kappa$.

More generally, we can also show the following.

Main Theorem 3 (H. 2022 [H1])

Let $p > n^2/2$. Let M^n be a closed n-manifold $(n \geq 2)$, g a C^2 Riem. met. on M, and (g_i) a sequence of C^2 Riem. met.s on M s.t. g_i converges to g on M in the $W^{1,p}$-sense as $i \to \infty$. Let m be a measure on M and set $e^{-f} d\mathrm{vol}_g := dm =: e^{-f_i} d\mathrm{vol}_{g_i}$. Assume the followings.

(1) $\exists \Lambda > 0$ s.t. f and f_i $(i \geq 0)$ are Λ-Lipschitz functions on M,

(2) $f_i \xrightarrow{C^0} f$ uniformly on M,

(3) $R(g_i) \geq 0$ on M for all i,

(4) $\int_M R(g_i)\, dm \geq \kappa \quad (\kappa \in \mathbb{R})$.

Then

$$\int_M R(g)\, dm \geq \kappa.$$

Corollary of Main thm 3

Let $p > n^2/2$. Let M be a closed n-manifold ($n \geq 2$) and g a C^2 Riemannian metric on M. Let κ be a positive continuous function on M. Let (g_i) be a sequence of metrics such that $g_i \in W^{1,p}$, $R(g_i) \geq \kappa$ in the distributional sense, and g_i converges to g in the $W^{1,p}$-sense. Then $R(g) \geq \kappa$ in the distributional sense.

Rem:

Since the limiting metric g is C^2, for any test function $\phi \in C^\infty(M)$, $\langle R_g, \phi \rangle = \int_M R_g \phi \, d\mathrm{vol}_g$.
Therefore, $R(g) \geq \kappa$ in the distributional sense $\Leftrightarrow$ $R(g) \geq \kappa$ in the classical sense.

A Gromov type of definition

Let M^n be a closed n-manifold $(n \geq 2)$ and g_0 a $W^{1,p}$ $(p > n^2/2)$ metric on M. Let κ be a positive continuous function on M. We say that g_0 is of $R(g_0) \geq \kappa$ if there exists a sequence of $W^{1,p}$ merics (g_i) on M such that

- $R(g_i) \geq \kappa$ in the distributional sense, and
- $g_i \to g_0$ with respect to the $W^{1,q}$ $(q > n^2/2)$ topology.

[G] and this definition

A difference of Gromov's definition and this one is that in this definition each metric in the approximate sequence can have some singularities. For example, on tori, there is NO metric g with $R(g) \geq \kappa > 0$ in the Gromov's sense ($\Leftrightarrow$ [BG]) from the resolution of Geroch's conjecture. In contrast, a metric g with $R(g) \geq \kappa > 0$ in the sense of this definition might exist on a torus. ($\rightsquigarrow$ Schoen's conjecture)

NOTE: The Morrey embedding says

$$C^1 \hookrightarrow W^{1,p} \hookrightarrow C^{0,1-\frac{n}{p}} \hookrightarrow C^0 \quad \text{if } p > n.$$

Therefore the same statement of Main Theorem 2 still holds even though one replace $W^{1,p}$ $(p > n)$ with $C^{0,\alpha}$ for all $\alpha \in (0, 1]$.

On the other hand, in Main Theorem 2, if we weaken the assumption from $W^{1,p}$ to C^0, then the same statement (without the assumption $R(g_i) \geq 0$) does NOT hold in general. Indeed, we will give some examples in the appendix.
All metrics g_i in such examples have sign-changing scalar curvatures, i.e., for each i, there are some points $x_i, y_i \in M$ s.t. $R(g_i)(x_i) < 0 < R(g_i)(y_i)$.

Questions

In dim 3, Main thm 2 follows from the fact that every orientable closed 3-mfd is papallelizable, and the Bochner identity for 1-forms. (As far as I know, the original idea is due to Lohkamp.)

Questions

- What is the relation between **parallelizability** and $W^{1,2}$-**convergence** of metric tensors?

  Fact M is closed, parallelizable $\Rightarrow \int_M R(g_i)\,\mathrm{dvol}_{g_i} \to \int_M R(g)\,\mathrm{dvol}_g$ as $g_i \xrightarrow{W^{1,2}} g$
  (Rem. Every oriented closed 3-mfd is parallelizable)
- Define $\int_M R(g)\,\mathrm{dvol}_g \geq \kappa$ for g with only $W^{1,p}$ $(p > n)$-regularity using a geometric flow and investigate its properties (cf. Burkhardt-Guim's work [BG] and [JSZ]).
- "$\int_M R(g)\,\mathrm{dvol}_g \geq \kappa$ in a dstributional sense" by Lee–LeFloch [LL] $\Rightarrow$ in the weaker sense defined as in the sense of [G] from Main thm 2 (for $g \in W^{1,p}$ $(p > n)$ with "$R(g) \geq 0$" in the sense of [G])?
- Is $p > n^2/2$ sharp in Main thm 3? (i.e., $\exists$counterexample for $p = n^2/2$?), $R(g) \rightsquigarrow$ "weighted scal." (in the integrand)?

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Fix a smooth background metric g_0 and denote the Levi-Civita connection $\bar{\nabla}$ of g_0.

Definition (Lee–LeFloch, [LL])

For $g \in L^\infty_{loc} \cap W^{1,2}_{loc}$ with $g^{-1} \in L^\infty_{loc}$, for every test function $u \in C_0^\infty(M)$, the *scalar curvature distribution* R_g is defined as

$$\langle R_g, u \rangle := \int_M \left(-V \cdot \overline{\nabla} \left(u \frac{\mathrm{dvol}_g}{\mathrm{dvol}_{g_0}} \right) + F u \frac{\mathrm{dvol}_g}{\mathrm{dvol}_{g_0}} \right) \mathrm{dvol}_{g_0},$$

where $V = (V^k) \in \Gamma(M)$ is given by $V^k := g^{ij}\Gamma^k_{ij} - g^{ik}\Gamma^j_{ji}$, F is a function as

$$F := R_{g_0} - \overline{\nabla}_k g^{ij} \Gamma^k_{ij} + \overline{\nabla}_k g^{ik} \Gamma^j_{ji} + g^{ij} \left(\Gamma^k_{kl}\Gamma^l_{ij} - \Gamma^k_{jl}\Gamma^l_{ik} \right)$$

and $\Gamma^k_{ij} := \frac{1}{2} g^{kl} \left(\overline{\nabla}_i g_{jl} + \overline{\nabla}_j g_{il} - \overline{\nabla}_l g_{ij} \right)$.
For $\kappa \in \mathbb{R}$, "$R(g) \geq \kappa$ in the distributional sense" if for any nonnegative test function $u \in C_0^\infty(M)$, $\langle R_g, u \rangle - \kappa \int_M u \, \mathrm{dvol}_g \geq 0$.

Def (Sormani–Tian–Wang, [STW])

The *total distributional scalar curvature* of metric g is defined as $\langle R_g, 1\rangle$.

Remark:

- $\langle R_g, u\rangle$ is independent of the choice of g_0, as long as g is in $C^0 \cap W^{1,2}_{\mathrm{loc}}$.
- For a metric g with the above regularity, one has

$$\Gamma^k_{ij} \in L^2_{loc},\ V \in L^2_{loc},\ F \in L^1_{loc} \text{ and } \frac{\mathrm{dvol}_g}{\mathrm{dvol}_{g_0}} \in L^\infty_{loc} \cap W^{1,2}_{loc}.$$

  Therefore

$$\int_M \left(-V \cdot \overline{\nabla}\left(\frac{\mathrm{dvol}_g}{\mathrm{dvol}_{g_0}}\right)\right) \mathrm{dvol}_{g_0} \quad \text{and} \quad \int_M \left(Fu\frac{\mathrm{dvol}_g}{\mathrm{dvol}_{g_0}}\right) \mathrm{dvol}_{g_0}$$

  are both finite.
- BUT the total distributional scalar curvature $\langle R_g, 1\rangle$ may be infinite.

Approximation lemma 1 ([GL, Lemma 4.1])

Let M^n be a cpt smooth mfd and g a $C^0 \cap W^{1,p}$ $(1 \leq p \leq \infty)$ metric on M, then $\exists$ $(g_\delta)_{\delta>0}$ s.t. g_δ converges to g both in the C^0-norm and in the $W^{1,p}$-norm as $\delta \to 0^+$.

Approximation lemma 2 ([JSZ, Lemma 2.2])

Let M^n be a cpt smooth mfd and g a $C^0 \cap W^{1,n}$ metric on M. Let (g_δ) be the approximation in the previous lemma. Then, $\exists \varepsilon > 0, \exists \delta_0 = \delta_0(g) > 0$ s.t.

$$|\langle R_{g_\delta}, u\rangle - \langle R_g, u\rangle| \leq \varepsilon ||u||_{W^{1,\frac{n}{n-1}}(M)}, \quad \forall u \in C^\infty(M), \ \forall \delta \in (0, \delta_0).$$

- When M is open, we can construct the following counterexamples.

Fix $r_0 > 0$. Consider $\left(\mathbb{R}^n,\ g_i := u_i^{\frac{4}{n-2}} \cdot g_{Eucl}\right)$ $(n \geq 3,\ i = 2, 3, \cdots)$. Here, $u_i : \mathbb{R}^n \to \mathbb{R}$ is defined by

$$u_i := \phi\left(i^{-1}\sin(ir^2)\right) + 1,$$

where $r(\cdot) := |o - \cdot|_{Eucl}$ and $\phi : \mathbb{R}^n \to [0, 1]$ is a smooth cut-off fct. s.t. $\phi \equiv 1$ on $\overline{B_{r_0}} := \{x \in \mathbb{R}^n |\ r(x) \leq r_0\}$ and $\phi \equiv 0$ outside of B_{2r_0}. Then,

  - u_i is smooth positive, $u_i - 1$ is compactly supported, and g_i is complete smooth $(i = 2, 3, \cdots)$,
  - $g_i \xrightarrow{C^0} g_{Eucl.}$ on $\mathbb{R}^n$ uniformly. **BUT** $g_i \overset{C^1}{\not\to} g_{Eucl.}$,
  - $\int_{\mathbb{R}^n} R(g_i)\, \mathrm{dvol}_{g_i} \geq \exists\kappa(n) > 0 = \int_{\mathbb{R}^n} R(g_{Eucl.})\, \mathrm{dvol}_{g_{Eucl.}}$.

- In the same manner, we can also construct the following example on a closed manifold:

When (M, g_0) is closed, $\exists g_i \in \mathcal{M}(M)$ $(n \geq 3)$ s.t. $g_i \overset{C^0}{\to} g_0$ on M. **BUT** $g_i \overset{C^1}{\not\to} g_0$ and

$$\int_M R(g_i)\, \mathrm{dvol}_{g_i} \geq \kappa + \int_M R(g_0)\, \mathrm{dvol}_{g_0} \quad \text{for some } \kappa = \kappa(n) > 0.$$

Counterexamples (Continuation)

Consider $\left(\mathbb{R}^n,\ g_i := u_i^{\frac{4}{n-2}} \cdot g_{Eucl}\right)$ $(n \geq 3,\ i = 2, 3, \cdots)$. Here, $u_i : \mathbb{R}^n \to \mathbb{R}$ is defined by

$$u_i := \phi_i \left(i^{-2} \sin(ir^2)\right) + 1,$$

where $\phi_i : \mathbb{R}^n \to [0, 1]$ is a smooth cut-off fct. s.t. $\phi_i \equiv 1$ on $\overline{B_{r_i}}$ $(r_i := i^{\frac{2}{n+2}})$ and $\phi_i \equiv 0$ outside of B_{2r_i}. Then,

- u_i is positive smooth $(i = 2, 3, \cdots)$, and $r_i \to \infty$,
- g_i complete smooth $(i = 2, 3, \cdots)$,
- $g_i \xrightarrow{C^1} g_{Eucl.}$ on $\mathbb{R}^n$ uniformly. **BUT** $g_i \overset{C^2}{\not\to} g_{Eucl.}$,
- $\int_{\mathbb{R}^n} R(g_i)\, \mathrm{dvol}_{g_i} \geq \exists \kappa(n) > 0 = \int_{\mathbb{R}^n} R(g_{Eucl.})\, \mathrm{dvol}_{g_{Eucl.}}$.

Counterexamples (Continuation)

Consider $\left(\mathbb{R}^2, e^{u_i} g_{Eucl}\right)$, where

$$u_i := e^{-ir^2} \sin\left(-\frac{i}{2} r^2\right) \quad (i = 1, 2, \cdots).$$

Then,

- g_i complete smooth $(i = 1, 2, \cdots)$,
- $g_i \xrightarrow{C^1} g_{Eucl.}$ on $\mathbb{R}^n$ uniformly. **BUT** $g_i \overset{C^2}{\nrightarrow} g_{Eucl.}$,
- $\int_{\mathbb{R}^2} R(g_i)\, \mathrm{dvol}_{g_i} = \frac{128\pi}{25} > 0 = \int_{\mathbb{R}^2} R(g_{Eucl.})\, \mathrm{dvol}_{g_{Eucl.}}$.

Note: All metrics g_i in the above examples have sign-changing scalar curvatures, i.e., for each i, there are points $x_i, y_i \in M$ s.t. $R(g_i)(x_i) < 0 < R(g_i)(y_i)$.

Further Questions

Llarull's Sphere Rigidity Theorem (1998 [Ll])

(M^n, g) : a closed spin Riemannian mfd. If $f : (M, g) \to (S^n, g_{\mathrm{std}})$ is a dist. decreasing map of non-zero degree and if $R(g) \geq n(n-1)$, then f must be a Riemannian isometry.

Lee–Tam (2022 [LT]) (A weak top. ver. of Llarull's thm)

M^n : a closed spin mfd and g_0 a C^0-met. on M with $R(g_0) \geq n(n-1)$ in the "Gromov sense". Suppose there is a 1-Lip. conti. map $f : (M, d_{g_0}) \to (S^n, d_{\mathrm{std}})$ with non-zero degree, then f is a dist. isometry.

- Recall: For a C^0-met. g and a conti. fct. κ, we say "$R(g) \geq \kappa$ *in the Gromov sense*" if $\exists C^2$-Riem. met's (g_i) s.t. $R(g_i) \geq \kappa$ and $g_i \xrightarrow{C^0} g$.
- A stability version of Llarull's theorem (Allen-Bryden-Kazaras [ABK]).

Further Questions

For example, the following is known.

Green (1963 [Gr])

Let (M^n, g) be a closed Riem. mfd w/ $(\mathrm{Vol}(M,g))^{-1}\int_M R(g)\,d\mathrm{vol}_g$ is at least $n(n-1)$. Then the conjugate radius $\mathrm{conj}(M,g)$ of (M,g) is $\leq \pi$. If "$=$", then (M,g) has $\sec \equiv 1$.

Very Rough Question

Does a weaker topology version of this hold? (This question, of course, also includes what the appropriate statement is.)

About the new definition

(Shi–Tam, [ST] Corollary 4.11)

Let (M^n, g) be a compact manifold s.t. M is the topological torus and g is smooth away from some compact set $\Sigma \subset M$ with codimension at least 2. Here, "codimension at least 2" means that $\mathrm{Vol}(\Sigma_\varepsilon, g) = O(\varepsilon^2)$, where $\Sigma_\varepsilon := \{x \in M \mid d_g(x, \Sigma)\}$. Moreover, assume $g \in W^{1,p}_{\mathrm{loc}}(M)$ for some $p > n$. Suppose $R(g) \geq 0$ in $M \setminus \Sigma$, then g must be flat.

(Jiang–Sheng–Zhang, [JSZ] Lemma A.1)

Let M^n be a smooth manifold and $g \in L^\infty \cap W^{1,p}_{\mathrm{loc}}(M)$ $(n \leq p \leq \infty)$ a metric on M. Let Σ be a closed subset of M. Assume that $g \in C^\infty(M \setminus \Sigma)$ (or $g \in C^2(M \setminus \Sigma)$). Suppose $\mathcal{H}^{n-\frac{p}{p-1}}(\Sigma) < +\infty$ if $n < p < \infty$, $\mathcal{H}^{n-1}(\Sigma) = 0$ if $p = \infty$, and $R(g) \geq \kappa$ on $M \setminus \Sigma$. Here, κ is a constant $\kappa \in \mathbb{R}$. Then, $R(g) \geq \kappa$ on M in the distributional sense.

Q. How do we know that a metric g is of $R(g) \geq \kappa$ in the distributional sense in general?

About the new definition

(Lee–LeFloch, [LL] Proposition 5.1)

Let M_i $(i = 1, 2)$ be smooth n-manifolds with boundaries, carrying C^2 (up to boundaries) merics g_i $(i = 1, 2)$ respectively. Assume $\exists \Phi : (\partial M_1, g_1) \to (\partial M_2, g_2)$ an isometry. Let $(M, g := g_1 \sqcup g_2)$ be tha manifold obtained by gluing these along Φ. Let Σ be athe identification of these boundaries in M. Let H_i $(i = 1, 2)$ be the (scalar) mean curvatures computed w.r.t. g_i $(i = 1, 2)$ respectively. (Locally, for a Fermi coordinates $(x_0, x_1, \cdots, x_{n-1})$ where $x_0 < 0$ corresponds to M_1, $H_1 = g_1 (\nabla_{\partial_i} \partial_i, -\partial_0)$ and $H_2 = g_2 (\nabla_{\partial_i} \partial_i, \partial_0)$.) Assume that $R(g_1)$, $R(g_2) \geq \kappa$, and that at each point Σ, $H_1 \geq H_2$.
Then, $R(g) \geq \kappa$ on M in the distributional sense.

(cf. P. Miao, Positive mass theorem on manifolds admitting corners along a hypersurface, Adv. Theor. Math. Phys. **6** (2002), 1163–1182.)

NOTE: Conversely, if there exists a point $p \in \Sigma$ at which $H_1(p) < H_2(p)$, then we can show that $R(g) \ngeq \kappa$ in the distributional sense.

About the new definition

- Burkhardt-Guim [BG] suggested a sequence of metrics such as the one in the definition of Gromov can be taken along a Ricci flow. Roughly speaking, this is based on the fact that Ricci flows preserve the minimum of the scalar curvatures.
- Jiang–Sheng–Zhang [JSZ] proved that such property is also true for every $W^{1,p}$ $(p > \mathrm{dim})$ metric, and thereby proved that $R(g) \geq \kappa$ in the distributional sense $\Rightarrow$ $R(g) \geq \kappa$ in the sense of [G]($\Leftrightarrow$ [BG]).
- BUT, such a Ricci flow $g(t)$ is smooth for $t > 0$. Therefore, it CANNOT be used to construct a NONTRIVIAL example of the new definition.

About the new definition

- For example, in the above situation of Lee–LeFloch (or Miao), it is conceivable to use a Ricci flow *with boundary*. However, some convexities of the boundary are required to obtain the above scal-preservation-property. (i.e., Such converxities are necessary to apply a maximum principle.)
- Unfortunately, as far as I know, such convexities are stronger than the Bartnik's boundary condition (i.e., the condition in the above statement of Lee–LeFloch).
- Hence, at least on a torus, this method cannot be used because there is no psc metric on the torus. (i.e., If one can construct a sequence of psc C^2-metric on $[0, 1] \times \mathbb{T}^{n-1}$ with Bartnik's boundary conditions, then from the result of Miao, one can also construct a psc metric on the whole torus $\mathbb{T}^n$ by gluing them. However, such a metric cannot exist from the resolution of Geroch's conjecture.)

About the new definition

- RF on mfds with boundary (related literatures)
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About the new definition

Q.: $\exists$PSC metrics on a torus with prescribed singular set of non-integer dim.?

- PSC metrics with prescribed singularities on $\mathbb{R}^n$ or $\mathbb{S}^n$(related literatures)

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