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The Atiyah–Patodi–Singer Index and Domain-Wall Fermion Dirac Operators

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Abstract: We introduce a *mathematician-friendly* formulation of the *physicist-friendly* derivation of the Atiyah–Patodi–Singer index. In a previous paper, motivated by the study of lattice gauge theory, the physicist half of the authors derived a formula expressing the Atiyah–Patodi–Singer index in terms of the eta invariant of *domain-wall fermion Dirac operators* when the base manifold is a flat 4-dimensional torus. In this paper, we generalise this formula to any even dimensional closed Riemannian manifolds, and prove it mathematically rigorously. Our proof uses a Witten localisation argument combined with a devised embedding into a cylinder of one dimension higher. Our viewpoint sheds some new light on the interplay among the Atiyah–Patodi–Singer boundary condition, domain-wall fermions, and edge modes.

1. Introduction

The Atiyah–Patodi–Singer index theorem [1–3], a generalisation of the Atiyah–Singer index theorem to manifolds with boundary, has been attracting attention in condensed matter physics. For example, Witten used it in [23] to describe the bulk-edge correspondence of symmetry-protected topological phases of matter and explain why boundary-localised modes must appear on the boundary of topological insulators. The boundary correction term of the Atiyah–Patodi–Singer index theorem, the eta invariant, appears as the phase of the edge mode partition function, and the Atiyah–Patodi–Singer index theorem suggests the existence of the bulk topological couplings to restore time-reversal symmetry. We refer the reader to [12,18,20,25–27] for related works.

It is, however, somewhat puzzling to relate the Atiyah–Patodi–Singer index and symmetry-protected topological phases of matter. Considering the Atiyah–Patodi–Singer index, we need a $\mathbb{Z}_2$-grading *chirality* operator; thus, we should consider massless fermions in the bulk and the *non-local* Atiyah–Patodi–Singer spectral boundary condition. In symmetry-protected topological phases of matter, by contrast, fermions are massive in the bulk and *local* boundary conditions are imposed. For example, using the Atiyah–Patodi–Singer boundary condition is justified in [24] by rotating the boundary to the temporal direction and regarding it as an intermediate state in the partition function of massive fermion systems. In a previous paper by the physicist half of the authors [8], we looked at the Atiyah–Patodi–Singer boundary condition in a different light of *domain-wall fermion Dirac operators*.

Domain-wall fermion Dirac operators [5,9,13,14] are a particular class of massive Dirac operators that have zero-eigenvalue solutions concentrated on small neighbourhoods of separating submanifolds, domain walls. Using these operators, without imposing any global boundary conditions, the physicist half of the authors gave a physically intuitive reformulation of the Atiyah–Patodi–Singer index in the previous paper [8]. We refer the reader to [21] for a different link between the Atiyah–Patodi–Singer index theorem and domain walls.

In this paper, we will pursue our investigation of the relation between the Atiyah–Patodi–Singer index theorem and domain-wall fermion Dirac operators. We will establish a mathematical formulation of [8] based on the embedding trick and a Witten localisation argument [10,22] with a new excision theorem of the index under very weak assumptions, which will localise the index to open submanifolds. See Sects. 3.4 and 2 respectively.

Before stating the main theorem, we begin with a formula relating the usual Atiyah–Singer index and the eta invariant. Let X be a closed oriented Riemannian manifold with $\dim X$ even. Let S be a $\mathbb{Z}_2$-graded hermitian vector bundle on X, and Γ_S its $\mathbb{Z}_2$-grading operator. Let $D\colon C^\infty(X; S) \to C^\infty(X; S)$ be a first-order, formally self-adjoint, elliptic partial differential operator. We assume that D is an *odd* operator in the sense that it anti-commutes with Γ_S. Thus, S is decomposed as a direct sum $S = S_+ \oplus S_-$, and we can write

$$D = \begin{pmatrix} 0 & D_- \\ D_+ & 0 \end{pmatrix}$$

in matrix form. We define the index of the odd, self-adjoint, elliptic operator D by

$$\mathrm{Ind}(D) := \dim \mathrm{Ker}\, D_+ - \dim \mathrm{Ker}\, D_- = \mathrm{tr}\left(\Gamma_S|_{\mathrm{Ker}\, D} \right).$$

In physics notation, $\Gamma_S = \gamma_5$ and $D = \gamma_5 \not{D}$ if $\dim X = 4$, and the index gives the chiral asymmetry of the number of independent left and right zero modes. Fix $m > 0$, and we consider another self-adjoint, elliptic operator $D + m\Gamma_S$. This is no longer an odd operator. The eta invariant describes the overall asymmetry of the spectrum of a self-adjoint operator. Let us recall its definition. Let λ_j run over the eigenvalues of $D + m\Gamma_S$. Note that $\lambda_j \neq 0$ for any j. The eta function of $D + m\Gamma_S$ is defined by

$$\eta(s) := \sum_{\lambda_j} \frac{\mathrm{sign}\, \lambda_j}{|\lambda_j|^s}$$

for $s \in \mathbb{C}$. This series is absolutely convergent in $\mathrm{Re}(s) > \dim X$ and admits a meromorphic extension to the whole complex plane. Atiyah–Patodi–Singer [1, (3.9)] showed that $\eta(s)$ is holomorphic at $s = 0$. The special value $\eta(0)$ is called the eta invariant of the operator $D + m\Gamma_S$ and denoted by $\eta(D + m\Gamma_S)$. The eta invariant $\eta(D - m\Gamma_S)$ is defined similarly. Now we have a formula

$$\mathrm{Ind}(D) = \frac{\eta(D + m\Gamma_S) - \eta(D - m\Gamma_S)}{2} \tag{1}$$

for any $m > 0$. This formula might be unfamiliar to the reader; however, we can prove it easily, for example, by diagonalising D^2 and Γ_S simultaneously.

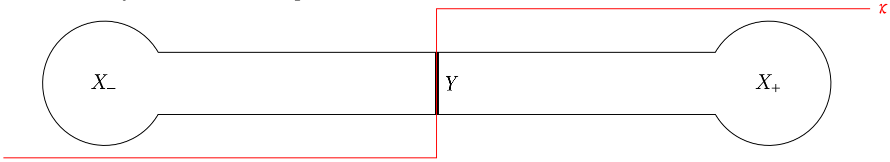
The previous paper [8] generalised this formula (1) to handle the Atiyah–Patodi–Singer index by considering the *domain-wall fermion Dirac operator*. Let us first recall the Atiyah–Patodi–Singer index. Let $Y \subset X$ be a separating submanifold that decomposes X into two compact manifolds X_+ and X_- with common boundary Y. We assume that Y has a collar neighbourhood isometric to the standard product $(-4, 4) \times Y$ and satisfying $((-4, 4) \times Y) \cap X_+ = [0, 4) \times Y$. The coordinate along $(-4, 4)$ is denoted by u. We also assume that S and D are standard in the following sense: there exist a hermitian bundle E on Y and a bundle isometry from $S|_{(-4,4)\times Y}$ to $\mathbb{C}^2 \otimes E$ such that, under this isometry, Γ_S corresponds to $\Gamma \otimes \mathrm{id}_E$ and D takes the form

$$D = c \otimes \partial_u + \epsilon \otimes A = \begin{pmatrix} 0 & \partial_u + A \\ -\partial_u + A & 0 \end{pmatrix},$$

where $A\colon C^\infty(Y; E) \to C^\infty(Y; E)$ is a formally self-adjoint, elliptic partial differential operator. In this paper, we will concentrate on the case when A has no zero eigenvalues, and assume this condition. Let $C^\infty(X_+; S_\pm|_{X_+} : P_A) := \{ f \in C^\infty(X_+; S_\pm|_{X_+}) \mid P_A(f|_Y) = 0 \}$, where $P_A\colon L^2(Y; E) \to L^2(Y; E)$ denotes the spectral projection onto the span of the eigensections of A with positive eigenvalues. We define the Atiyah–Patodi–Singer index of $D|_{X_+}$ by

$$\begin{aligned}\mathrm{Ind}_{\mathrm{APS}}(D|_{X_+}) := \dim &\left(\mathrm{Ker}\, D_+ \cap C^\infty(X_+; S_+|_{X_+} : P_A)\right) \\ &- \dim \left(\mathrm{Ker}\, D_- \cap C^\infty(X_+; S_-|_{X_+} : P_A)\right).\end{aligned}$$

To handle the Atiyah–Patodi–Singer index, let us next introduce the *domain-wall fermion Dirac operator*. Let $\kappa\colon X \to [-1, 1]$ be a step function such that $\kappa \equiv \pm 1$ on $X_\pm \backslash Y$ and $\kappa \equiv 0$ on Y, which is sometimes called a *domain-wall function*. We call $D + m\kappa\Gamma_S$ the *domain-wall fermion Dirac operator*.

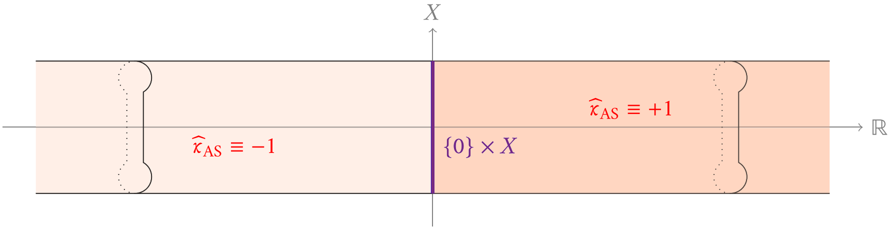


In [8], when X is a 4-dimensional flat manifold, a formula

$$\mathrm{Ind}_{\mathrm{APS}}(D|_{X_+}) = \frac{\eta(D + m\kappa\Gamma_S) - \eta(D - m\Gamma_S)}{2} \tag{2}$$

was derived by first expanding the right hand side with the Fujikawa method [7] and then identifying the result with the left hand side via the Atiyah–Patodi–Singer index theorem. Our new approach in this paper is mathematically rigorous and reveals the direct link between them; moreover, we will prove the formula (2) for any even-dimensional Riemannian manifolds. See Theorem 12.

As a warm-up, we will prove the formula (1) in the spirit of our proof of Theorem 12. The reader can skip this paragraph at first reading. Consider the cylinder $\mathbb{R} \times X$. The coordinate along $\mathbb{R}$ is denoted by s. We pull back the bundle S on X to $\mathbb{R} \times X$, which will be denoted by the same symbol. Let $\widehat{\kappa}_{\mathrm{AS}}\colon \mathbb{R} \times X \to [-1, 1]$ be a step function such that $\widehat{\kappa}_{\mathrm{AS}} \equiv 1$ on $(0, \infty) \times X$ and $\widehat{\kappa}_{\mathrm{AS}} \equiv -1$ on $(-\infty, 0) \times X$.



We introduce a self-adjoint operator $\widehat{D}_m : L^2(\mathbb{R} \times X; S \oplus S) \to L^2(\mathbb{R} \times X; S \oplus S)$ defined by

$$\widehat{D}_m := \begin{pmatrix} 0 & (D + m\widehat{\kappa}_{\mathrm{AS}}\Gamma_S) + \partial_s \\ (D + m\widehat{\kappa}_{\mathrm{AS}}\Gamma_S) - \partial_s & 0 \end{pmatrix}.$$

We will prove in Proposition 10 that this is a Fredholm operator and $\mathrm{Ind}(\widehat{D}_m) = \mathrm{Ind}(D)$. We also observe that the constant term in the asymptotic expansion of the heat kernel vanishes on such an odd-dimensional manifold as $\mathbb{R} \times X$. Thus, by the Atiyah–Patodi–Singer index theorem on cylinders, $\mathrm{Ind}(\widehat{D}_m)$ can be written only in terms of the eta invariant. See the discussion around (12). Note that $D + m\widehat{\kappa}_{\mathrm{AS}}(\pm 1, \cdot)\Gamma_S = D \pm m\Gamma_S$. Hence, we have

$$\mathrm{Ind}(D) = \mathrm{Ind}(\widehat{D}_m) = \frac{\eta(D + m\Gamma_S) - \eta(D - m\Gamma_S)}{2},$$

which proves the formula (1).

We will generalise the proof above to handle manifolds with boundary. In the book [10], one of the authors has modified the embedding proof of the Atiyah–Singer index theorem, using a localisation argument of Witten [22] with *supersymmetric harmonic oscillators*. In this paper, we will develop another Witten localisation argument with a particular embedding constructed in Sect. 3.4 and the *Jackiw–Rebbi solutions* of domain-wall fermionic Dirac operators instead of supersymmetric harmonic oscillators. We will introduce an operator (9) that interpolates domain-wall fermion Dirac operators and Atiyah–Patodi–Singer operators. Our localisation arguments will localise the index of this operator to open submanifolds. We emphasise here that the ideas behind the proof seem more interesting than the formula (2) itself and useful in other applications. We also remark that considering spectral flows of the family $\{D + m\widehat{\kappa}_{\mathrm{AS}}(s, \cdot)\Gamma_S\}_{s \in [-1,1]}$ seems more appropriate when dealing with various symmetries such as in the ten-fold way of topological insulators [15].

2. Excision

In this technical section, we will develop, under very weak assumptions, yet another excision formula of the index, which will be a technical basis to the rest of the paper and might be of independent interest.

Before going into details, we first explain some basic ideas underlying our proof of an excision formula, *Witten localisation arguments* [22]. Let $U_0 := \{x \in \mathbb{R}^d \mid |x| < 2\}$ and $U_1 := \{x \in \mathbb{R}^d \mid 1 < |x|\}$. We consider a Dirac-type operator D and a potential term h on $\mathbb{R}^d$. If D and h anti-commute on U_1, then we have $(D + mh)^2 = D^2 + m^2h^2$ on U_1 for any $m > 0$. When m is very large, the second term m^2h^2 is also very large. Hence, eigenmodes with small eigenvalues are very suppressed in this region U_1. In other words, a particle of quantum mechanics is rarely found in the region where the potential energy is very large. Thus, when m is large enough, the eigenmodes of $(D+mh)^2$ with small eigenvalues are localised and determined on U_0.

We set up notation. Let Z be a complete Riemannian manifold and S_Z a hermitian vector bundle on Z. We denote by $C_c^\infty(Z; S_Z)$ the space of compactly supported smooth sections of S_Z. Let $L\colon C_c^\infty(Z; S_Z) \to C_c^\infty(Z; S_Z)$ be a first-order, elliptic partial differential operator that is essentially self-adjoint on $L^2(Z; S_Z)$. We denote the symbol of L by $\sigma_L \in C^\infty(Z; \mathrm{Hom}(T^*Z, \mathrm{End}(S_Z)))$. Let h be a (not necessarily smooth but measurable) self-adjoint endomorphism of S_Z whose eigenvalues are uniformly bounded on Z. For each $m > 0$, we set $L_m := L + mh$. Throughout this section, $(\cdot, \cdot)$ will denote a pointwise hermitian inner product, $|\cdot|$ a pointwise norm, and $\|\cdot\|$ an L^2-norm, and denote the exterior derivative of a function f by df.

Lemma 1. *Let $\beta_0, \beta_1 \in C^\infty(Z; \mathbb{R})$ satisfy $\beta_0^2 + \beta_1^2 = 1$. We have a pointwise equality*

$$|L_m(\beta_0\Phi)|^2 + |L_m(\beta_1\Phi)|^2 = |L_m\Phi|^2 + |\sigma_L(d\beta_0)\Phi|^2 + |\sigma_L(d\beta_1)\Phi|^2$$

for any $\Phi \in C^\infty(Z; S_Z)$.

Proof. Fix $\Phi \in C^\infty(Z; S_Z)$. Since $L(\beta_0\Phi) = \beta_0(L\Phi) + \sigma_L(d\beta_0)\Phi$, we have $L_m(\beta_0\Phi) = \beta_0(L_m\Phi) + \sigma_L(d\beta_0)\Phi$. Hence, $|L_m(\beta_0\Phi)|^2 = \beta_0^2|L_m\Phi|^2 + |\sigma_L(d\beta_0)\Phi|^2 + 2\,\mathrm{Re}(\beta_0(L_m\Phi))\sigma_L(d\beta_0)\Phi$. Thus, we have

$$\begin{aligned} |L_m(\beta_0\Phi)|^2 + |L_m(\beta_1\Phi)|^2 &= |L_m\Phi|^2 + |\sigma_L(d\beta_0)\Phi|^2 + |\sigma_L(d\beta_1)\Phi|^2 \\ &\quad + 2\,\mathrm{Re}(\beta_0(L_m\Phi), \sigma_L(d\beta_0)\Phi) + 2\,\mathrm{Re}(\beta_1(L_m\Phi), \sigma_L(d\beta_1)\Phi). \end{aligned}$$

The assumption $\beta_0^2 + \beta_1^2 = 1$ implies $2(\beta_0(L_m\Phi), \sigma_L(d\beta_0)\Phi) + 2(\beta_1(L_m\Phi), \sigma_L(d\beta_1)\Phi) = 0$. □

Lemma 2. *Under the assumption of Lemma 1, suppose further that h is smooth and anti-commutes with L on* $\mathrm{supp}\,\beta_1$. *Then, we have an L^2-integral equality*

$$m^2\|h\beta_1\Phi\|^2 \leq \|L_m\Phi\|^2 + \|\sigma_L(d\beta_0)\Phi\|^2 + \|\sigma_L(d\beta_1)\Phi\|^2$$

for any $\Phi \in C_c^\infty(Z; S_Z)$.

Proof. Fix $\Phi \in C_c^\infty(Z; S_Z)$. By Lemma 1, we have an inequality

$$\int_Z |L_m(\beta_1\Phi)|^2\, d\mu \leq \int_Z \left(|L_m\Phi|^2 + |\sigma_L(d\beta_0)\Phi|^2 + |\sigma_L(d\beta_1)\Phi|^2\right) d\mu.$$

Since L and h anti-commute on $\mathrm{supp}\,\beta_1$, we deduce that

$$\begin{aligned} \int_Z (L(\beta_1\Phi), h\beta_1\Phi)\, d\mu &= \int_Z (\beta_1\Phi, L(h\beta_1\Phi))\, d\mu \\ &= \int_Z (\beta_1\Phi, -hL(\beta_1\Phi))\, d\mu = -\int_Z (h\beta_1\Phi, L(\beta_1\Phi))\, d\mu. \end{aligned}$$

Hence, $\mathrm{Re}\langle L(\beta_1\Phi), h\beta_1\Phi\rangle_{L^2} = 0$, and we have $\|L_m(\beta_1\Phi)\|^2 = \|L(\beta_1\Phi)\|^2 + \|mh\beta_1\Phi\|^2$. Thus,

$$\|mh\beta_1\Phi\|^2 \leq \|L_m(\beta_1\Phi)\|^2 \leq \|L_m\Phi\|^2 + \|\sigma_L(d\beta_0)\Phi\|^2 + \|\sigma_L(d\beta_1)\Phi\|^2,$$

as required. □

Lemma 3. *Let $Z = U_0 \cup U_1$ be an open covering of Z. Let $1 = \gamma_0^2 + \gamma_1^2$ be a smooth partition of unity subordinate to U_0 and U_1. We assume the following three conditions:*

(i) *h is smooth and anti-commutes with L on U_1.*
(ii) *the eigenvalues of h^2 are greater than or equal to 1 on U_1.*
(iii) *the eigenvalues of $\sigma_L(d\gamma_0)$ and $\sigma_L(d\gamma_1)$ are bounded on $U_0 \cap U_1$.*

Then, for any $\Lambda \geq 0$ and $\Phi \in C_c^\infty(Z; S_Z)$ with $\|L_m\Phi\|^2 \leq \Lambda^2\|\Phi\|^2$, we have

$$m^2\|\gamma_1\Phi\|^2 \leq (C_1^2 + \Lambda^2)\|\Phi\|^2,$$

where we set

$$\begin{aligned} C_1^2 &:= \sup_{x \in U_0 \cap U_1} \left(|\sigma_L(d\gamma_0)|^2 + |\sigma_L(d\gamma_1)|^2\right) \\ &= \sup_{x \in U_0 \cap U_1} \sup_{\phi \in S_x} \left(\frac{|\sigma_L(d\gamma_0)\phi|^2}{|\phi|^2} + \frac{|\sigma_L(d\gamma_1)\phi|^2}{|\phi|^2}\right). \end{aligned}$$

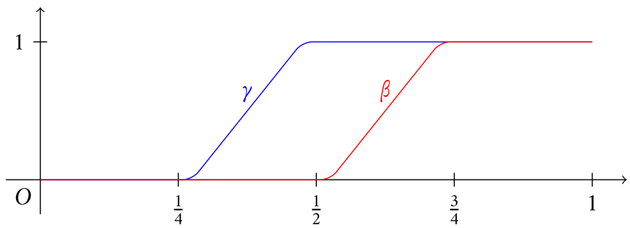
Proof. Fix $\Lambda \geq 0$ and $\Phi \in C_c^\infty(Z; S_Z)$ with $\|L_m\Phi\|^2 \leq \Lambda^2\|\Phi\|^2$. By assumption (ii), we have $m^2\|h\gamma_1\Phi\|^2 \geq m^2\|\gamma_1\Phi\|^2$. By definition of C_1, we have $\|\sigma_L(d\gamma_0)\Phi\|^2 + \|\sigma_L(d\gamma_1)\Phi\|^2 \leq C_1^2\|\Phi\|^2$. By assumption (i), we can use Lemma 2. Thus, we obtain

$$\begin{aligned} m^2\|\gamma_1\Phi\|^2 &\leq m^2\|h\gamma_1\Phi\|^2 \\ &\leq \|L_m\Phi\|^2 + \|\sigma_L(d\gamma_0)\Phi\|^2 + \|\sigma_L(d\gamma_1)\Phi\|^2 \\ &\leq \|L_m\Phi\|^2 + C_1^2\|\Phi\|^2 \leq \Lambda^2\|\Phi\|^2 + C_1^2\|\Phi\|^2 = (C_1^2 + \Lambda^2)\|\Phi\|^2, \end{aligned}$$

as required. □

Lemma 4. *Let $Z = U_0 \cup U_1$ be an open covering of Z. Let $1 = \eta_0 + (1 - \eta_0)$ be a smooth partition of unity subordinate to U_0 and U_1. We assume that $|\sigma_L(d\eta_0)|$ is bounded on $U_0 \cap U_1$. Then, there exist smooth partitions of unity $1 = \beta_0^2 + \beta_1^2 = \gamma_0^2 + \gamma_1^2$ subordinate to U_0 and U_1 such that both $|\sigma_L(d\beta_0)|^2 + |\sigma_L(d\beta_1)|^2$ and $|\sigma_L(d\gamma_0)|^2 + |\sigma_L(d\gamma_1)|^2$ are bounded on $U_0 \cap U_1$, and that $\gamma_1 \equiv 1$ on $(\operatorname{supp} d\beta_0) = (\operatorname{supp} d\beta_1)$.*

Proof. Let $\beta\colon [0, 1] \to [0, 1]$ and $\gamma : [0, 1] \to [0, 1]$ be smooth cut-off functions such that $\beta^2 + (1 - \beta^2) = 1$ and $\gamma^2 + (1 - \gamma^2) = 1$, and that $\gamma \equiv 0$ on $[0, 1/4]$, $\beta \equiv 0$ on $[0, 1/2]$, $\gamma \equiv 1$ on $[1/2, 1]$, and $\beta \equiv 1$ on $[3/4, 1]$.



We set $\beta_1 := \beta \circ (1 - \eta_0)$ and $\gamma_1 := \gamma \circ (1 - \eta_0)$, which clearly satisfy the claimed properties. □

Proposition 5. *Let $Z = U_0 \cup U_1$ and $1 = \eta_0 + (1 - \eta_0)$ satisfy the assumptions of Lemma 4. Let $1 = \beta_0^2 + \beta_1^2 = \gamma_0^2 + \gamma_1^2$ be partitions of unity constructed in Lemma 4. We also assume that h satisfies the conditions (i) and (ii) of Lemma 3. Then, there exists* *a constant $C_0 > 0$ that depends only on η_0 and σ_L such that, for any $\Lambda \geq 0$ and $\Phi \in C_c^\infty(Z; S_Z)$ with $\|L_m\Phi\|^2 \leq \Lambda^2\|\Phi\|^2$, we have*

$$\|L_m(\beta_0\Phi)\|^2 \leq \left(\Lambda^2 + C_0^2\frac{C_0^2+\Lambda^2}{m^2}\right)\|\Phi\|^2 \tag{3}$$

and

$$\left(1 - \frac{C_0^2+\Lambda^2}{m^2}\right)\|\Phi\|^2 \leq \|\beta_0\Phi\|^2. \tag{4}$$

Proof. Fix $\Lambda \geq 0$ and $\Phi \in C_c^\infty(Z; S_Z)$ with $\|L_m\Phi\|^2 \leq \Lambda^2\|\Phi\|^2$. Set $C_1^2 := \sup_{U_0\cap U_1}\left(|\sigma_L(d\gamma_0)|^2 + |\sigma_L(d\gamma_1)|^2\right)$ and $C_2^2 := \sup_{U_0\cap U_1}\left(|\sigma_L(d\beta_0)|^2 + |\sigma_L(d\beta_1)|^2\right)$.

We first show (3). By Lemma 3, we have

$$m^2\|\gamma_1\Phi\|^2 \leq (C_1^2 + \Lambda^2)\|\Phi\|^2.$$

Since $\gamma_1 \equiv 1$ on $(\operatorname{supp} d\beta_0) = (\operatorname{supp} d\beta_1)$, we have

$$\|\sigma_L(d\beta_0)\Phi\|^2 + \|\sigma_L(d\beta_1)\Phi\|^2 \leq C_2^2\|\gamma_1\Phi\|^2.$$

Thus, we obtain

$$\|\sigma_L(d\beta_0)\Phi\|^2 + \|\sigma_L(d\beta_1)\Phi\|^2 \leq C_2^2\frac{C_1^2+\Lambda^2}{m^2}\|\Phi\|^2.$$

By Lemma 1, we have

$$\|L_m(\beta_0\Phi)\|^2 \leq \|L_m\Phi\|^2 + \|\sigma_L(d\beta_0)\Phi\|^2 + \|\sigma_L(d\beta_1)\Phi\|^2.$$

Consequently, we have

$$\|L_m(\beta_0\Phi)\|^2 \leq \Lambda^2\|\Phi\|^2 + C_2^2\frac{C_1^2+\Lambda^2}{m^2}\|\Phi\|^2.$$

Now set $C_0 := \max\{C_1, C_2\}$, which yields (3).

Next we prove (4). Since $\beta_0^2 + \beta_1^2 = 1$, we have $\|\beta_0\Phi\|^2 + \|\beta_1\Phi\|^2 = \|\Phi\|^2$. By Lemma 3, we have $m^2\|\beta_1\Phi\|^2 \leq (C_2^2 + \Lambda^2)\|\Phi\|^2 \leq (C_0^2 + \Lambda^2)\|\Phi\|^2$. This completes the proof. □

Proposition 6. *Let $(Z = U_0 \cup U_1, 1 = \eta_0 + (1-\eta_0), S_Z, L, h)$ and $(Z' = U_0' \cup U_1', 1 = \eta_0' + (1-\eta_0'), S'_{Z'}, L', h')$ be two sets of data as above that satisfy the assumptions of Proposition 5. We assume that L coincides with L' on $U_0 \cong U_0'$ in the sense that there exists an isometry $\tau: U_0 \to U_0'$ covered by a bundle isometry $\tilde{\tau}: S_Z|_{U_0} \to S'_{Z'}\big|_{U_0'}$ such that $\tilde{\tau}^{-1}\circ L'_m \circ \tilde{\tau} = L_m$. Then, there exists a constant $C > 0$ that depends only on η_0, η_0', σ_L, and $\sigma_{L'}$ such that the following holds. Fix $\Lambda_2 > \Lambda_1 > \Lambda_0 \geq 0$ and $m > 0$. If L_m has only discrete spectrum[1] in $[-\Lambda_0, \Lambda_0]$ and has spectral gaps*

$$\left(\operatorname{Spec} L_m\right) \cap \left(\left[-\Lambda_2, -\Lambda_0\right) \cup \left(\Lambda_0, \Lambda_2\right]\right) = \emptyset \tag{5}$$

[1] The discrete spectrum of a self-adjoint operator consists of isolated eigenvalues with finite multiplicity.

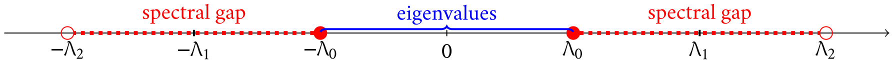
and if

$$m^2 > \max\left\{\frac{(C^2+\Lambda_2^2)(C^2+\Lambda_1^2)}{\Lambda_2^2-\Lambda_1^2}, \frac{(C^2+\Lambda_1^2)(C^2+\Lambda_0^2)}{\Lambda_1^2-\Lambda_0^2}, (C^2+\Lambda_2^2)\right\}, \tag{6}$$

then L'_m also has only discrete spectrum in $[-\Lambda_1, \Lambda_1]$ *and the number of eigenvalues of L'_m in* $[-\Lambda_1, \Lambda_1]$ *counted with multiplicity is equal to that of L_m in* $[-\Lambda_0, \Lambda_0]$.

Remark 7. The spectral gap condition (5) is only imposed on L_m.

Proof. Let E_0 be the span of the L^2-eigensections of L_m with eigenvalues in $[-\Lambda_0, \Lambda_0]$ and E_2 with eigenvalues in $[-\Lambda_2, \Lambda_2]$. Let $\Pi_0 \colon L^2(Z; S_Z) \to E_0$ and $\Pi_2 \colon L^2(Z; S_Z) \to E_2$ be the L^2-orthogonal projections. The spectral gap assumption (5) on L_m implies E_0 is a finite dimensional vector space and $E_0 = E_2$. Let $\Pi'_1 \colon L^2(Z'; S'_{Z'}) \to L^2(Z'; S'_{Z'})$ be the spectral projection for L'_m associated with $[-\Lambda_1, \Lambda_1]$. Let $E'_1 := \operatorname{Im} \Pi'_1$. We will show that $E_0 \cong E'_1$.



Let $1 = \beta_0^2 + \beta_1^2$ and $1 = (\beta'_0)^2 + (\beta'_1)^2$ be partitions of unity in Lemma 4. We regard, via τ and $\tilde{\tau}$, each section of S_Z supported in U_0 as a section of $S'_{Z'}$ supported in U'_0. We define a linear map $\rho \colon E_0 \to E'_1$ by

$$\Phi \mapsto \Pi'(\beta_0 \Phi)$$

and a linear map $\rho' \colon E'_1 \to E_2$ similarly. We will prove that ρ and ρ' are isomorphisms.

We first show that ρ is injective. Fix $\Phi \in \operatorname{Ker}(\rho)$. Assume that $\Phi \neq 0$. The assumption (6) and the inequality (4) implies that $\beta_0 \Phi \neq 0$. We define

$$C := \max\left\{C_0(\eta_0, \sigma_L), C_0(\eta'_0, \sigma_{L'})\right\},$$

where C_0 is the constant in Proposition 5. Then, by Proposition 5, we have

$$\begin{aligned}\|L_m(\beta_0\Phi)\|^2 &\le \left(\Lambda_0^2 + C^2\frac{C^2+\Lambda_0^2}{m^2}\right)\|\Phi\|^2 \\ &\le \left(\Lambda_0^2 + C^2\frac{C^2+\Lambda_0^2}{m^2}\right)\left(1-\frac{C^2+\Lambda_0^2}{m^2}\right)^{-1}\|\beta_0\Phi\|^2.\end{aligned}$$

The assumption (6) implies

$$\left(\Lambda_0^2 + C^2\frac{C^2+\Lambda_0^2}{m^2}\right)\left(1-\frac{C^2+\Lambda_0^2}{m^2}\right)^{-1} < \Lambda_1^2.$$

Hence, we have $\|L_m(\beta_0\Phi)\|^2 < \Lambda_1^2\|\beta_0\Phi\|^2$. Since $\beta_0\Phi$ is supported in U_0 and L_m coincides with L'_m on U_0, we have $\|L_m(\beta_0\Phi)\| = \|L'_m(\beta_0\Phi)\|$. Thus, we have

$$\|L'_m(\beta_0\Phi)\|^2 < \Lambda_1^2\|\beta_0\Phi\|^2,$$

which implies $\beta_0\Phi = 0$ by the definition of Π'. This contradicts the assumption. Thus, ρ is injective.

We next show that ρ' is injective. Fix $\Phi' \in \mathrm{Ker}(\rho')$. In the same way as above, we have

$$
\begin{aligned}
\|L_m(\beta_0'\Phi')\|^2 = \|L_m'(\beta_0'\Phi')\|^2 &\leq \left(\Lambda_1^2 + C^2\frac{C^2+\Lambda_1^2}{m^2}\right)\|\Phi'\|^2 \\
&\leq \left(\Lambda_1^2 + C^2\frac{C^2+\Lambda_1^2}{m^2}\right)\left(1-\frac{C^2+\Lambda_1^2}{m^2}\right)^{-1}\|\beta_0'\Phi'\|^2 \\
&< \Lambda_2^2\|\beta_0'\Phi'\|^2,
\end{aligned}
$$

which implies ρ' is also injective.

We have now shown that $\rho\colon E_0 \to E_1'$ and $\rho'\colon E_1' \to E_2$ are injective. Since E_0 is finite dimensional and $E_0 = E_2$, it follows that $E_0 \cong E_1'$. The proof is complete. □

Theorem 8. *Let* $(Z = U_0 \cup U_1, 1 = \eta_0 + (1-\eta_0), S_Z, L, h)$ *and* $(Z' = U_0' \cup U_1', 1 = \eta_0' + (1-\eta_0'), S_{Z'}', L', h')$ *be two sets of data as above. We make the following assumptions:*

(i) $|\sigma_L(d\eta_0)|$ *is bounded on* $U_0 \cap U_1$*, and* $|\sigma_{L'}(d\eta_0')|$ *is bounded on* $U_0' \cap U_1'$.
(ii) h *is smooth and anti-commutes with* L *on* U_1*, and* h' *is smooth and anti-commutes with* L' *on* U_1'.
(iii) *the eigenvalues of* h^2 *are greater than or equal to 1 on* U_1*, and the eigenvalues of* $(h')^2$ *are greater than or equal to 1 on* U_1'.
(iv) *there exists an isometry* $\tau\colon U_0 \to U_0'$ *covered by a bundle isometry* $\tilde{\tau}\colon\ S_Z|_{U_0} \to S_{Z'}'|_{U_0'}$ *such that* $\tilde{\tau}^{-1} \circ L_m' \circ \tilde{\tau} = L_m$.
(v) S_Z *and* $S_{Z'}'$ *are* $\mathbb{Z}_2$*-graded and that* L, L', h*, and* h' *are odd operators.*

Then, there exists a constant $C > 0$ *that depends only on* η_0, η_0', σ_L*, and* $\sigma_{L'}$ *such that the following holds. Fix* $\Lambda > 0$ *and* $m > 0$*. If* L_m *is a Fredholm operator with* $(\mathrm{Spec}\, L_m) \cap [-\Lambda, \Lambda] = \{0\}$ *and* $m > 2(C^2 + \Lambda^2)/\Lambda$*, then* L_m' *is also a Fredholm operator and we have*

$$\mathrm{Ind}(L_m) = \mathrm{Ind}(L_m').$$

Proof. Fix $\Lambda > 0$ and $m > 0$ with $(\mathrm{Spec}\, L_m) \cap [-\Lambda, \Lambda] = \{0\}$ and $m > 2(C^2+\Lambda^2)/\Lambda$. Set $\Lambda_2 := \Lambda$, $\Lambda_1 := \Lambda/\sqrt{2}$, and $\Lambda_0 := 0$. Then, $(\Lambda_2, \Lambda_1, \Lambda_0)$ satisfies (5) and (6) of Proposition 6. Note that spectral projections commute with grading operators. Thus, we conclude from Proposition 6 that $\mathrm{Ind}(L_m) = \mathrm{Ind}(L_m')$. □

3. The Main Theorem

3.1. Notation. We define c, ϵ, and Γ by

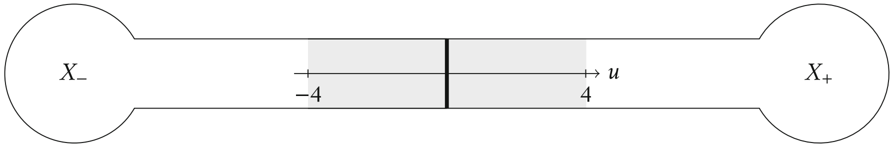
$$c = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad \epsilon = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \text{and } \Gamma = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

Then, $c^2 = -1$, $\epsilon^2 = \Gamma^2 = 1$, $\Gamma = c\epsilon$, and they anti-commute.

Let X be a closed oriented Riemannian manifold with $\dim X$ even. Let S be a $\mathbb{Z}_2$-graded hermitian vector bundle on X, and Γ_S its $\mathbb{Z}_2$-grading operator. Let $D\colon C^\infty(X; S) \to C^\infty(X; S)$ be a first-order, formally self-adjoint, elliptic partial differential operator that anti-commutes with Γ_S. Let $Y \subset X$ be a separating submanifold that decomposes X

into two compact manifolds X_+ and X_- with common boundary Y. Let $\kappa\colon X \to [-1, 1]$ be an L^∞-function such that $\kappa \equiv \pm 1$ on $X_\pm \setminus Y$.

We assume that Y has a collar neighbourhood isometric to the standard product $(-4, 4) \times Y$ and satisfying $((-4, 4) \times Y) \cap X_+ = [0, 4) \times Y$. The coordinate along $(-4, 4)$ is denoted by u.



We also assume that S and D are standard in the following sense: there exist a hermitian bundle E on Y and a bundle isometry from $S|_{(-4,4)\times Y}$ to $\mathbb{C}^2 \otimes E$ such that, under this isometry, Γ_S corresponds to $\Gamma \otimes \mathrm{id}_E$ and D takes the form

$$D = c \otimes \partial_u + \epsilon \otimes A = \begin{pmatrix} 0 & \partial_u + A \\ -\partial_u + A & 0 \end{pmatrix},$$

where $A\colon C^\infty(Y; E) \to C^\infty(Y; E)$ is a formally self-adjoint, elliptic partial differential operator. In this paper, we will concentrate on the case when A has no zero eigenvalues, and assume this condition.

3.2. Spectral gaps. As a first step, we will consider the spectral gap of domain-wall fermion Dirac operators. We begin with the one-dimensional operator

$$c\partial_t + m\,\mathrm{sgn}_0\,\epsilon = \begin{pmatrix} 0 & \partial_t + m\,\mathrm{sgn}_0 \\ -\partial_t + m\,\mathrm{sgn}_0 & 0 \end{pmatrix} : C_c^\infty(\mathbb{R}; \mathbb{C}^2) \to L^2(\mathbb{R}; \mathbb{C}^2),$$

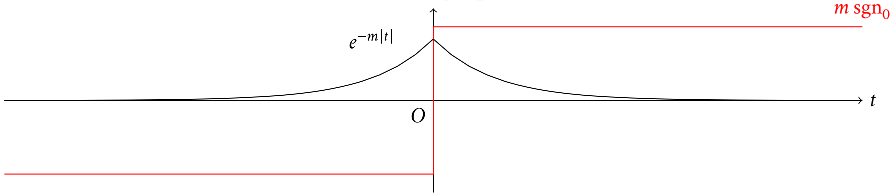
where $\mathrm{sgn}_0\colon \mathbb{R} \to \mathbb{R}$ is a sign function such that $\mathrm{sgn}_0(\pm t) = \pm 1$ for $t > 0$. It is well-known that the operator $(c\partial_t + m\,\mathrm{sgn}_0\,\epsilon)$ is essentially self-adjoint on $L^2(\mathbb{R}; \mathbb{C}^2)$ and that it has essential spectrum equal to $(-\infty, -m] \cup [m, \infty)$ and 0 is a unique and simple eigenvalue. See, for example [6, Theorem 4.2]. We note that

$$\frac{d}{dt} e^{-m|t|} = -m\,\mathrm{sgn}_0\, e^{-m|t|},$$

for any $m > 0$. Let $v_- = (0, 1)^T$, which satisfies $\Gamma v_- = -v_-$. Then, $e^{-m|t|} v_-$ satisfies

$$\big(c\partial_t + m\,\mathrm{sgn}_0\,\epsilon\big)\big(e^{-m|t|} v_-\big) = 0,$$

which is called the Jackiw–Rebbi solution [13].



We next consider the domain-wall fermion Dirac operator

$$(c \otimes \partial_u + \epsilon \otimes A) + \Gamma \otimes (m\,\mathrm{sgn}_0)\colon C_c^\infty(\mathbb{R} \times Y; \mathbb{C}^2 \otimes E) \to L^2(\mathbb{R} \times Y; \mathbb{C}^2 \otimes E)$$

on $\mathbb{R} \times Y$, which is also essentially self-adjoint on $L^2(\mathbb{R} \times Y; \mathbb{C}^2 \otimes E)$. Assume A has no zero eigenvalues. Let λ_A be the positive square root of the first non-zero eigenvalue of A^2. By the method of separation of variables [17]*Theorem VIII.33, we have

$$\mathrm{Spec}\left[(c \otimes \partial_u + \epsilon \otimes A) + \Gamma \otimes (m\,\mathrm{sgn}_0)\right] \cap (-\lambda_A, \lambda_A) = \emptyset$$

for any $m > \lambda_A$. We now proceed to the domain-wall fermion Dirac operator on X via Proposition 6. Recall that $D = c \otimes \partial_u + \epsilon \otimes A$ on the neck $(-4, 4) \times Y \subset X$.

Proposition 9. *Assume A has no zero eigenvalues. Let λ_A be the positive square root of the first non-zero eigenvalue of A^2. Then, there exists a constant $m_1 > 0$ that depends only on λ_A such that we have*

$$\mathrm{Spec}(D + m\kappa\Gamma_S) \cap \left(-\frac{\lambda_A}{2}, \frac{\lambda_A}{2}\right) = \emptyset$$

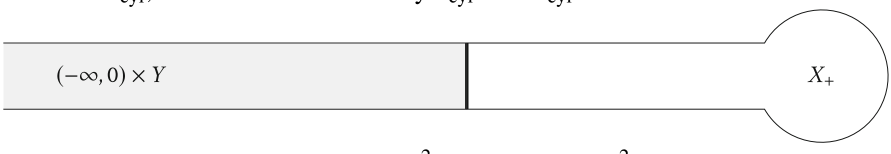
for any $m > m_1$.

Proof. We apply Proposition 6 for $\big((c \otimes \partial_u + \epsilon \otimes A) + \Gamma \otimes (m\,\mathrm{sgn}_0)\big)$ on $\mathbb{R} \times Y$ and $(D + m\kappa\Gamma_S)$ on X with $U_0 = U_0' = (-4, 4) \times Y$. Let $\Lambda_2 := \lambda_A$, $\Lambda_1 := \lambda_A/2$, and $\Lambda_0 := 0$, and we have $C > 0$ of Proposition 6. We set

$$m_1^2 := \max\left\{\frac{(C^2 + \Lambda_2^2)(C^2 + \Lambda_1^2)}{\Lambda_2^2 - \Lambda_1^2}, \frac{(C^2 + \Lambda_1^2)(C^2 + \Lambda_0^2)}{\Lambda_1^2 - \Lambda_0^2}, (C^2 + \Lambda_2^2)\right\},$$

which yields the conclusion. □

3.3. Product formula. Next, we will modify D on X_+. Let $X_{\mathrm{cyl}} := (-\infty, 0] \times Y \cup X_+$ with the standard cylindrical-end metric. The bundle S and the operator D naturally extends to X_{cyl}, which will be denoted by S_{cyl} and D_{cyl}.

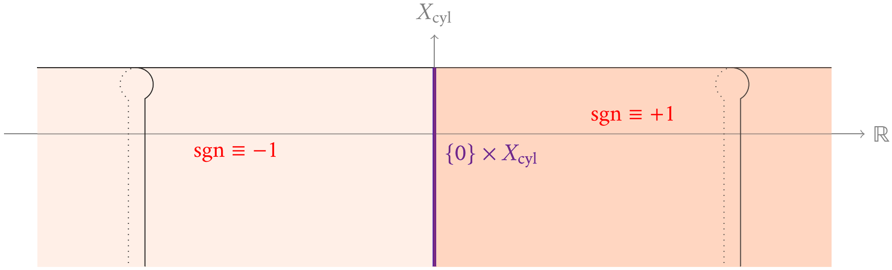


Recall [1, Corollary (3.14)] that $D_{\mathrm{cyl}} \colon L^2(X_{\mathrm{cyl}}; S_{\mathrm{cyl}}) \to L^2(X_{\mathrm{cyl}}; S_{\mathrm{cyl}})$ is a Fredholm operator if A has no zero eigenvalues; thus, there exists $\lambda_{D_{\mathrm{cyl}}} > 0$ such that $\mathrm{Spec}\, D_{\mathrm{cyl}} \cap (-\lambda_{D_{\mathrm{cyl}}}, \lambda_{D_{\mathrm{cyl}}}) = \{0\}$.

Let $\mathrm{sgn} \colon \mathbb{R} \times X_{\mathrm{cyl}} \to [-1, 1]$ be an L^∞-function such that $\mathrm{sgn} \equiv -1$ on $(-\infty, 0) \times X_{\mathrm{cyl}}$ and $\mathrm{sgn} \equiv 1$ on $(0, \infty) \times X_{\mathrm{cyl}}$. We consider a bundle $\mathbb{C}^2 \otimes S_{\mathrm{cyl}}$ on $\mathbb{R} \times X_{\mathrm{cyl}}$ equipped with a $\mathbb{Z}_2$-grading operator $\Gamma \otimes \mathrm{id}_S$ and an odd operator

$$\epsilon \otimes (D_{\mathrm{cyl}} + m\,\mathrm{sgn}\,\Gamma_S) + c \otimes \partial_t = \begin{pmatrix} 0 & (D_{\mathrm{cyl}} + m\,\mathrm{sgn}\,\Gamma_S) + \partial_t \\ (D_{\mathrm{cyl}} + m\,\mathrm{sgn}\,\Gamma_S) - \partial_t & 0 \end{pmatrix},$$

which is self-adjoint on $L^2(\mathbb{R} \times X_{\mathrm{cyl}}; \mathbb{C}^2 \otimes S_{\mathrm{cyl}})$. Note that this operator is a coordinate change of the graded tensor product of $(c\partial_t + m \operatorname{sgn}_0 \epsilon)$ and D_{cyl}.



Proposition 10. *If A has no zero eigenvalues, then the operator $\big(\epsilon \otimes (D_{\mathrm{cyl}} + m \operatorname{sgn} \Gamma_S) + c \otimes \partial_t\big)$ is also Fredholm, and we have*

$$\operatorname{Ind}(D_{\mathrm{cyl}}) = -\operatorname{Ind}\left[\epsilon \otimes (D_{\mathrm{cyl}} + m \operatorname{sgn} \Gamma_S) + c \otimes \partial_t\right]$$

for any $m > 0$, and

$$\operatorname{Spec}\left[\epsilon \otimes (D_{\mathrm{cyl}} + m \operatorname{sgn} \Gamma_S) + c \otimes \partial_t\right] \cap (-\lambda_{D_{\mathrm{cyl}}}, \lambda_{D_{\mathrm{cyl}}}) = \{0\}$$

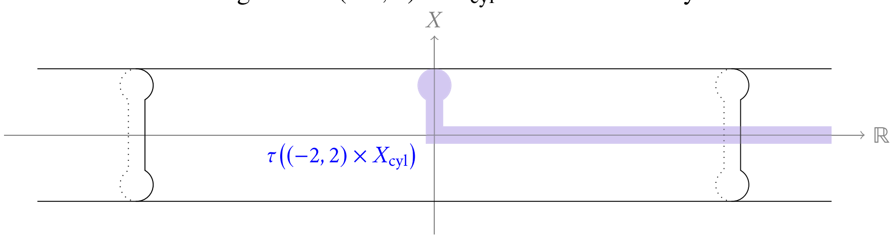
for any $m > \lambda_{D_{\mathrm{cyl}}}$.

Proof. Assume $D_{\mathrm{cyl}}\phi = 0$. Set $\phi_\pm := (\phi \pm \Gamma_S \phi)/2$. Recall that $\left(e^{-m|t|}\right)' = -m \operatorname{sgn}_0 e^{-m|t|}$ for any $m > 0$. Then, we have

$$\begin{pmatrix} 0 & (D_{\mathrm{cyl}} + m \operatorname{sgn} \Gamma_S) + \partial_t \\ (D_{\mathrm{cyl}} + m \operatorname{sgn} \Gamma_S) - \partial_t & 0 \end{pmatrix} \begin{pmatrix} e^{-m|t|}\phi_- \\ e^{-m|t|}\phi_+ \end{pmatrix} = 0.$$

The details are left to the reader. □

3.4. Embeddings into a cylinder. At the heart of this paper lies our next step, which constructs an embedding τ from $(-2, 2) \times X_{\mathrm{cyl}}$ into the infinite cylinder $\mathbb{R} \times X$.



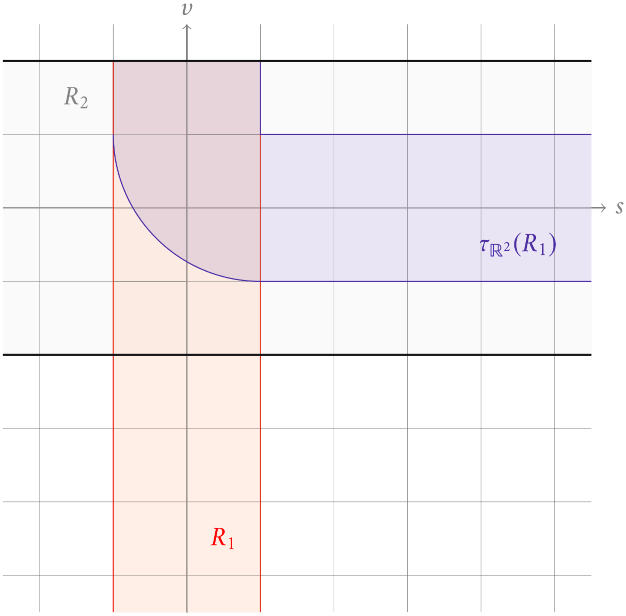
Let $R_1 := (-2, 2) \times (-\infty, 4)$ and $R_2 := \mathbb{R} \times (-4, 4)$. We denote the coordinates of R_1 by (t, u) and that of R_2 by (s, v). Fix an embedding $\tau_{\mathbb{R}^2} \colon R_1 \to R_2$ such that $\tau_{\mathbb{R}^2} \equiv \mathrm{id}$ for $2 \le u$ and

$$\begin{pmatrix} t \\ u \end{pmatrix} \mapsto \begin{pmatrix} -u \\ t \end{pmatrix}$$

for $u \le -100$. Since Y has a collar neighbourhood isometric to $(-4, 4) \times Y$, we can regard $R_1 \times Y$ and $R_2 \times Y$ as open subsets of $(-2, 2) \times X_{\mathrm{cyl}}$ and $\mathbb{R} \times X$ respectively. We define an embedding

$$\tau \colon (-2, 2) \times X_{\mathrm{cyl}} \to \mathbb{R} \times X$$

by $\tau \equiv \mathrm{id}_{\mathbb{R}} \times \mathrm{id}_X$ on $(-2,2) \times X_+$ and $\tau \equiv \tau_{\mathbb{R}^2} \times \mathrm{id}_Y$ on $R_1 \times Y$. By construction, τ is an isometry outside a compact set $\big((-2,2) \times (-100,2) \times Y\big)$.



We modify the Riemannian metric on $\mathbb{R} \times X$ so that τ becomes an isometry. Let g denote the product metric on $(-2,2) \times X_{\mathrm{cyl}}$ and g' on $\mathbb{R} \times X$. Let $\chi \colon \mathbb{R} \times X \to [0,1]$ be a bump function such that $\chi \equiv 1$ on $\tau\big((-1,1) \times X_{\mathrm{cyl}}\big)$ and $\chi \equiv 0$ outside $\tau\big((-2,2) \times X_{\mathrm{cyl}}\big)$. We define a family of Riemannian metrics g'_r connecting $g'_0 := g'$ to $g'_1 := \chi\big((\tau^{-1})^* g\big) + (1-\chi) g'$ by

$$g'_r := (1-r)g' + r\Big(\chi\big((\tau^{-1})^* g\big) + (1-\chi)g'\Big)$$

for $r \in [0,1]$. Now τ is an isometry from $\big((-1,1) \times X_{\mathrm{cyl}}, g\big)$ to $\big(\mathbb{R} \times X, g'_1\big)$. We also modify the hermitian metric on S conformally so that the L^2-norm on $C_c^\infty(\mathbb{R} \times X; \mathbb{C}^2 \otimes S)$ remains unchanged. Let $d\mu(g'_r)$ denote the volume form of $\big(\mathbb{R} \times X, g'_r\big)$, and we define $f_r \colon \mathbb{R} \times X \to \mathbb{R}$ by $d\mu(g'_r) = e^{2 f_r} d\mu(g')$. Let $(\cdot,\cdot)$ denote the hermitian metric on S. We define a family of hermitian metrics $(\cdot,\cdot)_r$ on S by

$$(\cdot,\cdot)_r := e^{-2 f_r} (\cdot,\cdot)$$

for $r \in [0,1]$. Now the L^2-norm on $C_c^\infty(\mathbb{R} \times X; \mathbb{C}^2 \otimes S)$ with $d\mu(g'_r)$ and $(\cdot,\cdot)_r$ remains unchanged. Using Spin(2)-action on $\mathbb{C}^2$, we can lift τ to $\tilde{\tau} \colon \mathbb{C}^2 \otimes S_{\mathrm{cyl}} \to \mathbb{C}^2 \otimes S$ so that

$$\tilde{\tau}^{-1} \circ \big[\epsilon \otimes D + c \otimes \partial_s\big] \circ \tilde{\tau} = \big[\epsilon \otimes D_{\mathrm{cyl}} + c \otimes \partial_t\big] \tag{7}$$

holds. Note that g'_r coincides with g' outside a compact set. We refer the reader to [4] for a related construction.

3.5. The eta invariant of domain-wall fermion Dirac operators. We will slightly extend the definition of the eta invariant to take care of discontinuities of domain-wall fermion Dirac operators.

Fix $m > 0$. Since $\mathrm{Ker}(D + m\kappa\Gamma_S) = \{0\}$, there exists a constant $C_m > 0$ such that

$$\mathrm{Ker}(D + m\kappa\Gamma_S + f) = \{0\}$$

for any $f \in L^2(Z; \mathrm{End}(S_Z))$ with $\|f\|_2 < C_m$.

Lemma 11. *Let $f_1 \in L^2(Z; \mathrm{End}(S_Z))$ with $\|f_1\|_2 < C_m$ and $f_2 \in L^2(Z; \mathrm{End}(S_Z))$ with $\|f_2\|_2 < C_m$. Assume that $m\kappa\Gamma_S + f_1$ and $m\kappa\Gamma_S + f_2$ are smooth operators. Then, we have*

$$\eta(D + m\kappa\Gamma_S + f_1) = \eta(D + m\kappa\Gamma_S + f_2).$$

Proof. This is a direct consequence of the variational formula of the eta invariant [11, Theorem 1.13.2]. □

We now define the eta invariant of domain-wall fermion Dirac operators by

$$\eta(D + m\kappa\Gamma_S) := \eta(D + m\kappa\Gamma_S + f) \tag{8}$$

for any $f \in L^2(Z; \mathrm{End}(S_Z))$ such that $\|f\|_2 < C_m$ and that $m\kappa\Gamma_S + f$ is a smooth operator. This definition is well defined by Lemma 11.

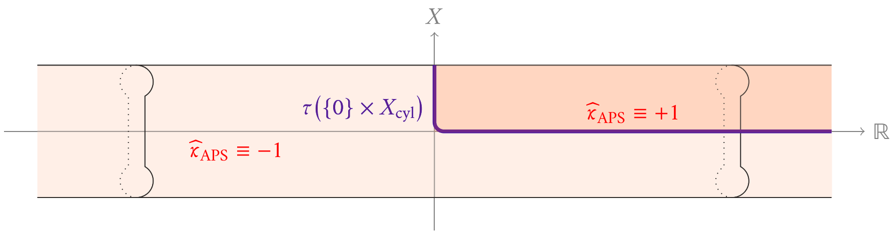
3.6. Main theorem.

Theorem 12. *If $A\colon C^\infty(Y; E) \to C^\infty(Y; E)$ has no zero eigenvalues, then there exists a constant $m_0 > 0$ that depends only on X, S, and D such that we have*

$$\mathrm{Ind}_{\mathrm{APS}}(D|_{X_+}) = \frac{\eta(D + m\kappa\Gamma_S) - \eta(D - m\Gamma_S)}{2}$$

for any $m > m_0$.

Proof. We begin with the observation that $\mathbb{R} \times X \backslash \tau\big(\{0\} \times X_{\mathrm{cyl}}\big)$ has two connected components; we will denote by $(\mathbb{R} \times X)_-$ the one containing $\{-10\} \times X_+$ and by $(\mathbb{R} \times X)_+$ the other half. Let $\widehat{\kappa}_{\mathrm{APS}}\colon \mathbb{R} \times X \to [-1, 1]$ be an L^∞-function such that $\widehat{\kappa}_{\mathrm{APS}} \equiv \pm 1$ on $(\mathbb{R} \times X)_\pm$.



Fix $m > 0$. We introduce an operator $\widehat{\mathfrak{D}}_m\colon C_c^\infty(\mathbb{R} \times X; \mathbb{C}^2 \otimes S) \to L^2(\mathbb{R} \times X; \mathbb{C}^2 \otimes S)$ defined by

$$\begin{aligned} \widehat{\mathfrak{D}}_m &:= \epsilon \otimes (D + m\widehat{\kappa}_{\mathrm{APS}}\Gamma_S) + c \otimes \partial_s \\ &= \begin{pmatrix} 0 & (D + m\widehat{\kappa}_{\mathrm{APS}}\Gamma_S) + \partial_s \\ (D + m\widehat{\kappa}_{\mathrm{APS}}\Gamma_S) - \partial_s & 0 \end{pmatrix}, \end{aligned} \tag{9}$$

which is essentially self-adjoint on $L^2(\mathbb{R} \times X; \mathbb{C}^2 \otimes S)$.

By (7), we have

$$\tilde{\tau}^{-1} \circ \widehat{\mathcal{D}}_m \circ \tilde{\tau} = \left[\epsilon \otimes (D_{\mathrm{cyl}} + m \operatorname{sgn} \Gamma_S) + c \otimes \partial_t\right].$$

By Proposition 10, we have

$$\operatorname{Spec}\left[\epsilon \otimes (D_{\mathrm{cyl}} + m \operatorname{sgn} \Gamma_S) + c \otimes \partial_t\right] \cap (-\lambda_{D_{\mathrm{cyl}}}, \lambda_{D_{\mathrm{cyl}}}) = \{0\}$$

for any $m > \lambda_{D_{\mathrm{cyl}}}$. Set $\Lambda := \lambda_{D_{\mathrm{cyl}}}$. We now apply Theorem 8 for $\left(\epsilon \otimes (D_{\mathrm{cyl}} + m \operatorname{sgn} \Gamma_S) + c \otimes \partial_t\right)$ on $\mathbb{R} \times X_{\mathrm{cyl}}$ and $\widehat{\mathcal{D}}_m$ on $\mathbb{R} \times X$ equipped with the modified metric g'_1. By homotopy invariance of the index, we can use either g' or g'_1 to compute $\operatorname{Ind}(\widehat{\mathcal{D}}_m)$. Thus, we have the constant $C > 0$ such that

$$\operatorname{Ind}\left[\epsilon \otimes (D_{\mathrm{cyl}} + m \operatorname{sgn} \Gamma_S) + c \otimes \partial_t\right] = \operatorname{Ind}(\widehat{\mathcal{D}}_m) \tag{10}$$

for $m > 2(C^2 + \Lambda^2)/\Lambda$. Note that $2(C^2 + \Lambda^2)/\Lambda > \Lambda = \lambda_{D_{\mathrm{cyl}}}$. By Proposition 10, we also have

$$\operatorname{Ind}(D_{\mathrm{cyl}}) = -\operatorname{Ind}\left[\epsilon \otimes (D_{\mathrm{cyl}} + m \operatorname{sgn} \Gamma_S) + c \otimes \partial_t\right] \tag{11}$$

for any $m > 0$.

To apply the Atiyah–Patodi–Singer index theorem, we will approximate $\widehat{\mathcal{D}}_m$ by an operator with smooth coefficients. Let $C_m > 0$ be the same constant as in Lemma 11. Let $\widehat{\kappa}^{\mathrm{sm}}_{\mathrm{APS}} \colon \mathbb{R} \times X \to [-1, 1]$ be a *smooth* function that approximates $\widehat{\kappa}_{\mathrm{APS}}$ such that $\widehat{\kappa}^{\mathrm{sm}}_{\mathrm{APS}} \equiv -1$ on $\{-10\} \times X$,

$$\| \widehat{\kappa}^{\mathrm{sm}}_{\mathrm{APS}}|_{\{10\}\times X} - \kappa \|_{L^2(\{10\}\times X)} < C_m,$$

and $\operatorname{Ind}(\widehat{\mathcal{D}}_m) = \operatorname{Ind}(\widehat{\mathcal{D}}^{\mathrm{sm}}_m)$, where

$$\widehat{\mathcal{D}}^{\mathrm{sm}}_m := \epsilon \otimes (D + m\widehat{\kappa}^{\mathrm{sm}}_{\mathrm{APS}}\Gamma_S) + c \otimes \partial_s = \begin{pmatrix} 0 & (D + m\widehat{\kappa}^{\mathrm{sm}}_{\mathrm{APS}}\Gamma_S) + \partial_s \\ (D + m\widehat{\kappa}^{\mathrm{sm}}_{\mathrm{APS}}\Gamma_S) - \partial_s & 0 \end{pmatrix}.$$

Note that

$$\widehat{\mathcal{D}}_m = \begin{cases} \epsilon \otimes (D - m\Gamma_S) + c \otimes \partial_s & \text{on } \{-10\} \times X \\ \epsilon \otimes \left(D + m\left(\widehat{\kappa}^{\mathrm{sm}}_{\mathrm{APS}}|_{\{10\}\times X}\right)\Gamma_S\right) + c \otimes \partial_s & \text{on } \{+10\} \times X. \end{cases}$$

By Proposition 9, the domain-wall fermion Dirac operator $D + m\kappa\Gamma_S$ has no zero eigenvalues if $m > m_1$;hence, neither does $D + m\left(\widehat{\kappa}^{\mathrm{sm}}_{\mathrm{APS}}|_{\{10\}\times X}\right)\Gamma_S$. The Atiyah–Patodi–Singer index theorem on the cylinder [1, (2.27)] yields

$$\operatorname{Ind}(\widehat{\mathcal{D}}^{\mathrm{sm}}_m) = -\frac{\eta\left(D + m\left(\widehat{\kappa}^{\mathrm{sm}}_{\mathrm{APS}}|_{\{10\}\times X}\right)\Gamma_S\right) - \eta(D - m\Gamma_S)}{2},$$

where we have used the assumption that $\dim(\mathbb{R} \times X)$ is odd and the fact [11, Lemma 1.8.2 (d)] that the constant term in the asymptotic expansion of the heat kernel of an elliptic *differential* operator on an odd-dimensional manifold vanishes.[2] By the definition (8), we

[2] This does not remain true for pseudodifferential operators [16, Theorem 13.12].

have $\eta\big(D+m\big(\widehat{\kappa}^{\mathrm{sm}}_{\mathrm{APS}}|_{\{10\}\times X}\big)\Gamma_S\big)=\eta(D+m\kappa\Gamma_S)$. By assumption, we have $\mathrm{Ind}(\widehat{\mathscr{D}}_m)=\mathrm{Ind}(\widehat{\mathscr{D}}^{\mathrm{sm}}_m)$. Thus, we get

$$\mathrm{Ind}(\widehat{\mathscr{D}}_m)=-\frac{\eta(D+m\kappa\Gamma_S)-\eta(D-m\Gamma_S)}{2}. \tag{12}$$

Now set $m_0:=\max\{m_1,2(C^2+\Lambda^2)/\Lambda\}$. Note that $\mathrm{Ind}_{\mathrm{APS}}(D|_{X_+})=\mathrm{Ind}(D_{\mathrm{cyl}})$ by [1, Proposition 3.11]. Combining (10), (11), and (12), we have

$$\mathrm{Ind}_{\mathrm{APS}}(D|_{X_+})=\frac{\eta(D+m\kappa\Gamma_S)-\eta(D-m\Gamma_S)}{2}$$

for $m>m_0$. The proof is complete. □

Remark 13. The particular choice of κ will not be essential in our arguments but clarify our idea. See Lemma 11. Our arguments extend easily to the case when κ satisfies $\kappa\equiv\kappa_\pm$ on $X_\pm\backslash(-2,2)\times Y$ for some $\kappa_\pm\in\mathbb{R}$ with $\kappa_+\kappa_-<0$. In particular, taking an extreme limit $(-\kappa_-)\gg\kappa_+$, we recover Shamir domain-wall fermions [9, 19]. See [8, IV.B].

Remark 14. We have used the Atiyah–Patodi–Singer theorem only on *cylinders*.

We conclude this paper with a problem. Although the proof above implies that there are no *edge-of-edge states* or *corner states* in our situation, we expect that corner states will emerge if we introduce extra domain walls. Let $\widehat{\kappa}_{\llcorner}:\mathbb{R}\times X\to[-1,1]$ be a bump function such that $\widehat{\kappa}_{\llcorner}\equiv 1$ on $[-10,10]\times X$ and $\widehat{\kappa}_{\llcorner}\equiv -1$ outside $[-10,10]\times X$. Fix $M\gg 0$. We consider yet another self-adjoint operator $\widehat{\mathscr{D}}_m+M\widehat{\kappa}_{\llcorner}(\Gamma\otimes\mathrm{id}_S)$ so that corner states would emerge around $\{10\}\times Y$. Then, perturbation arguments as in [8] lead us to ask whether

$$\mathrm{Ind}_{\mathrm{APS}}\big(\widehat{\mathscr{D}}_m \text{ on } [-10,10]\times X\big)=\frac{\eta\big(\widehat{\mathscr{D}}_m+M\widehat{\kappa}_{\llcorner}(\Gamma\otimes\mathrm{id}_S)\big)-\eta\big(\widehat{\mathscr{D}}_m-M(\Gamma\otimes\mathrm{id}_S)\big)}{2}$$

holds with some regularisation to define the right-hand side.

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