インフレーション宇宙における赤外発散問題とLargeゲージ変換

Takahiro Tanaka
Kyoto University

in collaboration with
Yuko Urakawa

JCAP 07 (2021) 051: arXiv:2101.05707
JHEP 1710 127:arXiv:1707.05485
JCAP 1606 020:xrXiv:1510.05059
PTEP2014 073E01 :arXiv:1402.2076
CQG 30 233001:arXiv:1306.4461
PTEP2013 063E02:arXiv:1301.3088
PTEP2013 083E01 :arXiv:1209.XXXX
PTP125 1067 arXiv:1009.2947
Phys.Rev.D82 121301 arXiv:1007.0468
PTP122 779 arXiv:0902.3209
PTP122 1207 arXiv:0904.4415

Various IR issues

IR divergence coming from k-integral

Secular growth in time $\propto (Ht)^n$

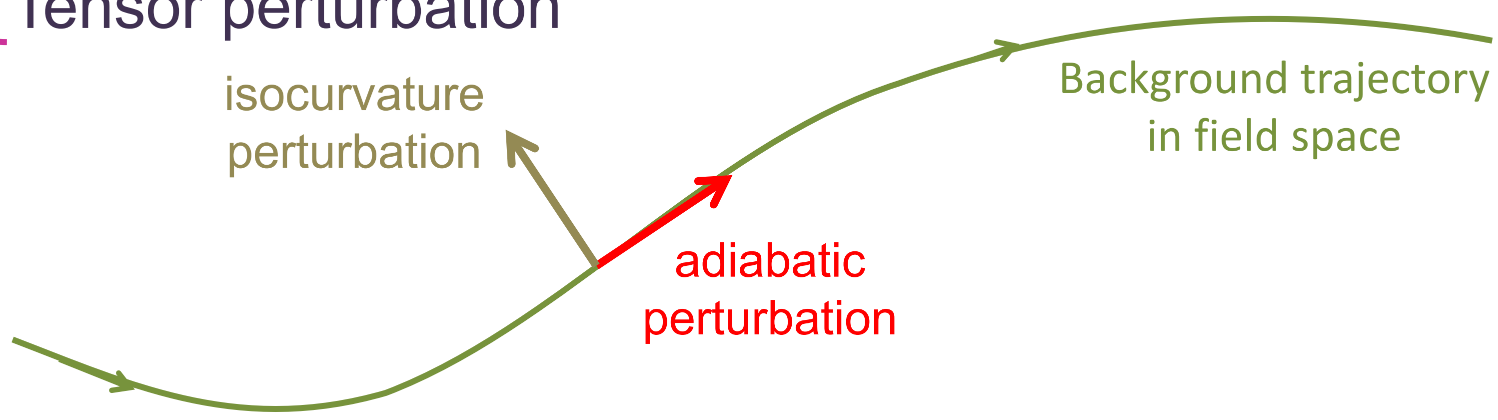
Adiabatic perturbation,

which can be locally absorbed by the choice of time slicing.

Isocurvature perturbation

$\approx$ field theory on a fixed curved background

Tensor perturbation



Isocurvature perturbation

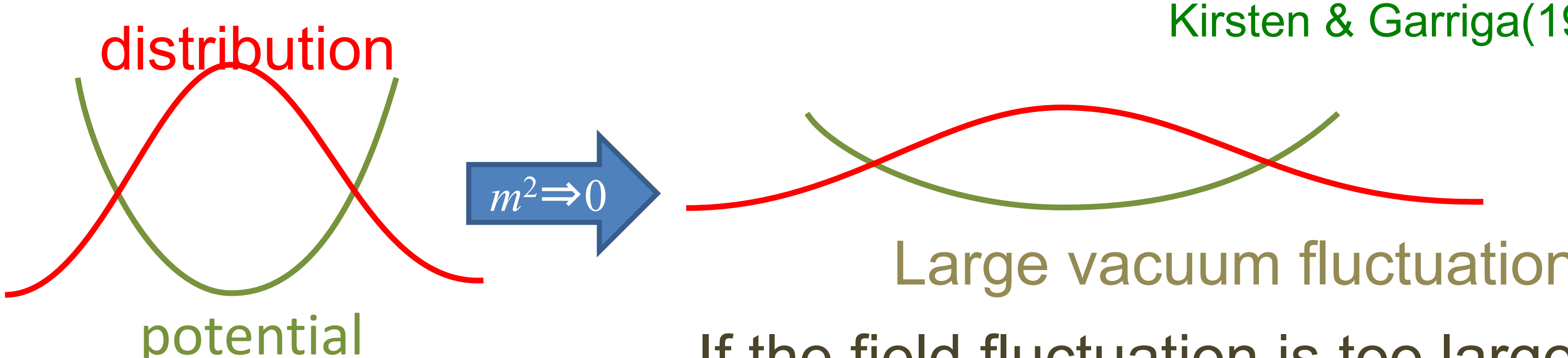
Subtle issue arises in the small mass limit.

Summing up only long wavelength modes beyond the Horizon scale

$$\left\langle \phi^2 \right\rangle_{reg} \approx \int_0^{aH} d^3k \, \frac{H^2}{k^3} \left(\frac{k}{aH} \right)^{\frac{2m^2}{3H^2}} \approx \frac{H^4}{m^2}$$

$\Leftrightarrow$ De Sitter inv. vac. state does not exist in the massless limit.

Allen & Folacci(1987)
Kirsten & Garriga(1993)



If the field fluctuation is too large, it is easy to imagine that a naïve perturbative analysis will break down once interaction is turned on.

Stochastic interpretation

(Starobinsky & Yokoyama (1994))

- Let's consider local average of ϕ :

$$\bar{\phi} = \int_0^{aH} d^3k\, \phi_k e^{ikx}$$

More and more short wavelength modes participate in $\bar{\phi}$ as time goes on.

Equation of motion for $\bar{\phi}$:

$$\frac{d^2\bar{\phi}}{dt^2} + 3H\frac{d\bar{\phi}}{dt} = -V'(\bar{\phi}) + f$$

Newly participating modes act as random fluctuation

$$\langle \phi_k \phi_{-k} \rangle \approx H^2/k^3$$

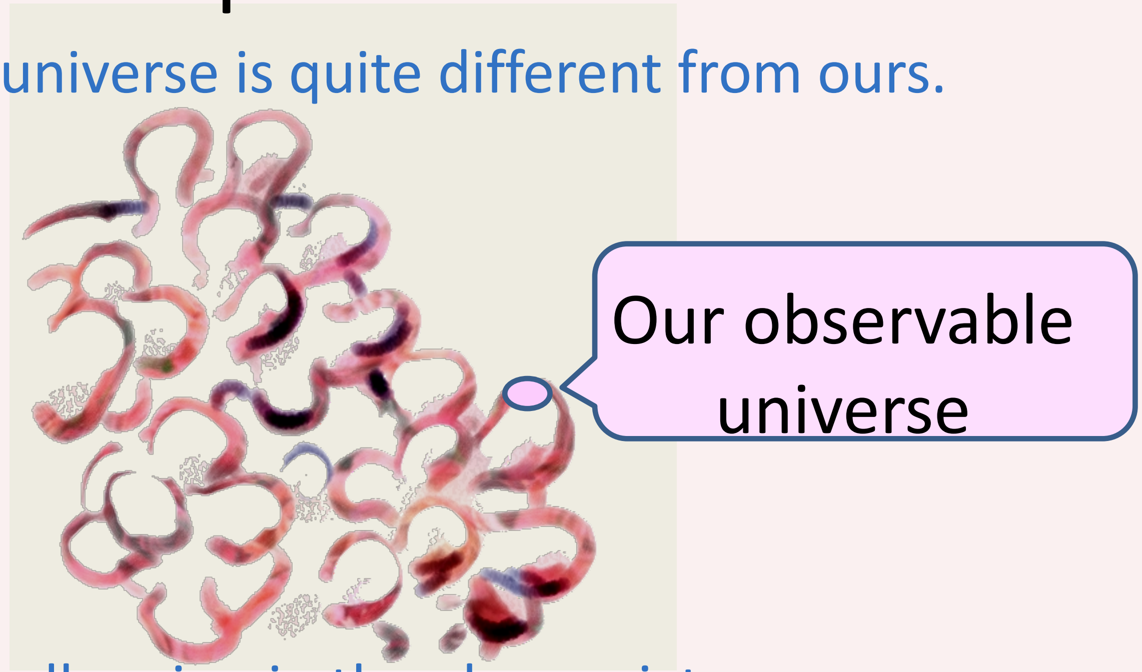
$$\Rightarrow \langle f(t) f(t') \rangle \approx H\delta(t-t')$$

In the case of massless $\lambda\phi^4$: $\langle \bar{\phi}^2 \rangle \to \dfrac{H^2}{\sqrt{\lambda}}$

Namely, in the end, thermal equilibrium is realized : $V \approx T^4 \approx H^4$

Wave function of the universe ~parallel universes

- Distant universe is quite different from ours.



- Each small region in the above picture gives one representation of many parallel universes.
- However: wave function of the universe = “a superposition of all the possible parallel universes”
  must be so to keep translational invariance of the wave fn. of the universe
- Do “simple expectation values give really observables for us?”

No!

§ IR divergence in single field inflation

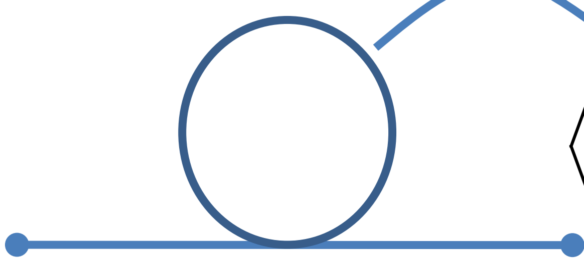
Setup: 4D Einstein gravity + minimally coupled scalar field

Single field case is special because broadening of averaged field can be absorbed by an appropriate choice of the time coordinate.

But naively we still have IR problem.

$$\left\{ \begin{array}{l} \gamma_{ij} = e^{2\rho+2\zeta} \exp(h)_{ij} \\ \delta\phi = 0 \end{array} \right.$$

Transverse traceless



Factor coming from this loop:

$$\langle \zeta(y)\zeta(y) \rangle \approx \int d^3k \, \underline{P(k)} \approx \log(aH / k_{\min})$$

$\propto 1/k^3$

scale invariant spectrum
- no typical mass scale

curvature perturbation in co-moving gauge.

Gauge issue in single field inflation

Yuko Urakawa and T.T., PTP122: 779 arXiv:0902.3209

– In conventional cosmological gauge invariant perturbation theory, gauge is not completely fixed.

Time slicing can be uniquely specified: $\delta\phi = 0$ OK!

but spatial coordinates are not.

$$h^j_j = 0 = h^j_{i,j}$$

Residual gauge:

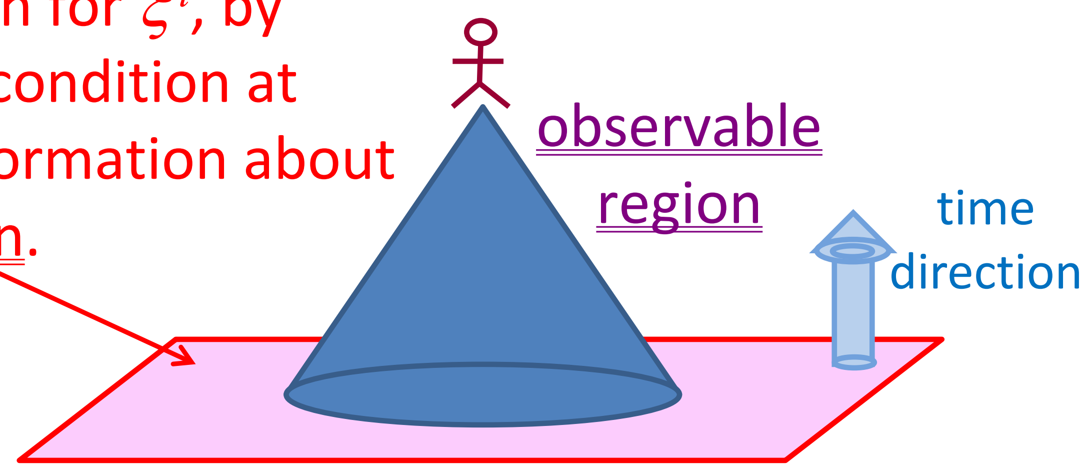
$$\delta_g h_{ij} = \xi_{i,j} + \xi_{j,i}$$

Elliptic-type differential equation for ξ^i.

$$\Delta \xi^i = \cdots$$

Not unique locally!

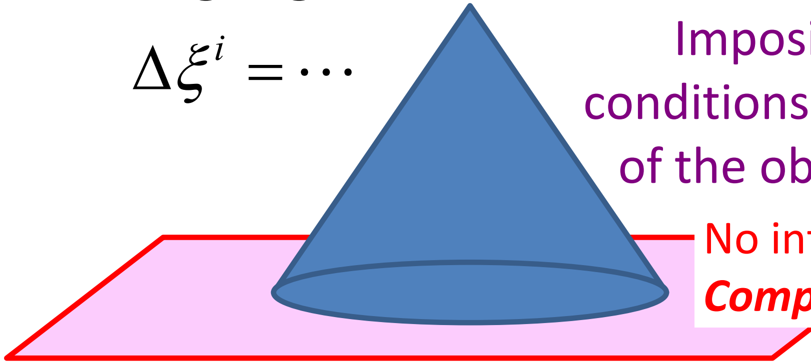
- To solve the equation for ξ^i, by imposing boundary condition at infinity, we need information about <u>un-observable region</u>.



Complete gauge fixing vs. Genuine gauge-invariant quantities

- Local gauge conditions.

$$\Delta \xi^i = \cdots$$



Imposing boundary conditions on the boundary of the observable region

No influence from outside

Complete gauge fixing ☺

But unsatisfactory?

The results depend on the choice of boundary conditions.

Translation invariance of the initial state is lost.

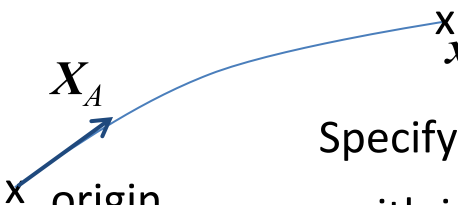
◆ ***Genuine*** coordinate-independent quantities.

Correlation functions for 3-d scalar curvature on ϕ =constant slice.

$\langle R(\boldsymbol{x}_1)\, R(\boldsymbol{x}_2)\rangle$ Coordinates do not have gauge invariant meaning.

(Giddings & Sloth 1005.1056)
(Byrnes et al. 1005.33307)

Use of geodesic coordinates:



$\boldsymbol{x}(\boldsymbol{X}_A, \lambda{=}1) = \boldsymbol{X}_A + \delta \boldsymbol{x}_A$

Specify the position by solving geodesic eq. $D^2 x^i / d\lambda^2 = 0$

with initial condition $Dx^i / d\lambda \big|_{\lambda=0} = X^i$

${}^{g}R(\boldsymbol{X}_A) := R(\boldsymbol{x}(\boldsymbol{X}_A, \lambda{=}1)) = R(\boldsymbol{X}_A) + \delta \boldsymbol{x}_A \nabla R(\boldsymbol{X}_A) + \ldots$

$\langle {}^{g}R(\boldsymbol{X}_1)\, {}^{g}R(\boldsymbol{X}_2)\rangle$ should be truly coordinate independent.

Extra requirement for IR regularity

If we compute two point correlation to one-loop order, we obtain

$$\left\langle {}^gR(\boldsymbol{x}_1)\, {}^gR(\boldsymbol{x}_2)\right\rangle \ni \underline{\left\langle \zeta_I^2 \right\rangle} \left\langle \Delta\left(2\mathcal{L}^{-1}e^{-2\rho}\Delta + \boldsymbol{x}\cdot\partial_{\boldsymbol{x}}\right)\zeta_I(\boldsymbol{x}_1) \times \Delta\left(2\mathcal{L}^{-1}e^{-2\rho}\Delta + \boldsymbol{x}\cdot\partial_{\boldsymbol{x}}\right)\zeta_I(\boldsymbol{x}_1)\right\rangle$$

IR divergent factor

with $\mathcal{L}^{-1}$ being the formal inverse of the linearised EOM

$$\mathcal{L} = \partial_t^2 + (3+\varepsilon_2)\dot{\rho}\,\partial_t - e^{-2\rho}\Delta$$

We can eliminate such IR singular terms by choosing the mode function to satisfy

$$-2k^2\mathcal{L}_k^{-1}e^{-2\rho}v_k = D_k e^{i\phi(\mathbf{k})}v_k$$

where

$$D_k \equiv k^{-3/2}e^{-i\phi(\mathbf{k})}\frac{d}{d\log k}k^{3/2}e^{i\phi(\mathbf{k})}$$

and

$$\zeta_I \equiv \int d^3k\,(e^{i\boldsymbol{k}x}\,v_{\boldsymbol{k}}(t)\,a_{\boldsymbol{k}} + h.c.),$$

Physical meaning of IR regularity condition

In addition to considering ${}^{g}R$, we need additional conditions

$$-2k^2 \mathcal{L}_k^{-1} e^{-2\rho} v_k = D_k v_k$$ and its higher order extension.

What is the physical meaning of these conditions?

Background gauge: $\tilde{\boldsymbol{x}} = e^s \boldsymbol{x} \qquad \tilde{\zeta}(\tilde{\boldsymbol{x}}) = \zeta(\boldsymbol{x})$

$$ds^2 = -dt^2 + e^{2\rho} d\boldsymbol{x}^2 \Longrightarrow d\tilde{s}^2 = -dt^2 + e^{2\rho - 2s} d\tilde{\boldsymbol{x}}^2$$

$$H = H_0[\zeta] + H_{\text{int}}[\zeta] \Longrightarrow \underline{\tilde{H} = H_0[\tilde{\zeta}] + H_{\text{int}}[\tilde{\zeta} - s]}$$

- Quadratic part in $\tilde{\zeta}$ and s is identical to $s = 0$ case.
- Interaction Hamiltonian is obtained just by replacing the argument ζ with $\tilde{\zeta} - s$.

Therefore, one can use

1) common mode functions for ζ_I and $\tilde{\zeta}_I$

$$\zeta_I \equiv \int d^3k \, (e^{ikx} v_{\boldsymbol{k}}(t)\, a_{\boldsymbol{k}} + h.c.) \Longrightarrow \tilde{\zeta}_I \equiv \int d^3k \, (e^{ikx} v_{\boldsymbol{k}}(t)\, \tilde{a}_{\boldsymbol{k}} + h.c.)$$

2) common iteration scheme.

$$\zeta = \zeta_I + \delta\zeta[\zeta_I] \Longrightarrow \tilde{\zeta} = \tilde{\zeta}_I + \delta\zeta[\tilde{\zeta}_I - s]$$

We may require

$$\langle 0|\zeta(\boldsymbol{x}_1)\zeta(\boldsymbol{x}_2)\cdots\zeta(\boldsymbol{x}_n)|0\rangle = \langle\tilde{0}|\tilde{\zeta}(\tilde{\boldsymbol{x}}_1)\tilde{\zeta}(\tilde{\boldsymbol{x}}_2)\cdots\tilde{\zeta}(\tilde{\boldsymbol{x}}_n)|\tilde{0}\rangle$$

$$\Longrightarrow \quad -2k^2\mathcal{L}_k^{-1}e^{-2\rho}v_k = D_k v_k$$

the previous condition for the absence of IR divergence

$$D_k \equiv k^{-3/2}e^{-i\phi(\mathbf{k})}\frac{d}{d\log k}k^{3/2}e^{i\phi(\mathbf{k})}$$

"Wave function must be homogeneous in the direction of background scale transformation"

It looks quite non-trivial to find consistent IR regular states.

However, the Euclidean vacuum state (defined by the regularity $\eta_0 \to \pm i\infty$) satisfies this condition.

Large gauge transformation

Nambu-Goldstone's theorem:
When global symmetry is spontaneously broken, a massless degree of freedom appears.

Gauge symmetry (functional d.o.f.)
≠ Global symmetry (finite parameters)

However, after fixing the gauge imposing local gauge conditions, we are left with finite residual gauge transformation.

$$h_{i\ ,j}^{\ j} = 0$$

$$x \to x' = e^s x$$

Global dilatation transformation is one of such residual gauge dof.

The reduced Lagrangian after gauge fixing still has the symmetry under such residual gauge transformations, which can be understood as global gauge transformation.

= "Large gauge transformation"

Dilatation charge

$$Q_\zeta = \frac{1}{2}\int d^3x \left[\Delta_s \zeta\, \pi + \sum_\alpha \Delta_s \varphi^{(\alpha)}\, \pi_{(\alpha)} + (h.c.) \right]$$

$$s\Delta_s \zeta = \zeta\left(t, e^{-s}x\right) - \left(s + \zeta(t,x)\right) \approx -s\left(1 + \boldsymbol{x}\cdot\partial_{\boldsymbol{x}}\zeta\right)$$

$$s\Delta_s \varphi^{(\alpha)} = \varphi^{(\alpha)}\left(t, e^{-s}x\right) - \varphi^{(\alpha)}\left(t,x\right) \approx -s\boldsymbol{x}\cdot\partial_{\boldsymbol{x}}\varphi^{(\alpha)}$$

$$\begin{cases} \gamma_{ij} = e^{2\rho+2\zeta}\exp(h)_{ij} \\ \delta\phi = 0 \end{cases}$$

Transverse traceless

Physics should be invariant under the action of dilatation

$\left[Q_\zeta, H\right] = 0$ --- <u>dilatation charge conservation</u>

We also request the invariance of the quantum state

$Q_\zeta |\Psi\rangle = 0$ --- ☆ What does this condition imply?

Averaged field: $\bar{\zeta} = L^{-3}\int d^3\boldsymbol{x}\,\zeta(\boldsymbol{x})$

$$|\Psi\rangle = \int d\bar{\zeta}^c \underline{\psi(\bar{\zeta}^c)}\,|\bar{\zeta}^c\rangle |\Psi\rangle_{\bar{\zeta}^c}$$

Real wave-function

eigenstate

$$\bar{\zeta}|\bar{\zeta}^c\rangle = \bar{\zeta}^c|\bar{\zeta}^c\rangle$$

$$iQ_\zeta|\bar{\zeta}^c\rangle = \frac{d}{d\bar{\zeta}^c}|\bar{\zeta}^c\rangle$$

$$☆\Rightarrow \quad 0 = \int d\bar{\zeta}^c \left[-\frac{d\psi(\bar{\zeta}^c)}{d\bar{\zeta}^c}|\bar{\zeta}^c\rangle|\Psi\rangle_{\bar{\zeta}^c} + \psi(\bar{\zeta}^c)|\bar{\zeta}^c\rangle\left(iQ_\zeta - \frac{d}{d\bar{\zeta}^c}\right)|\Psi\rangle_{\bar{\zeta}^c}\right]$$

$$\Rightarrow \quad \frac{d}{d\bar{\zeta}^c}\psi(\bar{\zeta}^c) = 0$$ ---(1) Flatness of wave fn. in the $\bar{\zeta}$ direction

$$iQ_\zeta|\Psi\rangle_{\bar{\zeta}^c} = \frac{d}{d\bar{\zeta}^c}|\Psi\rangle_{\bar{\zeta}^c}$$ ---(2)

Effect of the change of $\bar{\zeta}$ to other modes

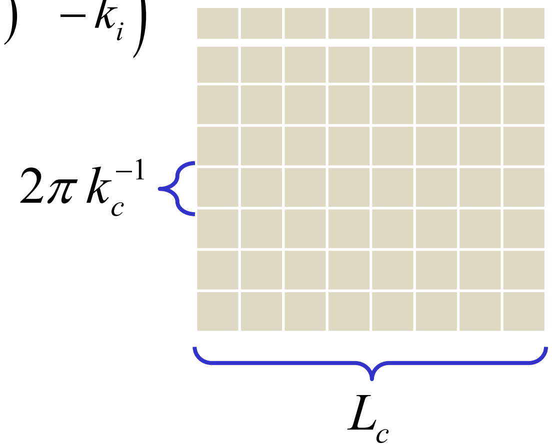
= Action of large gauge transformation

Extension of dilatation charge

Introduce discrete Fourier modes by focusing on finite size box:

$$\tilde{\zeta}_{\boldsymbol{k}}(t) := \int \frac{d^3\boldsymbol{k}'}{(2\pi)^3} W(\boldsymbol{k}-\boldsymbol{k}')\zeta_{\boldsymbol{k}'}(t) \qquad W(\boldsymbol{k}) = \prod_{i=1}^{3}\theta\left((2L_c)^{-1} - k_i\right)$$

We do not care about large scales beyond the observable region.



Inhomogeneous extension of dilatation charge:

$$Q_\zeta^W(\boldsymbol{k}) := \frac{L_c^3}{2}\int \frac{d^3\boldsymbol{k}'}{(2\pi)^3} W(\boldsymbol{k}-\boldsymbol{k}')\left[\Delta_s \zeta\, \pi + \cdots\right]_{\boldsymbol{k}'}$$

$$\left[iQ_\zeta^W(\boldsymbol{k}_L), \tilde{\zeta}_{\boldsymbol{p}_L}\right] \approx \delta_{\boldsymbol{k}_L, -\boldsymbol{p}_L} \qquad \left[iQ_\zeta^W(\boldsymbol{k}_L), \zeta_{\boldsymbol{p}_S}\right] \approx \partial_{\boldsymbol{p}_S}\left(\boldsymbol{p}_S \zeta_{\boldsymbol{p}_S + \boldsymbol{k}_L}\right)$$

Extension of charge conservation:

$$\left[iQ_\zeta^W(\boldsymbol{k}_L), H\right] \approx 0$$

This is not trivial, once the system is reduced by eliminating spatial derivative constraints.

When the constraint equation is like

$$\Delta\left(\delta N - f(\zeta)\right) = 0 \qquad \delta N_{k=0} = 0 \Leftrightarrow \lim_{k\to 0}\delta N_k = f_{k=0}(\zeta)$$

discontinuity

Eigenstate of soft modes

$$\tilde{\zeta}_{\boldsymbol{k}_L}\left|\tilde{\zeta}^c_L\right\rangle=\tilde{\zeta}^c_{\boldsymbol{k}_L}\left|\tilde{\zeta}^c_L\right\rangle$$

Decomposition of wave function

$$|\Psi\rangle=\int D\tilde{\zeta}^c_L\,\psi\left(\tilde{\zeta}^c_L\right)\left|\tilde{\zeta}^c_L\right\rangle|\Psi\rangle_{\tilde{\zeta}^c_L}$$

Locality condition

$$iQ^W_\zeta(\boldsymbol{k}_L)|\Psi\rangle_{\tilde{\zeta}^c_L}=\frac{\partial}{\partial\tilde{\zeta}^c_{\boldsymbol{k}_L}}|\Psi\rangle_{\tilde{\zeta}^c_L}\quad\Longleftarrow\quad iQ_\zeta|\Psi\rangle_{\bar{\zeta}^c}=\frac{d}{d\bar{\zeta}^c}|\Psi\rangle_{\bar{\zeta}^c}\qquad\text{---(2)}$$

Effect of the change of soft modes to hard modes

= Action of the inhomogeneous extension of dilatation

$$Q^W_\zeta(\boldsymbol{x}_L):=\left(\frac{2\pi}{k_cL_c}\right)^3\sum_{\boldsymbol{k}_L}e^{i\boldsymbol{k}_L\boldsymbol{x}_L}Q^W_\zeta(\boldsymbol{k}_L)$$

$$\zeta_L(x):=\sum_{\boldsymbol{k}_L}e^{i\boldsymbol{k}'\boldsymbol{x}}\tilde{\zeta}_{\boldsymbol{k}'}(t)\approx\int_{k'<k_c}\frac{d^3\boldsymbol{k}'}{(2\pi)^3}e^{i\boldsymbol{k}'\boldsymbol{x}}\tilde{\zeta}_{\boldsymbol{k}'}(t)\ \Rightarrow\ \tilde{\zeta}_{\boldsymbol{k}_L}(t)=\left(\frac{2\pi}{k_cL_c}\right)^3\sum_{\boldsymbol{x}_L}e^{i\boldsymbol{k}_L\boldsymbol{x}_L}\zeta_L(x_L)$$

$$\Rightarrow\ iQ^W_\zeta(\boldsymbol{x}_L)|\Psi\rangle_{\tilde{\zeta}^c_L}=\left(\frac{2\pi}{k_cL_c}\right)^3\sum_{\boldsymbol{k}_L}e^{i\boldsymbol{k}_L\boldsymbol{x}_L}\frac{\partial}{\partial\tilde{\zeta}^c_{\boldsymbol{k}_L}}|\Psi\rangle_{\tilde{\zeta}^c_L}=\frac{\partial}{\partial\tilde{\zeta}^c_L(\boldsymbol{x}_L)}|\Psi\rangle_{\tilde{\zeta}^c_L}$$

Locality condition= “Local action of dilatation to the short wavelength modes is identical to the change of embedding in the wave function of the whole universe”

$$iQ^W_\zeta(\boldsymbol{k}_L)|\Psi\rangle\neq0$$

completely different from the dilatation symmetry.

Another formulation of consistency relation

(Soft theorem in cosmology)

$$-P_\zeta(\boldsymbol{k}_L)\langle\Psi|\partial_{\boldsymbol{p}_S}(\boldsymbol{p}_S O_S(\boldsymbol{p}_S+\boldsymbol{k}_L))|\Psi\rangle = \langle\Psi|\tilde{\zeta}_{\boldsymbol{k}_L} O_S(\boldsymbol{p}_S+\boldsymbol{k}_L)|\Psi\rangle$$

conventional consistency relations

- ***Dilatation invariant quantities***

Correlation functions for 3-d scalar curvature on ϕ =constant slice.

$\langle R(\boldsymbol{x}_1)\, R(\boldsymbol{x}_2)\ldots\, R(\boldsymbol{x}_\mathrm{n})\,\rangle$

Coordinates do not have dilatation invariance.

Use of rescaled coordinates:

$${}^{(g)}O_S(\{\boldsymbol{x}_i\}) := O_S\left(\left\{e^{\tilde{\zeta}_L}\boldsymbol{x}_i\right\}\right) = O_S(\{\boldsymbol{x}_i\}) + \tilde{\zeta}_L \sum_j \boldsymbol{x}_j \cdot \nabla_j O_S(\{\boldsymbol{x}_i\})$$

$\langle{}^{(g)}O\,\rangle$ should be invariant under dilatation.

More accurate consistency relation:

$$\left\langle f\left(\left\{\tilde{\zeta}_L\right\}\right){}^{(g)}O_S(\{\boldsymbol{x}_i\})\right\rangle = \left\langle f\left(\left\{\tilde{\zeta}_L\right\}\right)\right\rangle\left\langle{}^{(g)}O_S(\{\boldsymbol{x}_i\})\right\rangle$$

If we expand ${}^{(g)}O_S(\{\boldsymbol{x}_\mathrm{i}\})$, setting $f\left(\left\{\tilde{\zeta}_L\right\}\right) = \tilde{\zeta}_{\boldsymbol{k}_L}$,
we recover the conventional consistency relations.

Locality condition $\Longleftrightarrow$ Consistency relations

$$\left|\Psi\right\rangle = \int D\tilde{\zeta}_L^c D\varphi_L^{(a)}\ \psi\left(\tilde{\zeta}_L^c, \varphi_L^{(a)}\right) \left|\tilde{\zeta}_L^c, \varphi_L^{(a)}\right\rangle \left|\Psi\right\rangle_{\tilde{\zeta}_L^c, \varphi_L^{(a)}}$$

Below, we suppress other isocurvature modes, $\varphi^{(a)}$, for simplicity.

$$\left\langle f\left(\left\{\tilde{\zeta}_L^c\right\}\right) {}^{(g)}O_S\left(\{x_i\}\right)\right\rangle = \int D\tilde{\zeta}_L^c f\left(\left\{\tilde{\zeta}_L^c\right\}\right) \psi^2\left(\tilde{\zeta}_L^c\right) \underline{{}_{\tilde{\zeta}_L^c}\left\langle\Psi\right| \left\langle\tilde{\zeta}_L^c\right| {}^{(g)}O_S \left|\tilde{\zeta}_L^c\right\rangle \left|\Psi\right\rangle_{\tilde{\zeta}_L^c}}$$

($\Rightarrow$) $$\underline{\frac{\partial}{\partial\tilde{\zeta}_{k_L}^c} {}_{\tilde{\zeta}_L^c}\left\langle\Psi\right| \left\langle\tilde{\zeta}_L^c\right| {}^{(g)}O_S \left|\tilde{\zeta}_L^c\right\rangle \left|\Psi\right\rangle_{\tilde{\zeta}_L^c}} = {}_{\tilde{\zeta}_L^c}\left\langle\Psi\right| \left\langle\tilde{\zeta}_L^c\right| \underbrace{\left[{}^{(g)}O_S, iQ_\zeta^W\left(k_L\right)\right]}_{\approx 0} \left|\tilde{\zeta}_L^c\right\rangle \left|\Psi\right\rangle_{\tilde{\zeta}_L^c}$$

($\Leftarrow$) If $\left\langle f\left(\left\{\tilde{\zeta}_L^c\right\}\right) {}^{(g)}O_S\left(\{x_i\}\right)\right\rangle = \left\langle f\left(\left\{\tilde{\zeta}_L^c\right\}\right)\right\rangle \left\langle {}^{(g)}O_S\left(\{x_i\}\right)\right\rangle$ holds for arbitrary f and ${}^{(g)}O_S$

$$\Rightarrow \frac{\partial}{\partial\tilde{\zeta}_{k_L}^c} {}_{\tilde{\zeta}_L^c}\left\langle\Psi\right| \left\langle\tilde{\zeta}_L^c\right| {}^{(\mathrm{di})}O_S \left|\tilde{\zeta}_L^c\right\rangle \left|\Psi\right\rangle_{\tilde{\zeta}_L^c} = 0 \quad -\bigstar$$

∵ Substitute $f = \dfrac{\partial\delta\left(\tilde{\zeta}_{k_L}^c - b\right)}{\psi^2 \partial\tilde{\zeta}_{k_L}^c}$ to obtain $\int D\tilde{\zeta}_L^c \delta'\left(\tilde{\zeta}_{k_L}^c - b\right) {}_{\tilde{\zeta}_L^c}\left\langle\Psi\right| \left\langle\tilde{\zeta}_L^c\right| {}^{(\mathrm{di})}O_S \left|\tilde{\zeta}_L^c\right\rangle \left|\Psi\right\rangle_{\tilde{\zeta}_L^c} = 0$

$$\frac{\partial}{\partial\tilde{\zeta}_{k_L}^c} \left|\Psi\right\rangle_{\tilde{\zeta}_L^c} \equiv i\underline{\hat{Q}_S} \left|\Psi\right\rangle_{\tilde{\zeta}_L^c} \quad \bigstar \quad \Rightarrow \quad {}_{\tilde{\zeta}_L^c}\left\langle\Psi\right| \left\langle\tilde{\zeta}_L^c\right| \left[{}^{(g)}O_S, \underline{\left(iQ_L + i\hat{Q}_S\right)}\right] \left|\tilde{\zeta}_L^c\right\rangle \left|\Psi\right\rangle_{\tilde{\zeta}_L^c} = 0$$

Unknown Hermitian op.

$$= i\hat{Q}_S - iQ_S =: i\Delta Q_S$$

$$\Rightarrow 0 = \left[\left\langle\tilde{\zeta}_L^c\right| {}^{(g)}O_S \left|\tilde{\zeta}_L^c\right\rangle, i\Delta Q_S\right]$$

$$Q_\zeta^W\left(k_L\right) = Q_L + Q_S$$

long wavelength part — short wavelength part

If $\Delta Q_S \neq \lambda I$, $\left[{}^{\exists}O_S, i\Delta Q_S\right] \neq 0$.

Then one can show $\left[\left\langle\tilde{\zeta}_L^c\right| {}^{(g)}O_S \left|\tilde{\zeta}_L^c\right\rangle, i\Delta Q_S\right] \neq 0$,

which is a contradiction. So, ΔQ_S=0.

IR regularity

"Wave function must be homogeneous in the direction of background scale transformation"

which was nothing but the locality conditions.

$$iQ_{\zeta}^{W}\left(\boldsymbol{k}_L\right)\left|\Psi\right\rangle_{\tilde{\zeta}_L^c} = \frac{\partial}{\partial \tilde{\zeta}_{\boldsymbol{k}_L}^{c}}\left|\Psi\right\rangle_{\tilde{\zeta}_L^c}$$

Genuine gauge invariant operator: ${}^{(g)}O$

$$\left[iQ_{\zeta}, {}^{(g)}O\right] = 0 \quad \Rightarrow \quad \left[iQ_{\zeta}^{W}\left(\mathbf{x}_L\right), {}^{(g)}O_S\right] = 0$$

as long as ${}^{(g)}O$ is an localized operator

$$\frac{\partial}{\partial \tilde{\zeta}_L^c\left(\mathbf{x}\right)}\, {}_{\tilde{\zeta}_L^c}\left\langle\Psi\right|{}^{(g)}O_S\left|\Psi\right\rangle_{\tilde{\zeta}_L^c} = -\,{}_{\tilde{\zeta}_L^c}\left\langle\Psi\right|\left[iQ_{\zeta}^{W}\left(\mathbf{x}_L\right), {}^{(g)}O_S\right]\left|\Psi\right\rangle_{\tilde{\zeta}_L^c} = 0$$

$$\left\langle\Psi\right|{}^{(g)}O_S\left|\Psi\right\rangle = \int D\tilde{\zeta}_L^c\, \psi^2\left(\tilde{\zeta}_L^c\right)\, {}_{\tilde{\zeta}_L^c}\left\langle\Psi\right|{}^{(g)}O_S\left|\Psi\right\rangle_{\tilde{\zeta}_L^c}$$

$$= {}_{\tilde{\zeta}_L^c=0}\left\langle\Psi\right|{}^{(g)}O_S\left|\Psi\right\rangle_{\tilde{\zeta}_L^c=0} \underline{\int D\tilde{\zeta}_L^c\, \psi^2\left(\tilde{\zeta}_L^c\right)}$$

IR divergent factor is completely factorized. Notice that IR wave function is infinitely broad.

Assumption

Dilatation invariance $\left[Q_\zeta, H\right]=0$

Ex. of DI violation: Solid inflation

for one particular quantum state

for arbitrary dilatation invariant short mode excitations.

Consistency relation

Locality condition

Existence of Constant ζ solution

$$\left\langle f\left(\left\{\tilde{\zeta}_L\right\}\right)^{(g)} O_S\left(\left\{\boldsymbol{x}_i\right\}\right)\right\rangle = \left\langle f\left(\left\{\tilde{\zeta}_L\right\}\right)\right\rangle \left\langle^{(g)} O_S\left(\left\{\boldsymbol{x}_i\right\}\right)\right\rangle$$

$$iQ_\zeta^W\left(\boldsymbol{x}_L\right)\left|\Psi\right\rangle_{\tilde{\zeta}_L^c} = \frac{\partial}{\partial \zeta_L^c\left(\boldsymbol{x}_L\right)}\left|\Psi\right\rangle_{\tilde{\zeta}_L^c}$$

If $\left|\Psi\right\rangle$ is a solution,

$\left|\Psi\right\rangle\Big|_{\tilde{\zeta}_L^c \to \tilde{\zeta}_L^c + s}$ is also a solution.

Ex. of LC violation: Sudden turn-on of interaction

IR regularity

DI of effective action

IR divergence and/or
secular growth owing to the
growth of the variance of
$\tilde{\zeta}_L$ Is factorized.