JOURNAL OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 28, Number 1, January 2015, Pages 159–182
S 0894-0347(2014)00798-0
Article electronically published on May 22, 2014

BRODY CURVES AND MEAN DIMENSION

SHINICHIROH MATSUO AND MASAKI TSUKAMOTO

1. Introduction

1.1. **Main result.** Let $z = x + y\sqrt{-1} \in \mathbb{C}$ be the standard coordinate in the complex plane $\mathbb{C}$. Let $f = [f_0 : f_1 : \cdots : f_N] : \mathbb{C} \to \mathbb{C}P^N$ be a holomorphic map (f_i: holomorphic function). We define $|df|(z) \geq 0$ by

$$|df|^2(z) := \frac{1}{4\pi}\Delta \log(|f_0|^2 + |f_1|^2 + \cdots + |f_N|^2) \quad \left(\Delta := \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right).$$

$|df|(z)$ is classically called a spherical derivative. It evaluates the dilatation of the map f with respect to the Euclidean metric on $\mathbb{C}$ and the Fubini–Study metric on $\mathbb{C}P^N$. Namely, for a tangent vector $u \in T_z\mathbb{C}$, the norm of $df(u) \in T_{f(z)}\mathbb{C}P^N$ is given by $|df(u)| = |df|(z) \cdot |u|$. See the equation (6) in Section 4.2 for more details.

A holomorphic map $f : \mathbb{C} \to \mathbb{C}P^N$ is called a **Brody curve** ([3]) if it satisfies $|df|(z) \leq 1$ for all $z \in \mathbb{C}$ (i.e. f is 1-Lipschitz). When $N = 1$, Brody curves are meromorphic functions $f : \mathbb{C} \to \mathbb{C} \cup \{\infty\}$ satisfying

$$|df|(z) = \frac{|f'(z)|}{\sqrt{\pi}(1 + |f(z)|^2)} \leq 1.$$

The exponential function $f(z) = e^z$ satisfies this condition. If $f(z)$ is a rational function or elliptic function, then we can find $c > 0$ such that $f(cz)$ is a Brody curve. Hence all rational/exponential/elliptic functions become Brody curves under some scale changes. So Brody curves exhibit a quite wide variety of behaviors. Moreover we cannot expect any symmetry for general Brody curves. Therefore it is difficult to establish a deep structure theory of Brody curves.

In this paper we adopt a relatively new viewpoint (initiated by Gromov [11]) for the study of Brody curves. We study the dynamical system consisting of Brody curves. General Brody curves might have little structure, but there is a possibility that the system of all Brody curves has much more structure than individual Brody curves. The purpose of this paper is to show that it certainly has a beautiful structure.

Let $\mathcal{M}(\mathbb{C}P^N)$ be the space of Brody curves in $\mathbb{C}P^N$ endowed with the compact-open topology (the topology of uniform convergence on compact subsets). $\mathcal{M}(\mathbb{C}P^N)$ is a compact metrizable space, and it becomes a dynamical system under the following continuous $\mathbb{C}$-action:

$$\mathcal{M}(\mathbb{C}P^N) \times \mathbb{C} \to \mathcal{M}(\mathbb{C}P^N), \quad (f(z), a) \mapsto f(z + a).$$

Received by the editors June 28, 2012 and, in revised form, January 6, 2014.
2010 *Mathematics Subject Classification.* Primary 32H30, 54H20.
Key words and phrases. Brody curve, mean dimension, deformation theory.

We can easily prove that the dynamical system $\mathcal{M}(\mathbb{C}P^N)$ is infinite dimensional and has infinite topological entropy. So this is a very huge system. Gromov [11] introduced the notion **mean dimension** for the study of this kind of huge dynamical system. Mean dimension is an invariant of topological dynamical systems. We review its definition in Section 2.1. Our main subject in this paper is to calculate the mean dimension $\dim(\mathcal{M}(\mathbb{C}P^N):\mathbb{C})$ of the system $\mathcal{M}(\mathbb{C}P^N)$.

Let $f:\mathbb{C}\to\mathbb{C}P^N$ be a Brody curve. We define an **energy density** $\rho(f)$ by setting

$$\rho(f) := \lim_{R\to\infty}\frac{1}{\pi R^2}\left(\sup_{a\in\mathbb{C}}\int_{|z-a|<R}|df|^2dxdy\right). \tag{1}$$

This limit always exists; see Section 2.2. Let $\rho(\mathbb{C}P^N)$ be the supremum of $\rho(f)$ over $f\in\mathcal{M}(\mathbb{C}P^N)$.

The main result of this paper is the following:

Theorem 1.1.

$$2(N+1)\rho(\mathbb{C}P^N)\le\dim(\mathcal{M}(\mathbb{C}P^N):\mathbb{C})\le 4N\rho(\mathbb{C}P^N).$$

Corollary 1.2.

$$\dim(\mathcal{M}(\mathbb{C}P^1):\mathbb{C}) = 4\rho(\mathbb{C}P^1).$$

The formula $\dim(\mathcal{M}(\mathbb{C}P^1):\mathbb{C}) = 4\rho(\mathbb{C}P^1)$ was conjectured in [21, p. 1643, (4)]. This formula is very surprising (at least for the authors) because the definitions of the left-hand side and the right-hand side are totally different.

The upper bound $\dim(\mathcal{M}(\mathbb{C}P^N):\mathbb{C})\le 4N\rho(\mathbb{C}P^N)$ was already proved in [18, Theorem 1.5] by using the Nevanlinna theory. (Remark: A different definition of the energy density was used in the papers [18, 21]. For this point, see Section 2.2.) The subject of the present paper is to prove the lower bound $\dim(\mathcal{M}(\mathbb{C}P^N):\mathbb{C})\ge 2(N+1)\rho(\mathbb{C}P^N)$.

1.2. **Non-degenerate Brody curves.** For $a\in\mathbb{C}$ and $r>0$ we set $D_r(a):=\{z\in\mathbb{C}\,|\,|z-a|\le r\}$. The following is a key notion of the paper. This notion was first introduced by Yosida [23]. Gromov [11, p. 399] also discussed it in a more general situation. See also Eremenko [4, Section 4] and Remark 1.4 below.

Definition-Lemma 1.3. Let $f:\mathbb{C}\to\mathbb{C}P^N$ be a Brody curve. Then the following two conditions are equivalent.

(i) Any constant curve does not belong to the closure of the $\mathbb{C}$-orbit of f. In other words, for any sequence of complex numbers $\{a_n\}_{n\ge1}$, the sequence of Brody curves $\{f(z+a_n)\}_{n\ge1}$ does not converge to a constant curve.

(ii) There exist $\delta>0$ and $R>0$ such that for all $a\in\mathbb{C}$ we have $\|df\|_{L^\infty(D_R(a))}\ge\delta$.

f is said to be **non-degenerate** if it satisfies one of (and hence both) the above conditions.

Proof. The following argument is given in [23]. Suppose that the condition (ii) fails. Then for any $n\ge1$ there is $a_n\in\mathbb{C}$ such that $\|df\|_{L^\infty(D_1(a_n))}\le 1/n$. Taking a subsequence, we can assume that the sequence $\{f(z+a_n)\}_{n\ge1}$ converges to a Brody curve $g(z)$. Then $\|dg\|_{L^\infty(D_1(0))}=0$. This implies that g is a constant curve.

Suppose the condition (ii) holds. Let $\{a_n\}_{n\geq 1}$ be a sequence of complex numbers. If $\{f(z+a_n)\}_{n\geq 1}$ converges to $g(z)$, then $\|dg\|_{L^\infty(D_R(0))}\geq \delta$. Hence $g(z)$ is not a constant curve. This proves the condition (i). $\square$

Remark 1.4. The above argument also proves that the conditions in Definition-Lemma 1.3 are equivalent to the following:

(ii') For any $R>0$ there exists $\delta>0$ such that for all $a\in\mathbb{C}$ we have $\|df\|_{L^\infty(D_R(a))}\geq\delta$.

Yosida [23, Theorem 4] proved (i) $\Leftrightarrow$ (ii') for the case of $N=1$. In [23] Brody curves $f:\mathbb{C}\to\mathbb{C}P^1$ satisfying (i) are called meromorphic functions of 1st category. In Eremenko [4, Section 4] Brody curves $f:\mathbb{C}\to\mathbb{C}P^N$ satisfying (i) are called binormal curves. Gromov [11, p. 399] used the terminology "uniformly nondegenerate."

Example 1.5. If $f(z)\in\mathcal{M}(\mathbb{C}P^1)$ is a rational or exponential function, then it is a degenerate (i.e. not non-degenerate) Brody curve. A non-constant elliptic function $f(z)\in\mathcal{M}(\mathbb{C}P^1)$ is a non-degenerate Brody curve.

In our viewpoint, non-degenerate Brody curves are "non-singular points" of the system $\mathcal{M}(\mathbb{C}P^N)$, and they behave very nicely for the calculation of the mean dimension.

Theorem 1.6. *Let $f:\mathbb{C}\to\mathbb{C}P^N$ be a non-degenerate Brody curve with $\|df\|_{L^\infty(\mathbb{C})}<1$. Then*

$$\dim(\mathcal{M}(\mathbb{C}P^N):\mathbb{C})\geq 2(N+1)\rho(f).$$

Next we show that there are "sufficiently many" non-degenerate Brody curves.

Theorem 1.7. *Let $f:\mathbb{C}\to\mathbb{C}P^N$ be a holomorphic map with $\|df\|_{L^\infty(\mathbb{C})}<1$. Then for any $\varepsilon>0$ there exists a non-degenerate Brody curve $g:\mathbb{C}\to\mathbb{C}P^N$ satisfying $\|dg\|_{L^\infty(\mathbb{C})}<1$ and $\rho(g)\geq\rho(f)-\varepsilon$.*

Proof of Theorem 1.1*, assuming Theorems* 1.6 *and* 1.7. The upper bound $\dim(\mathcal{M}(\mathbb{C}P^N):\mathbb{C})\leq 4N\rho(\mathbb{C}P^N)$ was already proved in [18, Theorem 1.5]. Here we prove the lower bound. Let $f:\mathbb{C}\to\mathbb{C}P^N$ be a Brody curve. Let $0<c<1$ and set $f_c(z)=f(cz)$. Then $|df_c|(z)=c|df|(cz)$ and $\rho(f_c)=c^2\rho(f)$. Since $\|df_c\|_{L^\infty(\mathbb{C})}\leq c<1$, we can apply Theorem 1.7 to f_c. Then for any $\varepsilon>0$ there exists a non-degenerate Brody curve $g:\mathbb{C}\to\mathbb{C}P^N$ satisfying $\|dg\|_{L^\infty(\mathbb{C})}<1$ and $\rho(g)\geq\rho(f_c)-\varepsilon=c^2\rho(f)-\varepsilon$. By Theorem 1.6

$$\dim(\mathcal{M}(\mathbb{C}P^N):\mathbb{C})\geq 2(N+1)\rho(g)\geq 2(N+1)(c^2\rho(f)-\varepsilon).$$

Let $\varepsilon\to 0$ and $c\to 1$. We get $\dim(\mathcal{M}(\mathbb{C}P^N):\mathbb{C})\geq 2(N+1)\rho(f)$. Taking the supremum over $f\in\mathcal{M}(\mathbb{C}P^N)$, we get $\dim(\mathcal{M}(\mathbb{C}P^N):\mathbb{C})\geq 2(N+1)\rho(\mathbb{C}P^N)$. $\square$

2. Some preliminaries

2.1. **Review of mean dimension.** In this subsection we review the definition of mean dimension. For the detail, see Gromov [11] and Lindenstrauss–Weiss [13]. For some related works, see also Lindenstrauss [12] and Gournay [6–8, 10].

Let (X,d) be a compact metric space, and let Y be a topological space. Let $\varepsilon>0$. A continuous map $f:X\to Y$ is called an ε-embedding if $\mathrm{Diam}f^{-1}(y)\leq\varepsilon$

for all $y \in Y$. Here $\mathrm{Diam} f^{-1}(y)$ is the supremum of $d(x_1, x_2)$ over $x_1, x_2 \in f^{-1}(y)$. We define $\mathrm{Widim}_\varepsilon(X, d)$ as the minimum integer $n \geq 0$ such that there are an n-dimensional polyhedron P and an ε-embedding $f : X \to P$.

For example, let $X = [0,1] \times [0,\varepsilon]$ with the Euclidean distance. Then the projection $\pi : X \to [0,1]$ is an ε-embedding, and we have $\mathrm{Widim}_\varepsilon(X, \mathrm{Euclid}) = 1$. The following example is very important in the later argument. This was given by Gromov [11, p. 333]. For the detailed proof, see Gournay [10, Lemma 2.5] or Tsukamoto [21, Appendix].

Example 2.1. Let V be a finite dimensional Banach space over $\mathbb{R}$, and set $B_r(V) := \{x \in V \mid \|x\| \leq r\}$ for $r > 0$. For $0 < \varepsilon < r$,

$$\mathrm{Widim}_\varepsilon(B_r(V), \|\cdot\|) = \dim V.$$

Here we consider the norm distance on $B_r(V)$.

For a subset $\Omega \subset \mathbb{C}$ and $r > 0$, we define $\partial_r \Omega$ as the set of $a \in \mathbb{C}$ satisfying $D_r(a) \cap \Omega \neq \emptyset$ and $D_r(a) \cap (\mathbb{C} \setminus \Omega) \neq \emptyset$. Let Ω_n $(n \geq 1)$ be a sequence of bounded Borel subsets of $\mathbb{C}$. It is called a Følner sequence if for all $r > 0$

$$\frac{\mathrm{Area}(\partial_r \Omega_n)}{\mathrm{Area}(\Omega_n)} \to 0 \quad (n \to \infty).$$

For example, the sequence $\Omega_n := D_n(0)$ is a Følner sequence. The sequence $\Omega_n := [0,n] \times [0,n]$ is also Følner. We need the following "Ornstein–Weiss lemma." For the proof, see Gromov [11, pp. 336–338].

Lemma 2.2. *Let $h : \{\text{bounded Borel subsets of } \mathbb{C}\} \to \mathbb{R}_{\geq 0}$ be a map satisfying the following three conditions:*

(i) *If $\Omega_1 \subset \Omega_2$, then $h(\Omega_1) \leq h(\Omega_2)$.*
(ii) *$h(\Omega_1 \cup \Omega_2) \leq h(\Omega_1) + h(\Omega_2)$.*
(iii) *For any $a \in \mathbb{C}$ and any bounded Borel subset $\Omega \subset \mathbb{C}$, we have $h(a + \Omega) = h(\Omega)$ where $a + \Omega := \{a + z \in \mathbb{C} \mid z \in \Omega\}$.*

Then for any Følner sequence Ω_n $(n \geq 1)$ in $\mathbb{C}$, the limit of the sequence

$$\frac{h(\Omega_n)}{\mathrm{Area}(\Omega_n)} \quad (n \geq 1)$$

exists, and its value is independent of the choice of a Følner sequence.

Suppose that the Lie group $\mathbb{C}$ continuously acts on a compact metric space X. Here we don't assume that the distance is invariant under the group action. For a subset $\Omega \subset \mathbb{C}$, we define a new distance d_Ω on X by

$$d_\Omega(x, y) := \sup_{a \in \Omega} d(a.x, a.y).$$

It is easy to see that the map $\Omega \mapsto \mathrm{Widim}_\varepsilon(X, d_\Omega)$ satisfies the three conditions in Lemma 2.2 for each $\varepsilon > 0$. So we define a mean dimension $\dim(X : \mathbb{C})$ by

$$\dim(X : \mathbb{C}) := \lim_{\varepsilon\to+0}\left(\lim_{n\to\infty}\frac{\mathrm{Widim}_\varepsilon(X, d_{\Omega_n})}{\mathrm{Area}(\Omega_n)}\right)$$

where Ω_n ($n \geq 1$) is a Følner sequence in $\mathbb{C}$. The value of the mean dimension $\dim(X : \mathbb{C})$ is independent of the choice of a Følner sequence, and it is a topological invariant. (That is, it is independent of the choice of a distance on X compatible with the topology.) For example, we have

$$\begin{aligned}\dim(X : \mathbb{C}) &= \lim_{\varepsilon\to+0}\left(\lim_{R\to\infty}\frac{\mathrm{Widim}_\varepsilon(X, d_{D_R(0)})}{\pi R^2}\right)\\ &= \lim_{\varepsilon\to+0}\left(\lim_{R\to\infty}\frac{\mathrm{Widim}_\varepsilon(X, d_{[0,R]\times[0,R]})}{R^2}\right).\end{aligned} \tag{2}$$

2.2. **Energy density.** Here we explain some basic properties of the energy density $\rho(f)$ introduced in (1). Let $f : \mathbb{C} \to \mathbb{C}P^N$ be a Brody curve. Then the map

$$\Omega \mapsto \sup_{a\in\mathbb{C}}\int_{a+\Omega}|df|^2dxdy$$

clearly satisfies the three conditions in Lemma 2.2, where $\Omega \subset \mathbb{C}$ is a bounded Borel subset. Therefore we can define the energy density $\rho(f)$ by

$$\rho(f) := \lim_{n\to\infty}\frac{1}{\mathrm{Area}(\Omega_n)}\left(\sup_{a\in\mathbb{C}}\int_{a+\Omega_n}|df|^2dxdy\right),$$

where Ω_n ($n \geq 1$) is a Følner sequence in $\mathbb{C}$. In particular, we have

$$\begin{aligned}\rho(f) &= \lim_{R\to\infty}\frac{1}{\pi R^2}\left(\sup_{a\in\mathbb{C}}\int_{|z-a|<R}|df|^2dxdy\right)\\ &= \lim_{R\to\infty}\frac{1}{R^2}\left(\sup_{a,b\in\mathbb{R}}\int_{[a,a+R]\times[b,b+R]}|df|^2dxdy\right).\end{aligned} \tag{3}$$

From [19], the quantity $\rho(\mathbb{C}P^N) = \sup_{f\in\mathcal{M}(\mathbb{C}P^N)}\rho(f)$ satisfies

$$0 < \rho(\mathbb{C}P^N) < 1, \quad \lim_{N\to\infty}\rho(\mathbb{C}P^N) = 1.$$

Moreover

$$\frac{2\pi}{\sqrt{3}}\left(\int_1^\infty\frac{dx}{\sqrt{x^3-1}}\right)^{-2} \leq \rho(\mathbb{C}P^1) \leq 1 - 10^{-100}.$$

See [21, Section 1.2] for the lower bound, and see [19, Section 5] for the upper bound. Here

$$\frac{2\pi}{\sqrt{3}}\left(\int_1^\infty\frac{dx}{\sqrt{x^3-1}}\right)^{-2} = 0.6150198678198\cdots.$$

In the papers [18,21] a different definition of energy density was used. Let $T(r, f)$ be the Nevanlinna–Shimizu–Ahlfors characteristic function:

$$T(r, f) := \int_1^r\left(\int_{|z|<t}|df|^2dxdy\right)\frac{dt}{t}.$$

From the Brody condition $|df| \leq 1$, we have $T(r,f) \leq \pi r^2/2$. We define $\rho_{\mathrm{NSA}}(f)$ by

$$\rho_{\mathrm{NSA}}(f) := \limsup_{r\to\infty} \frac{2}{\pi r^2} T(r,f).$$

It is easy to see $\rho_{\mathrm{NSA}}(f) \leq \rho(f)$. This quantity $\rho_{\mathrm{NSA}}(f)$ was used in the papers [18, 21]. (We used the notation $e(f)$ for $\rho_{\mathrm{NSA}}(f)$ in [18, 21].) Let $\rho_{\mathrm{NSA}}(\mathbb{C}P^N)$ be the supremum of $\rho_{\mathrm{NSA}}(f)$ over $f \in \mathcal{M}(\mathbb{C}P^N)$. Trivially we have $\rho_{\mathrm{NSA}}(\mathbb{C}P^N) \leq \rho(\mathbb{C}P^N)$, but indeed we can prove the equality:

$$\rho_{\mathrm{NSA}}(\mathbb{C}P^N) = \rho(\mathbb{C}P^N).$$

This is proved in [22].

3. Proof of Theorem 1.6

In this section we prove Theorem 1.6 assuming Propositions 3.1 and 3.2 below. Theorem 1.7 will be proved in Section 6. Let $T\mathbb{C}P^N$ be the tangent bundle of $\mathbb{C}P^N$. It naturally admits a structure of a holomorphic vector bundle. We consider the Fubini–Study metric on it. Let $f : \mathbb{C} \to \mathbb{C}P^N$ be a Brody curve, and let $f^*T\mathbb{C}P^N$ be the pull-back of $T\mathbb{C}P^N$ by f. $f^*T\mathbb{C}P^N$ is a holomorphic vector bundle over the complex plane $\mathbb{C}$, and its Hermitian metric is given by the pull-back of the Fubini–Study metric. Let H_f be the space of holomorphic sections $u : \mathbb{C} \to f^*T\mathbb{C}P^N$ satisfying $\|u\|_{L^\infty(\mathbb{C})} < +\infty$. $(H_f, \|\cdot\|_{L^\infty(\mathbb{C})})$ is a complex Banach space (possibly infinite dimensional). We set $B_r(H_f) := \{u \in H_f |\ \|u\|_{L^\infty(\mathbb{C})} \leq r\}$ for $r \geq 0$.

Proposition 3.1. *Let $f : \mathbb{C} \to \mathbb{C}P^N$ be a non-degenerate Brody curve with $\|df\|_{L^\infty(\mathbb{C})} < 1$. Then there exist $\delta > 0$ and a map*

$$B_\delta(H_f) \to \mathcal{M}(\mathbb{C}P^N), \quad u \mapsto f_u,$$

satisfying the following two conditions:

(i) $f_0 = f$.

(ii) *For all $u, v \in B_\delta(H_f)$ and $z \in \mathbb{C}$*

$$|d(f_u(z), f_v(z)) - |u(z) - v(z)|| \leq \frac{1}{8} \|u - v\|_{L^\infty(\mathbb{C})}.$$

Here $d(\cdot,\cdot)$ is the distance on $\mathbb{C}P^N$ defined by the Fubini–Study metric, and $|u(z) - v(z)|$ is the fiberwise norm of $f^*T\mathbb{C}P^N$.

Let $R > 0$ and $\Lambda \subset \mathbb{C}$. Λ is said to be an R-square if $\Lambda = [a, a+R] \times [b, b+R]$ for some $a, b \in \mathbb{R}$.

Proposition 3.2. *Let $f : \mathbb{C} \to \mathbb{C}P^N$ be a non-degenerate Brody curve. Then for any R-square $\Lambda \subset \mathbb{C}$ with $R > 2$ there exists a finite dimensional complex linear subspace $V \subset H_f$ satisfying the following two conditions:*

(i)

$$\dim_{\mathbb{C}} V \geq (N+1) \int_\Lambda |df|^2 dxdy - C_f R.$$

Here C_f is a positive constant depending only on f (and independent of R, Λ).

(ii) *For all $u \in V$ we have $\|u\|_{L^\infty(\mathbb{C})} \leq 2 \|u\|_{L^\infty(\Lambda)}$.*

Propositions 3.1 and 3.2 will be proved later (Sections 4 and 5). Here we prove Theorem 1.6, assuming them.

Proof of Theorem 1.6. We define a distance on $\mathcal{M}(\mathbb{C}P^N)$ by

$$\mathrm{dist}(g,h) := \sum_{n=0}^{\infty} \frac{1}{10^n} \sup_{|z|\le n} d(g(z),h(z)), \quad (g,h\in\mathcal{M}(\mathbb{C}P^N)).$$

Then $|\mathrm{dist}(g,h) - d(g(0),h(0))| \le (1/9)\sup_{z\in\mathbb{C}} d(g(z),h(z))$. Hence for $\Omega\subset\mathbb{C}$

$$|\mathrm{dist}_\Omega(g,h) - \sup_{z\in\Omega} d(g(z),h(z))| \le \frac{1}{9}\sup_{z\in\mathbb{C}} d(g(z),h(z)). \tag{4}$$

Let $\delta>0$ be the positive constant introduced in Proposition 3.1. Let $\Lambda\subset\mathbb{C}$ be an R-square ($R>2$). By Proposition 3.2, there exists $V=V_\Lambda\subset H_f$ satisfying the conditions (i) and (ii) in Proposition 3.2. We investigate the map $B_\delta(H_f)\to\mathcal{M}(\mathbb{C}P^N)$, $u\mapsto f_u$, (given by Proposition 3.1) and its restriction to $B_\delta(V) := V\cap B_\delta(H_f)$.

From the condition (ii) of Proposition 3.1, for $u,v\in B_\delta(H_f)$, we have $\sup_{z\in\mathbb{C}} d(f_u(z),f_v(z)) \le (9/8)\|u-v\|_{L^\infty(\mathbb{C})}$. Hence $(B_\delta(H_f),\|\cdot\|_{L^\infty(\mathbb{C})})\to\mathcal{M}(\mathbb{C}P^N)$ is continuous. For $u,v\in B_\delta(H_f)$

$$\begin{aligned}
&\left|\mathrm{dist}_\Lambda(f_u,f_v) - \sup_{z\in\Lambda}|u(z)-v(z)|\right| \\
&\le \left|\mathrm{dist}_\Lambda(f_u,f_v) - \sup_{z\in\Lambda} d(f_u(z),f_v(z))\right| + \left|\sup_{z\in\Lambda} d(f_u(z),f_v(z)) - \sup_{z\in\Lambda}|u(z)-v(z)|\right| \\
&\le \frac{1}{9}\sup_{z\in\mathbb{C}} d(f_u(z),f_v(z)) + \frac{1}{8}\|u-v\|_{L^\infty(\mathbb{C})} \quad \text{(by Proposition 3.1 (ii) and (4))} \\
&\le \frac{1}{4}\|u-v\|_{L^\infty(\mathbb{C})}.
\end{aligned}$$

Thus

$$\|u-v\|_{L^\infty(\Lambda)} \le \mathrm{dist}_\Lambda(f_u,f_v) + \frac{1}{4}\|u-v\|_{L^\infty(\mathbb{C})}.$$

For $u,v\in B_\delta(V) = V\cap B_\delta(H_f)$, we have $\|u-v\|_{L^\infty(\mathbb{C})} \le 2\|u-v\|_{L^\infty(\Lambda)}$ by Proposition 3.2 (ii). Hence

$$\|u-v\|_{L^\infty(\mathbb{C})} \le 4\,\mathrm{dist}_\Lambda(f_u,f_v), \quad (u,v\in B_\delta(V)).$$

Hence for $\varepsilon<\delta/4$,

$$\begin{aligned}
&\mathrm{Widim}_\varepsilon(\mathcal{M}(\mathbb{C}P^N),\mathrm{dist}_\Lambda) \\
&\quad \ge \mathrm{Widim}_{4\varepsilon}(B_\delta(V),\|\cdot\|_{L^\infty(\mathbb{C})}) \\
&\quad = \dim_{\mathbb{R}} V \quad \text{(by Example 2.1)} \\
&\quad \ge 2(N+1)\int_\Lambda |df|^2 dxdy - 2C_f R \quad \text{(by Proposition 3.2 (i))}.
\end{aligned}$$

Since $\mathrm{Widim}_\varepsilon(\mathcal{M}(\mathbb{C}P^N),\mathrm{dist}_\Lambda) = \mathrm{Widim}_\varepsilon(\mathcal{M}(\mathbb{C}P^N),\mathrm{dist}_{[0,R]\times[0,R]})$, for $\varepsilon<\delta/4$, the quantity $\mathrm{Widim}_\varepsilon(\mathcal{M}(\mathbb{C}P^N),\mathrm{dist}_{[0,R]\times[0,R]})$ is bounded from below by

$$2(N+1)\left(\sup_\Lambda \int_\Lambda |df|^2 dxdy\right) - 2C_f R.$$

Here Λ runs over all R-squares. Dividing this by R^2 and letting $R\to\infty$, we get

$$\lim_{R\to\infty}\left(\frac{1}{R^2}\mathrm{Widim}_\varepsilon(\mathcal{M}(\mathbb{C}P^N),\mathrm{dist}_{[0,R]\times[0,R]})\right) \ge 2(N+1)\rho(f).$$

Here we have used (3). Let $\varepsilon \to 0$. Then $\dim(\mathcal{M}(\mathbb{C}P^N) : \mathbb{C}) \geq 2(N+1)\rho(f)$ by (2). $\square$

Remark 3.3. The above argument also gives the lower bound on the **local mean dimension** $\dim_f(\mathcal{M}(\mathbb{C}P^N) : \mathbb{C})$. Local mean dimension is a notion introduced in [15]. The readers can skip this remark.

Let $f : \mathbb{C} \to \mathbb{C}P^N$ be a non-degenerate Brody curve with $\|df\|_{L^\infty(\mathbb{C})} < 1$. Let $B_r(f)_{\mathbb{C}} \subset \mathcal{M}(\mathbb{C}P^N)$ $(r > 0)$ be the set of $g \in \mathcal{M}(\mathbb{C}P^N)$ satisfying $\mathrm{dist}_{\mathbb{C}}(f, g) \leq r$. Since $f_0 = f$, if $(4/5)r \leq \delta$ then $u \in B_{(4/5)r}(H_f)$ satisfies $f_u \in B_r(f)_{\mathbb{C}}$. Let $\Lambda \subset \mathbb{C}$ be an R-square $(R > 2)$. As in the above proof, for $4\varepsilon < (4/5)r \leq \delta$, we get

$$\mathrm{Widim}_\varepsilon(B_r(f)_{\mathbb{C}}, \mathrm{dist}_\Lambda) \geq 2(N+1)\int_\Lambda |df|^2 dxdy - 2C_f R.$$

Hence

$$\begin{aligned}\dim_f(\mathcal{M}(\mathbb{C}P^N) : \mathbb{C}) &:= \lim_{r \to +0}\left\{\lim_{\varepsilon \to +0}\left(\lim_{R \to \infty}\frac{1}{R^2}\sup_{\Lambda : R\text{-square}}\mathrm{Widim}_\varepsilon(B_r(f)_{\mathbb{C}}, \mathrm{dist}_\Lambda)\right)\right\} \\ &\geq 2(N+1)\rho(f).\end{aligned}$$

Then $\dim_{loc}(\mathcal{M}(\mathbb{C}P^N) : \mathbb{C}) := \sup_{f \in \mathcal{M}(\mathbb{C}P^N)} \dim_f(\mathcal{M}(\mathbb{C}P^N) : \mathbb{C})$ satisfies

$$2(N+1)\rho(\mathbb{C}P^N) \leq \dim_{loc}(\mathcal{M}(\mathbb{C}P^N) : \mathbb{C}) \leq \dim(\mathcal{M}(\mathbb{C}P^N) : \mathbb{C}) \leq 4N\rho(\mathbb{C}P^N).$$

The proof is the same as the proof of Theorem 1.1. In particular we get

$$\dim_{loc}(\mathcal{M}(\mathbb{C}P^1) : \mathbb{C}) = \dim(\mathcal{M}(\mathbb{C}P^1) : \mathbb{C}).$$

4. Proof of Proposition 3.1

In this section we prove Proposition 3.1.

4.1. **Analytic preliminaries.** Let $f : \mathbb{C} \to \mathbb{C}P^N$ be a Brody curve. As in Section 3, let $T\mathbb{C}P^N$ be the tangent bundle of $\mathbb{C}P^N$ with the natural holomorphic vector bundle structure, and let $E := f^*T\mathbb{C}P^N$ be the pull-back of $T\mathbb{C}P^N$. E is a holomorphic vector bundle over the complex plane $\mathbb{C}$. Its Hermitian metric h is given by the pull-back of the Fubini–Study metric. E is equipped with the unitary connection ∇ defined by the holomorphic structure and the metric h.

Let $1 < p < \infty$ be a real number, and $k \geq 0$ be an integer. Let $a \in L^p_{k,loc}(\Lambda^{0,i}(E))$ $(i = 0, 1)$ be a locally L^p_k-section of $\Lambda^{0,i}(E)$ (the $\mathcal{C}^\infty$-vector bundle of $(0,i)$-forms valued in E). For a subset $\Omega \subset \mathbb{C}$, we set

$$\|a\|_{L^p_k(\Omega)} := \left(\sum_{n=0}^k \int_\Omega |\nabla^n a|^p dxdy\right)^{1/p}.$$

We define the $\ell^\infty L^p_k$-norm $\|a\|_{\ell^\infty L^p_k}$ by

$$\|a\|_{\ell^\infty L^p_k} := \sup_{z \in \mathbb{C}} \|a\|_{L^p_k(D_1(z))}.$$

Let $\ell^\infty L^p_k(\Lambda^{0,i}(E))$ be the Banach space of all $a \in L^p_{k,loc}(\Lambda^{0,i}(E))$ satisfying $\|a\|_{\ell^\infty L^p_k} < +\infty$.

Lemma 4.1. (i) *For* $a \in L^2_{2,loc}(\Lambda^{0,i}(E))$,

$$\|a\|_{L^\infty(\mathbb{C})} \leq \text{const}\,\|a\|_{\ell^\infty L^2_2}.$$

Precisely speaking, if the right-hand side is finite then the left-hand side is also finite and satisfies the inequality.

(ii) *If* $a \in L^p_{2,loc}(\Lambda^{0,i}(E))$ *with* $p > 2$, *then*

$$\|a\|_{L^\infty(\mathbb{C})} + \|\nabla a\|_{L^\infty(\mathbb{C})} \leq \text{const}_p\,\|a\|_{\ell^\infty L^p_2}.$$

Proof. Since $\mathcal{M}(\mathbb{C}P^N)$ is compact, there are $\delta > 0$ and $\text{const}_k > 0$ ($k \geq 0$) such that for every $z \in \mathbb{C}$ there is a trivialization u of the holomorphic vector bundle E over a neighborhood of $D_\delta(z)$ such that $u_* h = (h_{\alpha\bar\beta})_{\alpha\beta}$ (the Hermitian matrix representing h under the trivialization u) satisfies $\|h_{\alpha\bar\beta}\|_{\mathcal{C}^k(D_\delta(z))}, \|h^{\alpha\bar\beta}\|_{\mathcal{C}^k(D_\delta(z))} \leq \text{const}_k$. Here $(h^{\alpha\bar\beta}) = (h_{\alpha\bar\beta})^{-1}$. Then the norms $\|a\|_{L^p_k(D_\delta(z))}$ and $\|a\|_{L^\infty(D_\delta(z))}$ are equivalent to $\|u\circ a\|_{L^p_k(D_\delta(z))}$ and $\|u\circ a\|_{L^\infty(D_\delta(z))}$ uniformly in $z \in \mathbb{C}$ respectively. (We consider $u \circ a$ as a $\mathbb{C}^N$-valued $(0,i)$-form in $D_\delta(z)$.) Hence the Sobolev embedding theorem (Gilbarg–Trudinger [5, Chapter 7.7]) implies

$$\|a\|_{L^\infty(D_\delta(z))} \leq \text{const}\,\|a\|_{L^2_2(D_\delta(z))}.$$

Here the important point is that const is independent of $z \in \mathbb{C}$. Thus $\|a\|_{L^\infty(\mathbb{C})} \leq \text{const}\,\|a\|_{\ell^\infty L^2_2}$. (ii) can be proved in the same way. □

Let $\varphi : \mathbb{C} \to \mathbb{R}$ be a $\mathcal{C}^\infty$-function satisfying $\|\varphi\|_{\mathcal{C}^k(\mathbb{C})} < +\infty$ for all $k \geq 0$. We set $\bar\partial^*_\varphi(a) := e^{-\varphi}\bar\partial^*(e^\varphi a)$ for $a \in \Omega^{0,1}(E)$. Here $\bar\partial^*$ is the formal adjoint of the Dolbeault operator $\bar\partial : \Omega^0(E) \to \Omega^{0,1}(E)$ with respect to the Hermitian metric h. $\bar\partial^*_\varphi$ is the formal adjoint of $\bar\partial$ with respect to the metric $e^\varphi h$. We define the operator $\Box_\varphi : \Omega^{0,i}(E) \to \Omega^{0,i}(E)$ by setting

$$\Box_\varphi a := \bar\partial^*_\varphi \bar\partial a \quad (i = 0), \quad \Box_\varphi a := \bar\partial\bar\partial^*_\varphi a \quad (i = 1).$$

Lemma 4.2. *For* $a \in \ell^\infty L^p_{k+2}(\Lambda^{0,i}(E))$,

$$\|a\|_{\ell^\infty L^p_{k+2}} \leq \text{const}_{p,k,\varphi}\left(\|a\|_{\ell^\infty L^p} + \|\Box_\varphi a\|_{\ell^\infty L^p_k}\right).$$

More precisely, if $a \in L^p_{k+2,loc}(\Lambda^{0,i}(E))$ *and the right-hand side of the above is finite then* $a \in \ell^\infty L^p_{k+2}$ *and satisfies the above inequality.*

Proof. We use the trivialization u of E introduced in the proof of Lemma 4.1. Since $\|\varphi\|_{\mathcal{C}^l(\mathbb{C})} < +\infty$ for all $l \geq 0$, under the trivialization u, the operator $\Box_\varphi$ is represented as

$$\Box_\varphi = (-1/2)\Delta + A\frac{\partial}{\partial x} + B\frac{\partial}{\partial y} + C$$

over a neighborhood of $D_\delta(z)$ where the $\mathcal{C}^l$-norms ($l \geq 0$) of the matrices A, B, C over $D_\delta(z)$ are bounded uniformly in $z \in \mathbb{C}$. Then from the L^p-estimate (Gilbarg–Trudinger [5, Chapter 9.5])

$$\|a\|_{L^p_{k+2}(D_{\delta/2}(z))} \leq \text{const}_{p,k,\varphi}\left(\|a\|_{L^p(D_\delta(z))} + \|\Box_\varphi a\|_{L^p_k(D_\delta(z))}\right).$$

The desired estimate follows from this. □

4.2. **Perturbation of the Hermitian metric.** Here we develop a perturbation technique of a Hermitian metric (Lemma 4.5 below). Gromov also discussed it in [11, p. 399]. Tsukamoto [21, Section 4.3] studied an easier situation.

Lemma 4.3. *Let $g : \mathbb{C} \to \mathbb{R}_{\geq 0}$ be a non-negative smooth function with $\|g\|_{\mathcal{C}^k(\mathbb{C})} < +\infty$ for all $k \geq 0$. We suppose that the following non-degeneracy condition holds: There exist $\delta > 0$ and $R > 0$ such that for all $p \in \mathbb{C}$ we have $\|g\|_{L^\infty(D_R(p))} \geq \delta$. Then there exists a smooth function $\varphi : \mathbb{C} \to \mathbb{R}$ satisfying*

$$(-\Delta + 1)\varphi = -g, \quad \|\varphi\|_{\mathcal{C}^k(\mathbb{C})} < +\infty \quad (\forall k \geq 0), \quad \sup_{z\in\mathbb{C}} \varphi(z) < 0.$$

Here $\Delta = \partial^2/\partial x^2 + \partial^2/\partial y^2$.

Proof. We need the following sublemma.

Sublemma 4.4. *Let $\varphi : \mathbb{C} \to \mathbb{R}$ be a function of class $\mathcal{C}^2$ ($\varphi \in \mathcal{C}^2_{loc}$). Suppose that the norms $\|\varphi\|_{L^\infty(\mathbb{C})}$ and $\|(-\Delta + 1)\varphi\|_{L^\infty(\mathbb{C})}$ are both finite. Then*

$$\|\varphi\|_{L^\infty(\mathbb{C})} \leq 4 \|(-\Delta + 1)\varphi\|_{L^\infty(\mathbb{C})} .$$

Proof. Take $z_0 \in \mathbb{C}$ such that $|\varphi(z_0)| \geq \|\varphi\|_{L^\infty(\mathbb{C})}/2$. For simplicity, we suppose $z_0 = 0$. Moreover we suppose $\varphi(0) \geq 0$. (If $\varphi(0) < 0$, then we apply the following argument to $-\varphi$.) We define $w : \mathbb{C} \to \mathbb{R}$ by

$$w(z) := \frac{1}{2\pi} \int_0^{2\pi} e^{(x\cos\theta + y \sin\theta)/\sqrt{2}} d\theta.$$

w satisfies

$$(-\Delta + 1/2)w = 0, \quad \min_{z\in\mathbb{C}} w(z) = w(0) = 1, \quad w(z) \to +\infty \quad (|z| \to +\infty).$$

Then $(-\Delta + 1)w = w/2 \geq 1/2$. For $\varepsilon > 0$, set $M := 2 \|(-\Delta + 1)\varphi\|_{L^\infty(\mathbb{C})} + \varepsilon > 0$.

$$(-\Delta+1)(Mw - \varphi) \geq M/2 - (-\Delta+1)\varphi = \|(-\Delta + 1)\varphi\|_{L^\infty(\mathbb{C})} + \varepsilon/2 - (-\Delta+1)\varphi \geq \varepsilon/2.$$

Since the function $Mw - \varphi$ is positive for $|z| \gg 1$, the weak minimum principle (Gilbarg–Trudinger [5, Chapter 3.1, Corollary 3.2]) implies that this function is non-negative everywhere. Hence

$$\|\varphi\|_{L^\infty(\mathbb{C})}/2 \leq \varphi(0) \leq Mw(0) = M = 2\|(-\Delta + 1)\varphi\|_{L^\infty(\mathbb{C})} + \varepsilon.$$

Let $\varepsilon \to 0$. We get

$$\|\varphi\|_{L^\infty(\mathbb{C})} \leq 4 \|(-\Delta + 1)\varphi\|_{L^\infty(\mathbb{C})} .$$

□

Let $\phi_n : \mathbb{C} \to [0, 1]$ ($n \geq 1$) be a cut-off function such that $\phi_n = 1$ over $D_n(0)$ and $\mathrm{supp}(\phi_n) \subset D_{n+1}(0)$. We want to solve the equation $(-\Delta + 1)\varphi = -\phi_n g$. The following is a standard L^2-argument.

Let $L^2_1(\mathbb{C})$ be the space of L^2-functions $\varphi : \mathbb{C} \to \mathbb{R}$ satisfying $\partial\varphi/\partial x, \partial\varphi/\partial y \in L^2$ with the inner product $\langle\varphi, \varphi'\rangle_{L^2_1} := \langle\varphi, \varphi'\rangle_{L^2} + \langle\partial\varphi/\partial x, \partial\varphi'/\partial x\rangle_{L^2} + \langle\partial\varphi/\partial y, \partial\varphi'/\partial y\rangle_{L^2}$. Consider the bounded linear functional:

$$L^2_1(\mathbb{C}) \to \mathbb{R}, \quad \varphi \mapsto -\langle\varphi, \phi_n g\rangle_{L^2}.$$

From the Riesz representation theorem, there uniquely exists $\varphi_n \in L^2_1(\mathbb{C})$ satisfying $\langle\varphi, \varphi_n\rangle_{L^2_1} = -\langle\varphi, \phi_n g\rangle_{L^2}$ for all $\varphi \in L^2_1(\mathbb{C})$. This implies $(-\Delta + 1)\varphi_n = -\phi_n g$ as a distribution. From the local elliptic regularity, φ_n is smooth and $\|\varphi_n\|_{L^\infty(\mathbb{C})} < +\infty$. Then we can apply Sublemma 4.4 to φ_n and get

$$\|\varphi_n\|_{L^\infty(\mathbb{C})} \le 4\,\|\phi_n g\|_{L^\infty(\mathbb{C})} \le 4\,\|g\|_{L^\infty(\mathbb{C})} < +\infty.$$

By the local elliptic regularity, for every compact subset $K \subset \mathbb{C}$ and $k \ge 0$, the sequence $\|\varphi_n\|_{\mathcal{C}^k(K)}$ $(n \ge 1)$ is bounded. Then we can choose a subsequence $n_1 < n_2 < n_3 < \cdots$ such that φ_{n_k} converges to some φ in $\mathcal{C}^\infty$ over every compact subset of $\mathbb{C}$. φ satisfies $(-\Delta + 1)\varphi = -g$ and $\|\varphi\|_{L^\infty(\mathbb{C})} \le 4\,\|g\|_{L^\infty(\mathbb{C})}$. By the elliptic regularity, $\|\varphi\|_{\mathcal{C}^k(\mathbb{C})} < +\infty$ for all $k \ge 0$.

Note that we have not used the non-degeneracy condition of the function g so far. We need it for the proof of the condition $\sup_{z\in\mathbb{C}} \varphi(z) < 0$.

Set $M := \sup_{z\in\mathbb{C}} \varphi(z)$. There are $z_n \in \mathbb{C}$ $(n \ge 1)$ such that $\varphi(z_n) \to M$. Set $\varphi_n(z) := \varphi(z + z_n)$ and $g_n(z) := g(z + z_n)$. Then

$$(-\Delta + 1)\varphi_n = -g_n.$$

The sequences $\|\varphi_n\|_{\mathcal{C}^k(\mathbb{C})}$ and $\|g_n\|_{\mathcal{C}^k(\mathbb{C})}$ $(n \ge 1)$ are bounded for every $k \ge 0$. Hence by choosing a subsequence (denoted also by φ_n and g_n), we can assume that φ_n and g_n converge to φ_∞ and g_∞ respectively in $\mathcal{C}^\infty$ over every compact subset of $\mathbb{C}$. They satisfy

$$g_\infty \ge 0, \quad (-\Delta + 1)\varphi_\infty = -g_\infty \le 0, \quad \varphi_\infty(z) \le \varphi_\infty(0) = M.$$

From the non-degeneracy condition of g, the function g_∞ is not zero. Hence if φ_∞ is a constant, then $\varphi_\infty = -g_\infty$ is a negative constant function and $M < 0$. If φ_∞ is not a constant, then the strong maximum principle [5, Chapter 3.2, Theorem 3.5] implies that φ_∞ cannot achieve a non-negative maximum value. Hence $M = \varphi_\infty(0) = \max_{z\in\mathbb{C}} \varphi_\infty(z) < 0$. $\square$

Recall that $f : \mathbb{C} \to \mathbb{C}P^N$ is a Brody curve and $E = f^*T\mathbb{C}P^N$. For $a \in \Omega^{0,1}(E)$ we have the Weintzenböck formula:

$$\bar\partial\bar\partial^* a = \frac{1}{2}\nabla^*\nabla a + \Theta a, \tag{5}$$

where $\Theta := [\nabla_{\partial/\partial z}, \nabla_{\partial/\partial \bar z}]$ is the curvature operator. The crucial fact for the analysis of this paper is that the holomorphic bisectional curvature of the Fubini–Study metric is positive. From this, there exists a positive constant c such that

$$h(\Theta a, a) \ge c|df|^2|a|^2.$$

This means that the curvature operator is positive where $|df|$ is positive. The non-degeneracy condition of the map f enters into the argument through this point; see the condition (ii) of Definition-Lemma 1.3. In the next lemma we will prove that if f is non-degenerate then we can perturb the Hermitian metric h so that the curvature is uniformly positive.

Lemma 4.5. *Let $f : \mathbb{C} \to \mathbb{C}P^N$ be a non-degenerate Brody curve. There is a smooth function $\varphi : \mathbb{C} \to \mathbb{R}$ with $\|\varphi\|_{\mathcal{C}^k(\mathbb{C})} < +\infty$ $(\forall k \ge 0)$ satisfying the following. Let Θ_φ be the curvature of the Hermitian metric $h_\varphi := e^\varphi h$. Then there is $c' > 0$ such that*

$$h_\varphi(\Theta_\varphi a, a) \ge c'|a|^2_{h_\varphi}$$

for all $a \in \Omega^{0,1}(E)$.

Proof. We have $\Theta_\varphi a = \frac{-\Delta\varphi}{4}a + \Theta a$ for $a \in \Omega^{0,1}(E)$, and hence

$$h_\varphi(\Theta_\varphi a, a) = e^\varphi \left(\frac{-\Delta\varphi}{4}|a|_h^2 + h(\Theta a, a)\right) \geq e^\varphi \left(\frac{-\Delta\varphi}{4} + c|df|^2\right)|a|_h^2.$$

By the non-degeneracy of f and Lemma 4.3, there is a smooth function $\varphi : \mathbb{C} \to \mathbb{R}$ satisfying

$$(-\Delta + 1)\varphi = -4c|df|^2, \quad \|\varphi\|_{\mathcal{C}^k(\mathbb{C})} < +\infty \quad (\forall k \geq 0), \quad \sup_{z\in\mathbb{C}} \varphi(z) < 0.$$

Then

$$h_\varphi(\Theta_\varphi a, a) \geq e^\varphi(-\varphi/4)|a|_h^2 = (-\varphi/4)|a|_{h_\varphi}^2 \geq \left(-\sup_{z\in\mathbb{C}} \varphi(z)/4\right)|a|_{h_\varphi}^2.$$

Hence $c' := -\sup_{z\in\mathbb{C}} \varphi(z)/4 > 0$ satisfies the statement. ☐

In our convention, the Fubini–Study metric $g_{i\bar{j}}$ on $\mathbb{C}P^N$ is given by

$$g_{i\bar{j}} = \frac{1}{2\pi}\frac{\partial^2}{\partial z_i \partial \bar{z}_j} \log(1 + |z_1|^2 + \cdots + |z_N|^2)$$

over $\{[1 : z_1 : \cdots : z_N]\} \subset \mathbb{C}P^N$. The spherical derivative $|df|(z)$ for a holomorphic curve $f : \mathbb{C} \to \mathbb{C}P^N$ satisfies

$$f^*\left(\sqrt{-1}\sum g_{i\bar{j}} dz_i d\bar{z}_j\right) = |df|^2 dxdy. \tag{6}$$

The Fubini–Study metric $g_{i\bar{j}}$ satisfies the Kähler–Einstein equation

$$\mathrm{Ric}_{i\bar{j}} = -\frac{\partial^2}{\partial z_i \partial \bar{z}_j} \log(\det(g_{k\bar{l}})) = 2\pi(N+1)g_{i\bar{j}}.$$

From this, the curvature operator $\Theta = [\nabla_{\partial/\partial z}, \nabla_{\partial/\partial \bar{z}}]$ in (5) satisfies

$$\frac{\sqrt{-1}}{2\pi}\mathrm{tr}(\Theta) dz d\bar{z} = (N+1)|df|^2 dxdy \tag{7}$$

since $\mathrm{tr}(\Theta)dzd\bar{z} = f^*(\sum \mathrm{Ric}_{i\bar{j}} dz_i d\bar{z}_j)$. The equation (7) will be used in the proof of Proposition 5.1. Note that the form $(\sqrt{-1}/2\pi)\mathrm{tr}(\Theta)dzd\bar{z}$ is the Chern form representing $c_1(E)$ although we have $c_1(E) = 0$ because $H^2(\mathbb{C}; \mathbb{Z}) = 0$.

4.3. **L^∞-estimate.** Let $f : \mathbb{C} \to \mathbb{C}P^N$ be a non-degenerate Brody curve, and let $\varphi : \mathbb{C} \to \mathbb{R}$ be a smooth function introduced in Lemma 4.5. Propositions 4.6 and 4.7 below essentially use the positivity of the curvature Θ_φ.

The following L^∞-estimate was proved in [21, Proposition 4.2].

Proposition 4.6. *Let $a \in \Omega^{0,1}(E)$ be an E-valued $(0,1)$-form of class $\mathcal{C}^2$ ($a \in \mathcal{C}^2_{loc}$). Set $b := \Box_\varphi a$. If $\|a\|_{L^\infty(\mathbb{C})}, \|b\|_{L^\infty(\mathbb{C})} < +\infty$, then*

$$\|a\|_{L^\infty(\mathbb{C})} \leq \mathrm{const}_{f,\varphi} \|b\|_{L^\infty(\mathbb{C})}.$$

Proof. The proof is similar to the proof of Sublemma 4.4. For the detail, see [21, pp. 1648–1649]. ☐

Proposition 4.7. *Let $b \in L^2_{2,loc}(\Lambda^{0,1}(E))$ and suppose $\|b\|_{L^\infty(\mathbb{C})} < +\infty$. Then there uniquely exists $a \in L^2_{4,loc}(\Lambda^{0,1}(E))$ satisfying*

$$\Box_\varphi a = b, \quad \|a\|_{L^\infty(\mathbb{C})} < +\infty.$$

Moreover $\|a\|_{L^\infty(\mathbb{C})} + \|\nabla a\|_{L^\infty(\mathbb{C})} \leq \mathrm{const}_{f,\varphi} \|b\|_{L^\infty(\mathbb{C})}$.

Proof. The uniqueness follows from Proposition 4.6. (Note the Sobolev embedding $L^2_{4,loc} \hookrightarrow \mathcal{C}^2_{loc}$ in $\mathbb{R}^2$.) So the problem is the existence. We have the Weintzenböck formula: for $a \in \Omega^{0,1}(E)$

$$\Box_\varphi a = \frac{1}{2}\nabla^*_\varphi \nabla_\varphi a + \Theta_\varphi a,$$

where ∇_φ is the unitary connection on E with respect to the metric $h_\varphi = e^\varphi h$. Θ_φ satisfies the positivity condition in Lemma 4.5.

Let $\phi_n : \mathbb{C} \to [0,1]$ be a cut-off function such that $\phi_n = 1$ over $D_n(0)$ and $\mathrm{supp}(\phi_n) \subset D_{n+1}(0)$. From the positivity of the curvature, as in the proof of Lemma 4.3, a standard L^2-argument shows that there is $a_n \in L^2_1(\Lambda^{0,1}(E))$ (the space of L^2-sections a of $\Lambda^{0,1}(E)$ satisfying $\nabla_\varphi a \in L^2$) satisfying $\Box_\varphi a_n = \phi_n b$ as a distribution. For the detail, see [21, Lemma 5.3]. The local elliptic regularity implies $a_n \in L^2_{4,loc}$. By Lemmas 4.1 (i) and 4.2,

$$\begin{aligned}\|a_n\|_{L^\infty(\mathbb{C})} &\le \mathrm{const}\,\|a_n\|_{\ell^\infty L^2_2} \le \mathrm{const}_\varphi\left(\|a_n\|_{\ell^\infty L^2} + \|\Box_\varphi a_n\|_{\ell^\infty L^2}\right)\\ &\le \mathrm{const}_\varphi\left(\|a_n\|_{L^2} + \|\phi_n b\|_{L^\infty(\mathbb{C})}\right) < +\infty.\end{aligned}$$

By Proposition 4.6 we have $\|a_n\|_{L^\infty(\mathbb{C})} \le \mathrm{const}_{f,\varphi}\,\|\phi_n b\|_{L^\infty(\mathbb{C})} \le \mathrm{const}_{f,\varphi}\,\|b\|_{L^\infty(\mathbb{C})}$. Then for any compact set $K \subset \mathbb{C}$ the sequence $\|a_n\|_{L^2_2(K)}$ $(n \ge 1)$ is bounded. By choosing a subsequence $n_1 < n_2 < n_3 < \cdots$, the sequence a_{n_k} converges to some a weakly in $L^2_2(D_R(0))$ (and hence strongly in $L^\infty(D_R(0))$) for every $R > 0$. a satisfies $\Box_\varphi a = b$, and $\|a\|_{L^\infty(\mathbb{C})} \le \sup_{n\ge 1}\|a_n\|_{L^\infty(\mathbb{C})} \le \mathrm{const}_{f,\varphi}\,\|b\|_{L^\infty(\mathbb{C})}$. By the local elliptic regularity $a \in L^2_{4,loc}$. By Lemmas 4.1 (ii) and 4.2

$$\begin{aligned}\|a\|_{L^\infty(\mathbb{C})} + \|\nabla a\|_{L^\infty(\mathbb{C})} &\le \mathrm{const}\,\|a\|_{\ell^\infty L^3_2} \le \mathrm{const}_\varphi(\|a\|_{\ell^\infty L^3} + \|b\|_{\ell^\infty L^3})\\ &\le \mathrm{const}_{f,\varphi}\,\|b\|_{L^\infty(\mathbb{C})}.\end{aligned}$$

□

4.4. **Deformation theory.** Let $f : \mathbb{C} \to \mathbb{C}P^N$ be a non-degenerate Brody curve with $\|df\|_{L^\infty(\mathbb{C})} < 1$. In this subsection we study a deformation of f and prove Proposition 3.1. Gromov [11, pp. 399–400, Projective interpolation theorem] studied a different kind of deformation theory. Our argument is a generalization of the deformation theory of elliptic Brody curves developed in [21].

Consider the following map (see McDuff–Salamon [16, p. 40]):

$$\Phi : \ell^\infty L^2_3(E) \to \ell^\infty L^2_2(\Lambda^{0,1}(E)), \quad u \mapsto P_u(\bar\partial \exp u) \otimes d\bar z.$$

Here $\exp u = \exp_{f(z)} u(z)$ is defined by the exponential map of the Fubini–Study metric, and

$$\bar\partial \exp u := \frac{1}{2}\left(\frac{\partial}{\partial x}\exp u + J\frac{\partial}{\partial y}\exp u\right) \quad (J\text{: complex structure of } \mathbb{C}P^N).$$

$P_{u(z)} : T_{\exp_{f(z)} u(z)}\mathbb{C}P^N \to T_{f(z)}\mathbb{C}P^N$ is the parallel translation along the geodesic $\exp_{f(z)}(tu(z))$ $(0 \le t \le 1)$.

Φ is a smooth map between the Banach spaces. $\Phi(0) = 0$ and the derivative of Φ at the origin is equal to the Dolbeault operator:

$$d\Phi_0 = \bar\partial : \ell^\infty L^2_3(E) \to \ell^\infty L^2_2(\Lambda^{0,1}(E)).$$

Proposition 4.8. *There is a bounded linear operator $Q : \ell^\infty L^2_2(\Lambda^{0,1}(E)) \to \ell^\infty L^2_3(E)$ satisfying $\bar\partial \circ Q = 1$.*

Proof. We will prove that the map

$$\Box_\varphi = \bar{\partial}\bar{\partial}_\varphi^* : \ell^\infty L_4^2(\Lambda^{0,1}(E)) \to \ell^\infty L_2^2(\Lambda^{0,1}(E)) \tag{8}$$

is an isomorphism. ($\varphi : \mathbb{C} \to \mathbb{R}$ is a smooth function introduced in Lemma 4.5.) Then $Q := \bar{\partial}_\varphi^* \Box_\varphi^{-1} : \ell^\infty L_2^2(\Lambda^{0,1}(E)) \to \ell^\infty L_3^2(\Lambda^{0,1}(E))$ becomes a right inverse of $\bar{\partial}$. The injectivity of the map (8) directly follows from the L^∞-estimate in Proposition 4.6.

On the other hand, by Proposition 4.7, for every $b \in \ell^\infty L_2^2(\Lambda^{0,1}(E))$ there is $a \in L^\infty \cap L_{4,loc}^2(\Lambda^{0,1}(E))$ satisfying $\Box_\varphi a = b$. By Lemma 4.2, $a \in \ell^\infty L_4^2$. Thus the map (8) is surjective. $\Box$

Let H_f be the Banach space of all L^∞-holomorphic sections of E introduced in Section 3. H_f is equal to the kernel of the map $\bar{\partial} : \ell^\infty L_3^2(E) \to \ell^\infty L_2^2(\Lambda^{0,1}(E))$ by Lemmas 4.1 and 4.2. Moreover the norms $\|\cdot\|_{\ell^\infty L_k^2}$ ($k \geq 0$) are all equivalent to the norm $\|\cdot\|_{L^\infty(\mathbb{C})}$ over H_f.

From Proposition 4.8 and the implicit function theorem, there are $r > 0$ and a smooth map $\alpha : \{u \in H_f | \, \|u\|_{L^\infty(\mathbb{C})} < r\} \to \mathrm{Im}Q$ ($\mathrm{Im}Q \subset \ell^\infty L_3^2(E)$ is a closed subspace) such that

$$\Phi(u + \alpha(u)) = 0, \quad \alpha(0) = 0, \quad d\alpha_0 = 0.$$

The first and second conditions imply that $f_u := \exp_f(u + \alpha(u))$ becomes a holomorphic curve with $f_0 = f$. The third condition implies that for any $\varepsilon > 0$ there exists $0 < \delta < r$ such that if $u, v \in H_f$ satisfy $\|u\|_{L^\infty(\mathbb{C})}, \|v\|_{L^\infty(\mathbb{C})} \leq \delta$, then $\|\alpha(u) - \alpha(v)\|_{L^\infty(\mathbb{C})} \leq \varepsilon \|u - v\|_{L^\infty(\mathbb{C})}$.

Proof of Proposition 3.1. Since $\|df\|_{L^\infty(\mathbb{C})} < 1$, if $\delta \ll 1$, the holomorphic curves f_u ($u \in B_\delta(H_f)$) satisfy $\|df_u\|_{L^\infty(\mathbb{C})} \leq 1$. We will prove that if $0 < \delta < r$ is sufficiently small, then the map

$$B_\delta(H_f) \ni u \mapsto f_u \in \mathcal{M}(\mathbb{C}P^N)$$

satisfies the conditions in Proposition 3.1. The condition (i) ($f_0 = f$) is OK. So we want to prove the condition (ii).

We choose $0 < \delta < r$ sufficiently small so that all $u, v \in B_\delta(H_f)$ satisfy

$$\|\alpha(u) - \alpha(v)\|_{L^\infty(\mathbb{C})} \leq (1/20) \|u - v\|_{L^\infty(\mathbb{C})},$$

and that if $v_1, v_2 \in T_p\mathbb{C}P^N$ are two tangent vectors satisfying $|v_1|, |v_2| \leq 2\delta$, then

$$|d(\exp(v_1), \exp(v_2)) - |v_1 - v_2|| \leq (1/20)|v_1 - v_2|.$$

The former condition comes from $d\alpha_0 = 0$, and the latter is just a standard property of the exponential map. Then all $u, v \in B_\delta(H_f)$ satisfy

$$\begin{aligned}
&|d(\exp(u + \alpha(u)), \exp(v + \alpha(v))) - |u + \alpha(u) - v - \alpha(v)|| \\
&\quad \leq (1/20)|u + \alpha(u) - v - \alpha(v)| \\
&\quad \leq (1/20) \|u - v\|_{L^\infty(\mathbb{C})} + (1/20) \|\alpha(u) - \alpha(v)\|_{L^\infty(\mathbb{C})} \\
&\quad \leq (1/20 + 1/400) \|u - v\|_{L^\infty(\mathbb{C})},
\end{aligned}$$

and

$$||u + \alpha(u) - v - \alpha(v)| - |u - v|| \leq |\alpha(u) - \alpha(v)| \leq (1/20) \|u - v\|_{L^\infty(\mathbb{C})}.$$

These inequalities imply the condition (ii):

$$|d(\exp(u+\alpha(u)), \exp(v+\alpha(v))) - |u-v|| \leq (1/8)\,\|u-v\|_{L^\infty(\mathbb{C})}\,.$$

□

5. Study of H_f: proof of Proposition 3.2

In this section we prove Proposition 3.2. Let $R > 0$, and let $\Lambda = [a_1, b_1] \times [a_2, b_2] \subset \mathbb{C}$ be an R-square (i.e. $b_1 = a_1 + R$ and $b_2 = a_2 + R$). For $0 < r < R/2$, we set

$$\partial_r\Lambda = \{([a_1, a_1+r) \cup (b_1-r, b_1]) \times [a_2, b_2]\} \cup \{[a_1, b_1] \times ([a_2, a_2+r) \cup (b_2-r, b_2])\}.$$

This notation is used only in this section. It conflicts with the notation $\partial_r\Omega$ introduced in Section 2.1. The following is a preliminary version of Proposition 3.2.

Proposition 5.1. *Let $f : \mathbb{C} \to \mathbb{C}P^N$ be a Brody curve. Let $\varepsilon > 0$, and let $\Lambda \subset \mathbb{C}$ be an R-square with $R > 2$. Then there exists a finite dimensional complex linear subspace $W \subset \Omega^0(E)$ (the space of $\mathcal{C}^\infty$-sections of $E = f^*T\mathbb{C}P^N$) satisfying the following three conditions:*

(i)

$$\dim_{\mathbb{C}} W \geq (N+1)\int_\Lambda |df|^2 dxdy - C_\varepsilon R,$$

where C_ε is a constant depending only on ε. The important point is that it is independent of R.

(ii) *All $u \in W$ satisfy $u = 0$ outside of Λ.*

(iii) *All $u \in W$ satisfy $\|\bar\partial u\|_{L^\infty(\mathbb{C})} \leq \varepsilon\,\|u\|_{L^\infty(\mathbb{C})}$.*

Proof. Set $\Lambda = [a_1, b_1] \times [a_2, b_2]$. Let $\varphi_i : \mathbb{R} \to \mathbb{R}$ ($i = 1, 2$) be smooth functions such that $0 \leq \varphi_i' \leq 1$, $\varphi_i(x) = x$ over $[a_i + 1/2, b_i - 1/2]$, $\varphi_i(x) = \varphi_i(a_i + 1/4)$ over $x \leq a_i + 1/4$ and $\varphi_i(x) = \varphi_i(b_i - 1/4)$ over $x \geq b_i - 1/4$. Moreover we assume that, for $k \geq 1$, $|\varphi_i^{(k)}| \leq \mathrm{const}_k$ (depending only on $k \geq 1$).

We define a $\mathcal{C}^\infty$-map $\tilde f : \mathbb{C} \to \mathbb{C}P^N$ by $\tilde f(x+\sqrt{-1}y) := f(\varphi_1(x)+\sqrt{-1}\varphi_2(y))$. We have $|d\tilde f|(z) := \max_{u\in T_z\mathbb{C},\,|u|=1} |d\tilde f(u)| \leq 1$ for all $z \in \mathbb{C}$. Let $\tilde E := \tilde f^* T\mathbb{C}P^N$ be the pull-back of $T\mathbb{C}P^N$ by $\tilde f$. $\tilde E$ is a complex vector bundle over $\mathbb{C}$ with the Hermitian metric $\tilde h$ (the pull-back of the Fubini–Study metric) and the unitary connection $\tilde\nabla$ (the pull-back of the Levi-Civita connection on $T\mathbb{C}P^N$). From the definition of $\tilde f$, the connection $\tilde\nabla$ is flat over $\partial_{1/4}\Lambda$. Flat connections over $\partial_{1/4}\Lambda$ are classified by their holonomy maps $\pi_1(\partial_{1/4}\Lambda) \to U(N)$. Hence there is a bundle trivialization (as a Hermitian vector bundle) g of $\tilde E$ over $\partial_{1/4}\Lambda$ such that $g(\tilde\nabla) = d + A$ (A: connection matrix) satisfies

$$\|A\|_{\mathcal{C}^k(\partial_{1/4}\Lambda)} \leq \mathrm{const}_k \quad (k \geq 0).$$

Here const_k are universal constants depending only on k. The important point is that they are independent of R. Let $\psi : \Lambda \to [0, 1]$ be a cut-off function such that $\psi = 1$ over $\Lambda\setminus\partial_{1/5}\Lambda$, $\psi = 0$ over $\partial_{1/6}\Lambda$, and $\|\psi\|_{\mathcal{C}^k(\Lambda)} \leq \mathrm{const}_k$. We define a unitary connection ∇' on $\tilde E$ over Λ by $\nabla' := g^{-1}(d + \psi A)$. We have $\nabla' = \tilde\nabla$ over $\Lambda\setminus\partial_{1/5}\Lambda$. Under the trivialization g, the metric $\tilde h$ and the connection ∇' are equal to the standard metric and the product connection of $\partial_{1/6}\Lambda \times \mathbb{C}^N$ over $\partial_{1/6}\Lambda$.

Consider an elliptic curve $\mathbb{T} := \mathbb{C}/(R\mathbb{Z} + R\sqrt{-1}\mathbb{Z})$, and let $\pi : \mathbb{C} \to \mathbb{T}$ be the natural projection. We define a complex vector bundle E' over $\mathbb{T}$ as follows. $E' = \tilde{E}$ over $\pi(\Lambda\backslash\partial_{1/5}\Lambda) \cong \Lambda\backslash\partial_{1/5}\Lambda$, and $E'|_{\pi(\partial_{1/4}\Lambda)}$ is equal to the product bundle $\pi(\partial_{1/4}\Lambda) \times \mathbb{C}^N$. We glue these by the map g. The metric $\tilde{h}$ and the connection ∇' naturally descend to the metric and connection on E' (also denoted by $\tilde{h}$ and ∇').

Let $\Theta' := [\nabla'_{\partial/\partial z}, \nabla'_{\partial/\partial \bar{z}}]$ be the curvature of ∇'. From the definition, $\Theta' = [\nabla_{\partial/\partial z}, \nabla_{\partial/\partial \bar{z}}]$ over $\pi(\Lambda\backslash\partial_{1/2}\Lambda) \cong \Lambda\backslash\partial_{1/2}\Lambda$, and $|\Theta'| \leq \text{const}$ (a universal constant) all over $\mathbb{T}$. Then by (7)

$$\int_{\mathbb{T}} c_1(E') = \frac{\sqrt{-1}}{2\pi}\int_{\mathbb{T}} \operatorname{tr}(\Theta') dz d\bar{z} \geq (N+1)\int_{\Lambda} |df|^2 dxdy - \text{const}\cdot R. \tag{9}$$

Let $\bar{\partial}_{\nabla'} : \Omega^0(E') \to \Omega^{0,1}(E')$ be the Dolbeault operator over $\mathbb{T}$ twisted by the unitary connection ∇' (i.e. the $(0,1)$-part of the covariant derivative $\nabla' : \Omega^0(E) \to \Omega^1(E)$). Let $H^0_{\nabla'}$ be the space of $u \in \Omega^0(E')$ satisfying $\bar{\partial}_{\nabla'} u = 0$. From the Riemann–Roch formula and the above (9)

$$\dim_{\mathbb{C}} H^0_{\nabla'} \geq \int_{\mathbb{T}} c_1(E') \geq (N+1)\int_{\Lambda} |df|^2 dxdy - \text{const}\cdot R. \tag{10}$$

Lemma 5.2. *For all $u \in H^0_{\nabla'}$,*

$$\|\nabla' u\|_{L^\infty(\mathbb{T})} \leq K \|u\|_{L^\infty(\mathbb{T})}.$$

Here K is a universal constant (independent of f, R, Λ).

Proof. The connection ∇' has the following property: There is a universal constant $r > 0$ such that for every $p \in \mathbb{T}$ there is a bundle trivialization v of a Hermitian vector bundle E' over $D_r(p)$ satisfying $v(\nabla') = d + A'$ with

$$\|A'\|_{\mathcal{C}^k(D_r(p))} \leq \text{const}_k \quad (k \geq 0).$$

Then the result follows from the elliptic regularity. □

Let $\tau = \tau(\varepsilon) > 0$ be a small number which will be fixed later. We take points $p_1, \dots, p_M \in \pi(\partial_1\Lambda)$ with $M \leq \text{const}_\tau \cdot R$ such that for every $p \in \pi(\partial_1\Lambda)$ there is p_i satisfying $d(p, p_i) \leq \tau$. We define $V \subset H^0_{\nabla'}$ as the space of $u \in H^0_{\nabla'}$ satisfying $u(p_i) = 0$ for all $i = 1, \dots, M$. From (10),

$$\dim_{\mathbb{C}} V \geq \dim_{\mathbb{C}} H^0_{\nabla'} - \dim_{\mathbb{C}}\left(\bigoplus_{i=1}^{M} E'_{p_i}\right) \geq (N+1)\int_{\Lambda} |df|^2 dxdy - C_\varepsilon R. \tag{11}$$

Let $u \in V$ and $p \in \pi(\partial_1\Lambda)$. Take p_i satisfying $d(p, p_i) \leq \tau$. From $u(p_i) = 0$ and Lemma 5.2,

$$|u(p)| \leq \tau \|\nabla' u\|_{L^\infty(\mathbb{T})} \leq \tau K \|u\|_{L^\infty(\mathbb{T})}.$$

We choose $\tau > 0$ so that $\tau K < 1$. Then the maximum of $|u|$ is attained in $\mathbb{T}\backslash\pi(\partial_1\Lambda)$.

Let $\phi : \mathbb{C} \to [0,1]$ be a cut-off such that $\phi = 1$ over $\Lambda\backslash\partial_1\Lambda$, $\operatorname{supp}(\phi)$ is contained in the interior of $\Lambda\backslash\partial_{1/2}\Lambda$, and $|d\phi| \leq 10$. For $u \in V$, we set $u' := \phi u$. Here we identify the region $\Lambda\backslash\partial_{1/2}\Lambda$ with $\pi(\Lambda\backslash\partial_{1/2}\Lambda)$ where we have $E' = E$, and we consider u' as a section of E over the plane $\mathbb{C}$. Set $W := \{u' | u \in V\}$. We have $\|u'\|_{L^\infty(\mathbb{C})} = \|u\|_{L^\infty(\mathbb{T})}$. Hence, by (11), we get the condition (i):

$$\dim_{\mathbb{C}} W = \dim_{\mathbb{C}} V \geq (N+1)\int_{\Lambda} |df|^2 dxdy - C_\varepsilon R.$$

The condition (ii) is obviously satisfied. $\bar{\partial}u' = \bar{\partial}\phi \otimes u$ is supported in $\partial_1\Lambda$.

$$\left\|\bar{\partial}u'\right\|_{L^\infty(\mathbb{C})} \leq 10\left\|u\right\|_{L^\infty(\pi(\partial_1\Lambda))} \leq 10\tau K\left\|u\right\|_{L^\infty(\mathbb{T})} = 10\tau K\left\|u'\right\|_{L^\infty(\mathbb{C})}.$$

We choose $\tau > 0$ so that $10\tau K \leq \varepsilon$. Then the condition (iii) is satisfied. $\square$

Proof of Proposition 3.2. Let $\varepsilon > 0$ be a small number which will be fixed later. By Proposition 5.1, for this ε and any R-square Λ ($R > 2$), there is a finite dimensional complex linear subspace $W \subset \Omega^0(E)$ satisfying the conditions (i), (ii), (iii) in Proposition 5.1. By Proposition 4.7, there is a linear map

$$W \to \Omega^{0,1}(E), \quad u \mapsto a,$$

such that

$$\bar{\partial}\bar{\partial}^*_\varphi a = \bar{\partial}u, \quad \left\|\bar{\partial}^*_\varphi a\right\|_{L^\infty(\mathbb{C})} \leq C'_f\left\|\bar{\partial}u\right\|_{L^\infty(\mathbb{C})} \leq C'_f \cdot \varepsilon\left\|u\right\|_{L^\infty(\mathbb{C})}.$$

Set $u' := u - \bar{\partial}^*_\varphi a$. Then $\bar{\partial}u' = 0$ and $\left\|u'\right\|_{L^\infty(\mathbb{C})} \geq (1 - C'_f\varepsilon)\left\|u\right\|_{L^\infty(\mathbb{C})}$. We choose $\varepsilon > 0$ so that $1 - C'_f\varepsilon > 0$. We set $V := \{u' | u \in W\}$. Then $V \subset H_f$ and

$$\dim_{\mathbb{C}} V = \dim_{\mathbb{C}} W \geq (N+1)\int_\Lambda |df|^2 dxdy - C_\varepsilon R.$$

For $u \in W$ (recall $\mathrm{supp}(u) \subset \Lambda$)

$$\left\|u'\right\|_{L^\infty(\mathbb{C})} \leq (1 + C'_f\varepsilon)\left\|u\right\|_{L^\infty(\mathbb{C})} = (1 + C'_f\varepsilon)\left\|u\right\|_{L^\infty(\Lambda)},$$

$$\left\|u'\right\|_{L^\infty(\Lambda)} \geq (1 - C'_f\varepsilon)\left\|u\right\|_{L^\infty(\Lambda)}.$$

Hence

$$\left\|u'\right\|_{L^\infty(\mathbb{C})} \leq \frac{1 + C'_f\varepsilon}{1 - C'_f\varepsilon}\left\|u'\right\|_{L^\infty(\Lambda)}.$$

We choose $\varepsilon > 0$ so small that

$$\frac{1 + C'_f\varepsilon}{1 - C'_f\varepsilon} \leq 2.$$

$\square$

6. Infinite gluing: proof of Theorem 1.7

We prove Theorem 1.7 in this section. Our method is gluing; we glue infinitely many rational curves to a (possibly degenerate) Brody curve $f : \mathbb{C} \to \mathbb{C}P^N$, and construct a non-degenerate one.

A kind of "infinite gluing construction" is classically used for the proof of Mittag-Leffler's theorem. Probably another origin of infinite gluing construction is the shadowing lemma in dynamical system theory (for example, see Bowen [2, Chapter 3]). Angenent [1] developed a shadowing lemma for an elliptic PDE. Gromov [11, p. 403] suggested an idea of gluing infinitely many rational curves to a (pseudo-)holomorphic curve. Macrì–Nolasco–Ricciardi [14] developed gluing infinitely many selfdual vortices. Gournay [6, 9] studied an infinite gluing method for pseudo-holomorphic curves. Tsukamoto [17, 20] studied gluing infinitely many Yang–Mills instantons.

First we establish a result on gluing one rational curve.

Proposition 6.1. *There are $\delta_0 > 0$, $R_0 > 0$ and $K > 0$ satisfying the following statement. Let $f : \mathbb{C} \to \mathbb{C}P^N$ be a Brody curve. If f satisfies $\|df\|_{L^\infty(D_R(p))} < \delta_0$ for some $p \in \mathbb{C}$ and $R \geq R_0 + 1$, then there exists a holomorphic curve $g : \mathbb{C} \to \mathbb{C}P^N$ satisfying the following three conditions:*

(i) $\delta_0 \leq \|dg\|_{L^\infty(D_R(p))} \leq 2/3$.
(ii) $||dg|(z) - |df|(z)| \leq K/|z-p|^3$ *over* $|z-p| > R$.
(iii) $d(f(z), g(z)) \leq K/|z-p|^3$ *for* $z \neq p$.

Proof. The proof is just a calculation. It may be helpful for some readers to consider the case of $N = 1$ by themselves. Let $\varepsilon > 0$ be a sufficiently small number. δ_0, R_0, K and ε will be fixed later. Several conditions will be imposed on them through the argument, but basically they need to satisfy

$$\delta_0 \ll \frac{\varepsilon}{R_0}, \quad R_0 \gg 1, \quad \varepsilon \ll \frac{1}{R_0^4}.$$

Fix $a > 0$ so that the curve $q : \mathbb{C} \to \mathbb{C}P^N$ defined by $q(z) := [1 : a/z^3 : \cdots : a/z^3]$ satisfies $\|dq\|_{L^\infty(\mathbb{C})} = 1/12$. Here

$$|dq|(z) = \frac{3a\sqrt{N}r^2}{\sqrt{\pi}(r^6 + Na^2)} \quad (r = |z|).$$

We can suppose $\|dq\|_{L^\infty(D_{R_0}(0))} = 1/12$ since we choose $R_0 \gg 1$.

From the symmetry we can assume $p = 0$ and $f(0) = [1 : 0 : \cdots : 0]$. Let $f(z) = [1 : f_1(z) : \cdots : f_N(z)]$ where $f_i(z)$ are meromorphic functions in $\mathbb{C}$. Since $|df| \leq \delta_0$ over $|z| \leq R$ with $R \geq R_0 + 1$, if we choose δ_0 sufficiently small ($\delta_0 \ll \varepsilon/R_0$), we have

$$|f_i(z)| \leq \varepsilon, \quad |f_i'(z)| \leq \varepsilon \quad (|z| \leq R_0). \tag{12}$$

Set $g_i(z) := f_i(z) + a/z^3$, and we define $g : \mathbb{C} \to \mathbb{C}P^N$ by $g(z) := [1 : g_1(z) : \cdots : g_N(z)]$. We will prove that this map g satisfies the conditions (i), (ii), (iii).

First we study the condition (iii). The Fubini–Study metric is given by

$$ds^2 = \frac{\sum_{i=1}^N |dz_i|^2 + \sum_{1\leq i<j\leq N} |z_j dz_i - z_i dz_j|^2}{\pi(1 + \sum |z_i|^2)^2} \quad \text{on } \{[1 : z_1 : \cdots : z_N]\}.$$

$$ds^2 \leq \frac{\sum |dz_i|^2 + 2(\sum |z_i|^2)(\sum |dz_i|^2)}{\pi(1 + \sum |z_i|^2)^2} \leq \frac{2(1 + \sum |z_i|^2) \sum |dz_i|^2}{\pi(1 + \sum |z_i|^2)^2} \leq \frac{2}{\pi} \sum |dz_i|^2.$$

Hence $ds \leq \sqrt{2/\pi}\sqrt{\sum_{i=1}^N |dz_i|^2}$. Thus for $f(z) = [1 : f_1(z) : \cdots : f_N(z)]$ and $g(z) = [1 : f_1(z) + a/z^3 : \cdots : f_N(z) + a/z^3]$ we get

$$d(f(z), g(z)) \leq \sqrt{2/\pi}\sqrt{\sum_{i=1}^N |a/z^3|^2} = \frac{a\sqrt{2N/\pi}}{|z|^3}. \tag{13}$$

Next we study the conditions (i) and (ii). We have

$$|df|(z) = \frac{\sqrt{\sum |f_i'(z)|^2 + \sum_{i<j} |f_i'(z)f_j(z) - f_i(z)f_j'(z)|^2}}{\sqrt{\pi}(1 + \sum |f_i(z)|^2)},$$

$$|dg|(z) = \frac{\sqrt{\sum |g_i'(z)|^2 + \sum_{i<j} |g_i'(z)g_j(z) - g_i(z)g_j'(z)|^2}}{\sqrt{\pi}(1 + \sum |g_i(z)|^2)},$$

where

$$g_i' = f_i' - \frac{3a}{z^4}, \quad g_i' g_j - g_i g_j' = (f_i' f_j - f_i f_j') + \frac{3a}{z^4}(f_i - f_j) + \frac{a}{z^3}(f_i' - f_j').$$

Case 1: Suppose $r := |z| \le R_0$. We will prove $\delta_0 \le \|dg\|_{L^\infty(D_{R_0}(0))} \le 2/3$. From (12),

$$|g_i(z)| \le \varepsilon + \frac{a}{r^3} \le \frac{2a}{r^3}, \quad |g_i'(z)| \ge \frac{3a}{r^4} - \varepsilon \ge \frac{3a}{2r^4}.$$

Here we have supposed $\varepsilon \le \min(a/R_0^3, 3a/(2R_0^4))$. Then

$$|dg|(z) \ge \frac{\sqrt{N}(3a/(2r^4))}{\sqrt{\pi}(1+4Na^2/r^6)} = \frac{3a\sqrt{N}r^2}{2\sqrt{\pi}(r^6+4Na^2)} \ge \frac{3a\sqrt{N}r^2}{8\sqrt{\pi}(r^6+Na^2)} = \frac{|dq|(z)}{8}.$$

Hence $\|dg\|_{L^\infty(D_{R_0}(0))} \ge (1/8)\,\|dq\|_{L^\infty(D_{R_0}(0))} = 1/96 \ge \delta_0$. Here we have supposed $\delta_0 \le 1/96$. On the other hand,

$$|dg|(z) = \frac{\sqrt{\sum |3az^2 - z^6 f_i'|^2 + \sum_{i<j} |z^6(f_i' f_j - f_j' f_i) + 3az^2(f_i - f_j) + az^3(f_i' - f_j')|^2}}{\sqrt{\pi}(r^6 + \sum |a + z^3 f_i|^2)}.$$

From (12),

$$|a + z^3 f_i| \ge a - \varepsilon R_0^3 \ge \frac{a}{2}, \quad \text{(here we suppose } \varepsilon R_0^3 \le a/2).$$

$$r^6 + \sum |a + z^3 f_i|^2 \ge r^6 + \frac{Na^2}{4} \ge \frac{r^6 + Na^2}{4}.$$

$$|3az^2 - z^6 f_i'| \le 3ar^2 + r^6 \varepsilon \le r^2(3a + R_0^4 \varepsilon) \le 4ar^2, \quad \text{(we suppose } R_0^4\varepsilon \le a).$$

$$|z^6(f_i' f_j - f_j' f_i) + 3az^2(f_i - f_j) + az^3(f_i' - f_j')| \le r^2(2\varepsilon^2 R_0^4 + 6a\varepsilon + 2a\varepsilon R_0) \le \frac{ar^2}{\sqrt{\binom{N}{2}}}.$$

Here we have supposed $2\varepsilon^2 R_0^4 + 6a\varepsilon + 2a\varepsilon R_0 \le a/\sqrt{\binom{N}{2}}$. Then

$$|dg|(z) \le \frac{4ar^2\sqrt{16N+1}}{\sqrt{\pi}(r^6+Na^2)} \le \frac{24ar^2\sqrt{N}}{\sqrt{\pi}(r^6+Na^2)} = 8|dq|(z) \le \frac{2}{3}, \quad (|dq| \le 1/12).$$

Thus we get $\delta_0 \le \|dg\|_{L^\infty(D_{R_0}(0))} \le 2/3$.

Case 2: Suppose $|z| \ge R_0$. We will prove $||df|(z) - |dg|(z)| \le K/r^3$ for an appropriate $K > 0$. We have

$$\big||f_i|^2 - |g_i|^2\big| \le (|f_i| + |g_i|) \cdot |f_i - g_i| \le (2|f_i| + a/r^3)(a/r^3) \le (2|f_i| + a/R_0^3)(a/r^3),$$

$$\sum \big||f_i|^2 - |g_i|^2\big| \le \frac{a}{r^3}\left(2\sum |f_i| + \frac{Na}{R_0^3}\right) \le \frac{2a}{r^3}\left(1 + \sum |f_i|\right)$$

$$\text{(we suppose } \frac{Na}{R_0^3} \le 2).$$

If $|f_i| \ge a/r^3$, then

$$|g_i|^2 \ge \left(|f_i| - a/r^3\right)^2 \ge \frac{|f_i|^2}{2} - \frac{a^2}{r^6} \ge \frac{|f_i|^2}{2} - \frac{a^2}{R_0^6}, \quad \left((x-y)^2 \ge \frac{x^2}{2} - y^2\right).$$

If $|f_i| < a/r^3$, then

$$|g_i|^2 \geq 0 > \frac{|f_i|^2}{2} - \frac{a^2}{r^6} \geq \frac{|f_i|^2}{2} - \frac{a^2}{R_0^6}.$$

Therefore we always have $|g_i|^2 \geq |f_i|^2/2 - a^2/R_0^6$.

$$1+\sum |g_i|^2 \geq \left(1 - \frac{Na^2}{R_0^6}\right) + \frac{1}{2}\sum |f_i|^2 \geq \frac{1}{2}\left(1 + \sum |f_i|^2\right) \quad \left(\text{we suppose } \frac{Na^2}{R_0^6} \leq \frac{1}{2}\right).$$

Hence

$$\begin{aligned} \left|\frac{1}{1+\sum |g_i|^2} - \frac{1}{1+\sum |f_i|^2}\right| &\leq \frac{\frac{4a}{r^3}\left(1+\sum |f_i|\right)}{\left(1+\sum |f_i|^2\right)^2} \leq \frac{4a\sqrt{N+1}\sqrt{1+\sum |f_i|^2}}{r^3\left(1+\sum |f_i|^2\right)^2} \\ &= \frac{4a\sqrt{N+1}}{r^3\left(1+\sum |f_i|^2\right)^{3/2}} \leq \frac{4a\sqrt{N+1}}{r^3(1+\sum |f_i|^2)}. \end{aligned} \tag{14}$$

Then, from $g_i' = f_i' - 3a/z^4$ and the above (14),

$$\begin{aligned} \left|\frac{|g_i'|}{1+\sum |g_k|^2} - \frac{|f_i'|}{1+\sum |f_k|^2}\right| &\leq \left|\frac{|g_i'|}{1+\sum |g_k|^2} - \frac{|g_i'|}{1+\sum |f_k|^2}\right| \\ &\quad + \left|\frac{|g_i'|}{1+\sum |f_k|^2} - \frac{|f_i'|}{1+\sum |f_k|^2}\right| \\ &\leq \frac{4a\sqrt{N+1}(|f_i'| + 3a/r^4)}{r^3(1+\sum |f_k|^2)} + \frac{3a}{r^4(1+\sum |f_k|^2)}. \end{aligned}$$

From $|df| \leq 1$, we have $|f_i'|/(1+\sum |f_k|^2) \leq \sqrt{\pi}$. Hence the above is bounded by

$$\frac{4a\sqrt{N+1}}{r^3}(\sqrt{\pi} + 3a/r^4) + 3a/r^4 \leq \frac{4a\sqrt{N+1}}{r^3}(\sqrt{\pi} + 3a) + \frac{3a}{r^3}.$$

Here we have supposed $r \geq R_0 \geq 1$. Set $K_a := 4a\sqrt{N+1}(\sqrt{\pi} + 3a) + 3a$. Then

$$\left|\frac{|g_i'|}{1+\sum |g_k|^2} - \frac{|f_i'|}{1+\sum |f_k|^2}\right| \leq \frac{K_a}{r^3}. \tag{15}$$

From (14), for $i < j$,

$$\begin{aligned} \left|\frac{|g_i'g_j - g_j'g_i|}{1+\sum |g_k|^2} - \frac{|f_i'f_j - f_j'f_i|}{1+\sum |f_k|^2}\right| &\leq \left|\frac{|g_i'g_j - g_j'g_i|}{1+\sum |g_k|^2} - \frac{|g_i'g_j - g_j'g_i|}{1+\sum |f_k|^2}\right| \\ &\quad + \left|\frac{|g_i'g_j - g_j'g_i|}{1+\sum |f_k|^2} - \frac{|f_i'f_j - f_j'f_i|}{1+\sum |f_k|^2}\right| \\ &\leq \frac{4a\sqrt{N+1}|g_i'g_j - g_j'g_i|}{r^3(1+\sum |f_k|^2)} \\ &\quad + \frac{|(g_i'g_j - g_j'g_i) - (f_i'f_j - f_j'f_i)|}{1+\sum |f_k|^2}. \end{aligned}$$

From $g_i'g_j - g_j'g_i = (f_i'f_j - f_j'f_i) + (3a/z^4)(f_i - f_j) + (a/z^3)(f_i' - f_j')$, this is bounded by

$$\begin{aligned} &\frac{4a\sqrt{N+1}}{r^3}\left(\frac{|f_i'f_j - f_j'f_i|}{1+\sum |f_k|^2} + \frac{3a(|f_i| + |f_j|)}{r^4(1+\sum |f_k|^2)} + \frac{a(|f_i'| + |f_j'|)}{r^3(1+\sum |f_k|^2)}\right) \\ &\quad + \frac{3a(|f_i| + |f_j|)}{r^4(1+\sum |f_k|^2)} + \frac{a(|f_i'| + |f_j'|)}{r^3(1+\sum |f_k|^2)}. \end{aligned} \tag{16}$$

From $|df| \leq 1$,

$$\frac{|f_i' f_j - f_j' f_i|}{1+\sum |f_k|^2} \leq \sqrt{\pi}, \quad \frac{|f_i'| + |f_j'|}{1+\sum |f_k|^2} \leq 2\sqrt{\pi}.$$

Since $i < j$,

$$\frac{|f_i| + |f_j|}{1+\sum |f_k|^2} \leq \frac{\sqrt{2}\sqrt{|f_i|^2 + |f_j|^2}}{1+\sum |f_k|^2} \leq \sqrt{2}.$$

Hence the above (16) is bounded by

$$\begin{aligned} &\frac{4a\sqrt{N+1}}{r^3}\left(\sqrt{\pi} + \frac{3a\sqrt{2}}{r^4} + \frac{2a\sqrt{\pi}}{r^3}\right) + \frac{3a\sqrt{2}}{r^4} + \frac{2a\sqrt{\pi}}{r^3} \\ &\leq \frac{4a\sqrt{N+1}}{r^3}(\sqrt{\pi} + 3a\sqrt{2} + 2a\sqrt{\pi}) + \frac{3a\sqrt{2}}{r^3} + \frac{2a\sqrt{\pi}}{r^3}. \end{aligned}$$

Here $r \geq R_0 \geq 1$. Set $K_a' := 4a\sqrt{N+1}(\sqrt{\pi}+3a\sqrt{2}+2a\sqrt{\pi})+3a\sqrt{2}+2a\sqrt{\pi}$. Then

$$\left|\frac{|g_i' g_j - g_j' g_i|}{1+\sum |g_k|^2} - \frac{|f_i' f_j - f_j' f_i|}{1+\sum |f_k|^2}\right| \leq \frac{K_a'}{r^3}.$$

From this and (15),

$$||dg|(z) - |df|(z)| \leq (1/\sqrt{\pi})\sqrt{N(K_a/r^3)^2 + \binom{N}{2}(K_a'/r^3)^2} = \frac{\sqrt{NK_a^2 + \binom{N}{2}(K_a')^2}}{\sqrt{\pi} r^3}.$$

Here we have used the inequality

$$\left|\sqrt{x_1^2 + \cdots + x_l^2} - \sqrt{y_1^2 + \cdots + y_l^2}\right| \leq \sqrt{(x_1 - y_1)^2 + \cdots + (x_l - y_l)^2}.$$

Set

$$K := \max\left(a\sqrt{2N/\pi}, \sqrt{NK_a^2 + \binom{N}{2}(K_a')^2}/\sqrt{\pi}\right).$$

(This K satisfies the condition (iii) by (13).) Then

$$||df|(z) - |dg|(z)| \leq \frac{K}{r^3} \quad (r \geq R_0).$$

Thus we have proved the condition (ii).

For $R_0 \leq |z| \leq R$,

$$|dg|(z) \leq \|df\|_{L^\infty(D_R(0))} + \frac{K}{R_0^3} \leq \delta_0 + \frac{1}{2} \leq \frac{2}{3},$$

where we have chosen R_0 and δ_0 so that $K/R_0^3 \leq 1/2$ and $\delta_0 \leq 1/6$. In Case 1, we proved $\delta_0 \leq \|dg\|_{L^\infty(D_{R_0}(0))} \leq 2/3$. Thus we get the condition (i):

$$\delta_0 \leq \|dg\|_{L^\infty(D_R(0))} \leq 2/3.$$

□

Proof of Theorem 1.7. Let $\|df\|_{L^\infty(\mathbb{C})} \leq 1-\tau$, $(0 < \tau \leq 1)$. Let δ_0, R_0, K be the positive numbers introduced in Proposition 6.1. For $\varepsilon > 0$, we set $\delta := \min(\delta_0, \sqrt{\varepsilon})$. Let $R = R(\varepsilon, \tau) \geq R_0 + 1$ be a large positive number which will be fixed later.

We index the elements of $\mathbb{Z}^2$ by natural numbers: $\mathbb{Z}^2 = \{(\alpha_1, \beta_1), (\alpha_2, \beta_2), (\alpha_3, \beta_3), \ldots\}$. For $n \geq 1$, we set $p_n := 2R(\alpha_n + \sqrt{-1}\beta_n)$ and $\Lambda_n := \{x + y\sqrt{-1} \in$

$\mathbb{C}|\,|x-2R\alpha_n|\leq R, |y-2R\beta_n|\leq R\}$. The squares Λ_n $(n\geq 1)$ become a tiling of the plane $\mathbb{C}$.

We inductively define the sequence of Brody curves $f_n:\mathbb{C}\to\mathbb{C}P^N$ $(n\geq 0)$ as follows. We set $f_0:=f$. Suppose we have defined f_n.

(1) If $\|df\|_{L^\infty(\Lambda_{n+1})}\geq\delta$, then we set $f_{n+1}:=f_n$.
(2) If $\|df\|_{L^\infty(\Lambda_{n+1})}<\delta$ and $\|df_n\|_{L^\infty(\Lambda_{n+1})}\geq\delta_0$, then we set $f_{n+1}:=f_n$.
(3) If $\|df\|_{L^\infty(\Lambda_{n+1})}<\delta$ and $\|df_n\|_{L^\infty(\Lambda_{n+1})}<\delta_0$, then we apply Proposition 6.1 to f_n and p_{n+1} (note $D_R(p_{n+1})\subset\Lambda_{n+1}$) and get a holomorphic map $f_{n+1}:\mathbb{C}\to\mathbb{C}P^N$ satisfying the following (i), (ii), (iii).
  (i) $\delta_0\leq\|df_{n+1}\|_{L^\infty(D_R(p_{n+1}))}\leq 2/3$.
  (ii) $\big||df_{n+1}|(z)-|df_n|(z)\big|\leq K/|z-p_{n+1}|^3$ over $|z-p_{n+1}|>R$.
  (iii) $d(f_n(z),f_{n+1}(z))\leq K/|z-p_{n+1}|^3$ for $z\neq p_{n+1}$.

For every $n\geq 1$, by (i) and (ii)

$$|df_n|(z)\leq\max(1-\tau,2/3)+\sum_{k:|z-p_k|>R}\frac{K}{|z-p_k|^3}\leq\max(1-\tau,2/3)+\frac{\mathrm{const}\cdot K}{R^3}.$$

Here const is a positive constant independent of n. We choose R so large that the right-hand side is bounded by $\max(1-\tau/2,3/4)<1$. Then all $f_n:\mathbb{C}\to\mathbb{C}P^N$ become Brody curves, and we can continue the above inductive construction infinitely many times. Moreover, for all $n\geq 1$,

$$\|df_n\|_{L^\infty(\mathbb{C})}\leq\max(1-\tau/2,3/4). \tag{17}$$

For any compact set $\Omega\subset\mathbb{C}$, by the condition (iii), there exists $n(\Omega)\geq 1$ such that

$$\sum_{n\geq n(\Omega)}\sup_{z\in\Omega}d(f_n(z),f_{n+1}(z))\leq\sum_{k:d(p_k,\Omega)\geq 1}\frac{K}{d(p_k,\Omega)^3}<+\infty.$$

Hence the sequence f_n converges to a holomorphic curve $g:\mathbb{C}\to\mathbb{C}P^N$ uniformly over every compact subset of $\mathbb{C}$. From (17) we have $\|dg\|_{L^\infty(\mathbb{C})}\leq\max(1-\tau/2,3/4)<1$. We will prove that g is non-degenerate and $\rho(g)\geq\rho(f)-\varepsilon$.

For proving the non-degeneracy of g, it is enough to show $\|dg\|_{L^\infty(\Lambda_n)}\geq\delta/2$ for all $n\geq 1$. See the condition (ii) of Definition-Lemma 1.3.

Case 1: If $|df|(z)\geq\delta$ for some $z\in\Lambda_n$, then

$$|dg|(z)\geq\delta-\sum_{k:k\neq n}\frac{K}{|z-p_k|^3}\geq\delta-\frac{\mathrm{const}\cdot K}{R^3}.$$

We can choose R so large that $\|dg\|_{L^\infty(\Lambda_n)}\geq\delta/2$.

Case 2: If $|df|(z)<\delta$ for all $z\in\Lambda_n$, then for some $k\in\{n-1,n\}$ and $w\in\Lambda_n$ we have $|df_k|(w)\geq\delta_0$. Hence

$$|dg|(w)\geq\delta_0-\sum_{l:l\neq n}\frac{K}{|w-p_l|^3}\geq\delta-\frac{\mathrm{const}\cdot K}{R^3}.$$

We can choose R so large that $\|dg\|_{L^\infty(\Lambda_n)}\geq\delta/2$.

We have proved that g is non-degenerate. Next we will prove $\rho(g)\geq\rho(f)-\varepsilon$. For this sake, it is enough to prove that for every $n\geq 1$

$$\frac{1}{(2R)^2}\int_{\Lambda_n}|dg|^2dxdy\geq\frac{1}{(2R)^2}\int_{\Lambda_n}|df|^2dxdy-\varepsilon. \tag{18}$$

Case 1: If $\|df\|_{L^\infty(\Lambda_n)} \geq \delta$, then for all $z \in \Lambda_n$

$$\left||dg|^2(z) - |df|^2(z)\right| \leq 2\left||dg|(z) - |df|(z)\right| \leq \sum_{k:k\neq n} \frac{2K}{|z-p_k|^3} \leq \frac{\text{const}\cdot K}{R^3} \leq \varepsilon$$

for sufficiently large R. Hence (18) holds if we choose R sufficiently large.

Case 2: If $\|df\|_{L^\infty(\Lambda_n)} < \delta$, then (recall $\delta = \min(\delta_0, \sqrt{\varepsilon})$)

$$\frac{1}{(2R)^2}\int_{\Lambda_n} |df|^2 dxdy \leq \delta^2 \leq \varepsilon.$$

Hence (18) holds trivially.

Thus we have proved $\rho(g) \geq \rho(f) - \varepsilon$. □

Acknowledgements

The first author was supported by a Grant-in-Aid for JSPS fellows (23·149), and the second author was supported by a Grant-in-Aid for Young Scientists (B) (21740048).

References

[1] Sigurd Angenent, *The shadowing lemma for elliptic PDE*, Dynamics of infinite-dimensional systems (Lisbon, 1986), NATO Adv. Sci. Inst. Ser. F Comput. Systems Sci., vol. 37, Springer, Berlin, 1987, pp. 7–22. MR921893 (89b:58165)

[2] Rufus Bowen, *Equilibrium states and the ergodic theory of Anosov diffeomorphisms*, Second revised edition, Lecture Notes in Mathematics, vol. 470, Springer-Verlag, Berlin, 2008. With a preface by David Ruelle; Edited by Jean-René Chazottes. MR2423393 (2009d:37038)

[3] Robert Brody, *Compact manifolds and hyperbolicity*, Trans. Amer. Math. Soc. **235** (1978), 213–219. MR0470252 (57 #10010)

[4] Alexandre Eremenko, *Normal holomorphic curves from parabolic regions to projective spaces*, available at `arXiv:0710.1281`.

[5] David Gilbarg and Neil S. Trudinger, *Elliptic partial differential equations of second order*, Classics in Mathematics, Springer-Verlag, Berlin, 2001. Reprint of the 1998 edition. MR1814364 (2001k:35004)

[6] Antoine Gournay, *Dimension moyenne et espaces d'applications pseudo-holomorphes*, Département de Mathématiques d'Orsay, 2008.

[7] Antoine Gournay, *On a Hölder covariant version of mean dimension* (English, with English and French summaries), C. R. Math. Acad. Sci. Paris **347** (2009), no. 23-24, 1389–1392, DOI 10.1016/j.crma.2009.10.014. MR2588787 (2011b:46041)

[8] Antoine Gournay, *A dynamical approach to von Neumann dimension*, Discrete Contin. Dyn. Syst. **26** (2010), no. 3, 967–987, DOI 10.3934/dcds.2010.26.967. MR2600725 (2011d:37008)

[9] A. Gournay, *Complex surfaces and interpolation on pseudo-holomorphic cylinder*, available at `arXiv:1006.1775`.

[10] Antoine Gournay, *Widths of ℓ^p balls*, Houston J. Math. **37** (2011), no. 4, 1227–1248. MR2875268

[11] Misha Gromov, *Topological invariants of dynamical systems and spaces of holomorphic maps. I*, Math. Phys. Anal. Geom. **2** (1999), no. 4, 323–415, DOI 10.1023/A:1009841100168. MR1742309 (2001j:37037)

[12] Elon Lindenstrauss, *Mean dimension, small entropy factors and an embedding theorem*, Inst. Hautes Études Sci. Publ. Math. **89** (1999), 227–262 (2000). MR1793417 (2001j:37033)

[13] Elon Lindenstrauss and Benjamin Weiss, *Mean topological dimension*, Israel J. Math. **115** (2000), 1–24, DOI 10.1007/BF02810577. MR1749670 (2000m:37018)

[14] Marta Macrì, Margherita Nolasco, and Tonia Ricciardi, *Asymptotics for selfdual vortices on the torus and on the plane: a gluing technique*, SIAM J. Math. Anal. **37** (2005), no. 1, 1–16, DOI 10.1137/040619843. MR2176921 (2006g:58039)

[15] Shinichiroh Matsuo and Masaki Tsukamoto, *Instanton approximation, periodic ASD connections, and mean dimension*, J. Funct. Anal. **260** (2011), no. 5, 1369–1427, DOI 10.1016/j.jfa.2010.11.008. MR2749431 (2012b:58019)

[16] Dusa McDuff and Dietmar Salamon, *J-holomorphic curves and symplectic topology*, American Mathematical Society Colloquium Publications, vol. 52, American Mathematical Society, Providence, RI, 2004. MR2045629 (2004m:53154)

[17] Masaki Tsukamoto, *Gluing an infinite number of instantons*, Nagoya Math. J. **188** (2007), 107–131. MR2371770 (2009c:53035)

[18] Masaki Tsukamoto, *Moduli space of Brody curves, energy and mean dimension*, Nagoya Math. J. **192** (2008), 27–58. MR2477610 (2009i:32022)

[19] Masaki Tsukamoto, *A packing problem for holomorphic curves*, Nagoya Math. J. **194** (2009), 33–68. MR2536526 (2010g:32024)

[20] Masaki Tsukamoto, *Gauge theory on infinite connected sum and mean dimension*, Math. Phys. Anal. Geom. **12** (2009), no. 4, 325–380, DOI 10.1007/s11040-009-9065-z. MR2551659 (2011g:58014)

[21] Masaki Tsukamoto, *Deformation of Brody curves and mean dimension*, Ergodic Theory Dynam. Systems **29** (2009), no. 5, 1641–1657, DOI 10.1017/S014338570800076X. MR2545021 (2011a:32033)

[22] Masaki Tsukamoto, *Remark on energy density of Brody curves*, Proc. Japan Acad. Ser. A Math. Sci. **88** (2012), no. 8, 127–131, DOI 10.3792/pjaa.88.127. MR2989063

[23] K. Yosida, *On a class of meromorphic functions*, Proc. Phys.-Math. Soc. Japan **16** (1934), 227–235.

Department of Mathematics, Osaka University, Toyonaka, Osaka 560-0043, Japan
E-mail address: `matsuo@math.sci.osaka-u.ac.jp`

Department of Mathematics, Kyoto University, Kyoto 606-8502, Japan
E-mail address: `tukamoto@math.kyoto-u.ac.jp`