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Instanton approximation, periodic ASD connections, and mean dimension ☆

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Abstract

We study a moduli space of ASD connections over $S^3 \times \mathbb{R}$. We consider not only finite energy ASD connections but also infinite energy ones. So the moduli space is infinite dimensional in general. We study the (local) mean dimension of this infinite dimensional moduli space. We show the upper bound on the mean dimension by using a "Runge-approximation" for ASD connections, and we prove its lower bound by constructing an infinite dimensional deformation theory of periodic ASD connections.

Keywords: Yang–Mills gauge theory; Instanton approximation; Infinite dimensional deformation theory; Periodic ASD connections; Mean dimension

1. Introduction

Since Donaldson [4] discovered his revolutionary theory, many mathematicians have intensively studied the Yang–Mills gauge theory. There are several astonishing results on the structures of the ASD moduli spaces and their applications. But most of them study only finite energy ASD connections and their finite dimensional moduli spaces. Almost nothing is known about infinite energy ASD connections and their infinite dimensional moduli spaces. (One of the authors

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struggled to open the way to this direction in [21,22].) This paper studies an infinite dimensional moduli space coming from the Yang–Mills theory over $S^3 \times \mathbb{R}$. Our main purposes are to prove estimates on its "mean dimension" (Gromov [14]) and to show that there certainly exists a non-trivial structure in this infinite dimensional moduli space. (Mean dimension is a "dimension of an infinite dimensional space averaged by a group action".)

The reason why we consider $S^3 \times \mathbb{R}$ is that it is one of the simplest non-compact anti-self-dual 4-manifolds of (uniformly) positive scalar curvature. (Indeed it is conformally flat.) These metrical conditions are used via the Weitzenböck formula (see Section 4.1). Recall that one of the important results of the pioneering work of Atiyah, Hitchin and Singer [1, Theorem 6.1] is the calculation of the dimension of the moduli space of (irreducible) self-dual connections over a compact self-dual 4-manifold of positive scalar curvature. So our work is an attempt to develop an infinite dimensional analogue of [1, Theorem 6.1].

Of course, the study of the mean dimension is just one step toward the full understanding of the structures of the infinite dimensional moduli space. (But the authors believe that "dimension" is one of the most fundamental invariants of spaces and that the study of mean dimension is a crucial step toward the full understanding.) So we need much more studies, and the authors hope that this paper becomes a stimulus to a further study of infinite dimensional moduli spaces in the Yang–Mills gauge theory.

Set $X := S^3 \times \mathbb{R}$. Throughout the paper, the variable t means the variable of the $\mathbb{R}$-factor of $X = S^3 \times \mathbb{R}$. (That is, $t : X \to \mathbb{R}$ is the natural projection.) $S^3 \times \mathbb{R}$ is endowed with the product metric of a positive constant curvature metric on S^3 and the standard metric on $\mathbb{R}$. (Therefore X is $S^3(r) \times \mathbb{R}$ for some $r > 0$ as a Riemannian manifold, where $S^3(r) = \{x \in \mathbb{R}^4 \mid |x| = r\}$.) Let $\boldsymbol{E} := X \times SU(2)$ be the product principal $SU(2)$-bundle over X. The additive Lie group $\mathbb{R}$ acts on X by $X \times \mathbb{R} \ni ((\theta, t), s) \mapsto (\theta, t + s) \in X$. This action trivially lifts to the action on $\boldsymbol{E}$ by $\boldsymbol{E} \times \mathbb{R} \ni ((\theta, t, u), s) \mapsto (\theta, t + s, u) \in \boldsymbol{E}$.

Let $d \geqslant 0$. We define $\mathcal{M}_d$ as the set of all gauge equivalence classes of ASD connections $\boldsymbol{A}$ on $\boldsymbol{E}$ satisfying

$$\left\| F(\boldsymbol{A}) \right\|_{L^\infty(X)} \leqslant d. \tag{1}$$

Here $F(\boldsymbol{A})$ is the curvature of $\boldsymbol{A}$. $\mathcal{M}_d$ is equipped with the topology of $\mathcal{C}^\infty$-convergence on compact subsets: a sequence $[\boldsymbol{A}_n]$ $(n \geqslant 1)$ converges to $[\boldsymbol{A}]$ in $\mathcal{M}_d$ if there exists a sequence of gauge transformations g_n of $\boldsymbol{E}$ such that $g_n(\boldsymbol{A}_n)$ converges to $\boldsymbol{A}$ as $n \to \infty$ in the $\mathcal{C}^\infty$-topology over every compact subset in X. $\mathcal{M}_d$ becomes a compact metrizable space by the Uhlenbeck compactness [24,25]. Note that the condition (1) is a "L^∞-condition", and that the L^2-norm of $F(\boldsymbol{A})$ can be infinite. Hence the covering (topological) dimension of the moduli space $\mathcal{M}_d$ is infinite in general.

The additive Lie group $\mathbb{R}$ continuously acts on $\mathcal{M}_d$ by

$$\mathcal{M}_d \times \mathbb{R} \to \mathcal{M}_d, \quad \big([\boldsymbol{A}], s\big) \mapsto \big[s^* \boldsymbol{A}\big],$$

where s^* is the pull-back by $s : \boldsymbol{E} \to \boldsymbol{E}$. Then we can consider the mean dimension $\dim(\mathcal{M}_d : \mathbb{R})$. Intuitively,

$$\dim(\mathcal{M}_d : \mathbb{R}) = \frac{\dim \mathcal{M}_d}{\mathrm{vol}(\mathbb{R})}.$$

(This is ∞/∞ in general. The precise definition will be given in Section 2.) Our first main result is the following estimate on the mean dimension.

Theorem 1.1. *The mean dimension* $\dim(\mathcal{M}_d : \mathbb{R})$ *is finite*:

$$\dim(\mathcal{M}_d : \mathbb{R}) < +\infty.$$

Moreover, $\dim(\mathcal{M}_d : \mathbb{R}) \to +\infty$ *as* $d \to +\infty$.

For an ASD connection $\boldsymbol{A}$ on $\boldsymbol{E}$, we define $\rho(\boldsymbol{A})$ by setting

$$\rho(\boldsymbol{A}) := \lim_{T\to+\infty} \frac{1}{8\pi^2 T} \sup_{t\in\mathbb{R}} \int_{S^3\times[t,t+T]} \left|F(\boldsymbol{A})\right|^2 d\mathrm{vol}. \tag{2}$$

This limit always exists because we have the following subadditivity.

$$\sup_{t\in\mathbb{R}} \int_{S^3\times[t,t+T_1+T_2]} \left|F(\boldsymbol{A})\right|^2 d\mathrm{vol} \leqslant \sup_{t\in\mathbb{R}} \int_{S^3\times[t,t+T_1]} \left|F(\boldsymbol{A})\right|^2 d\mathrm{vol} + \sup_{t\in\mathbb{R}} \int_{S^3\times[t,t+T_2]} \left|F(\boldsymbol{A})\right|^2 d\mathrm{vol}.$$

$\rho(\boldsymbol{A})$ is translation invariant; for $s \in \mathbb{R}$, we have $\rho(s^*\boldsymbol{A}) = \rho(\boldsymbol{A})$, where $s^*\boldsymbol{A}$ is the pull-back of $\boldsymbol{A}$ by the map $s : \boldsymbol{E} = S^3 \times \mathbb{R} \times SU(2) \to \boldsymbol{E}$, $(\theta, t, u) \mapsto (\theta, t+s, u)$. We define $\rho(d)$ as the supremum of $\rho(\boldsymbol{A})$ over all ASD connections $\boldsymbol{A}$ on $\boldsymbol{E}$ satisfying $\|F(\boldsymbol{A})\|_{L^\infty} \leqslant d$.

Let $\boldsymbol{A}$ be an ASD connection on $\boldsymbol{E}$. We call $\boldsymbol{A}$ a periodic ASD connection if there exist $T > 0$, a principal $SU(2)$-bundle $\underline{\mathrm{E}}$ over $S^3 \times (\mathbb{R}/T\mathbb{Z})$, and an ASD connection $\underline{\mathrm{A}}$ on $\underline{\mathrm{E}}$ such that $(\boldsymbol{E}, \boldsymbol{A})$ is gauge equivalent to $(\pi^*(\underline{\mathrm{E}}), \pi^*(\underline{\mathrm{A}}))$ where $\pi : S^3 \times \mathbb{R} \to S^3 \times (\mathbb{R}/T\mathbb{Z})$ is the natural projection. (Here $S^3 \times (\mathbb{R}/T\mathbb{Z})$ is equipped with the metric induced by the covering map π.) Then we have

$$\rho(\boldsymbol{A}) = \frac{1}{8\pi^2 T} \int_{S^3\times[0,T]} \left|F(\boldsymbol{A})\right|^2 d\mathrm{vol} = \frac{c_2(\underline{\mathrm{E}})}{T}. \tag{3}$$

We define $\rho_{peri}(d)$ as the supremum of $\rho(\boldsymbol{A})$ over all periodic ASD connections $\boldsymbol{A}$ on $\boldsymbol{E}$ satisfying $\|F(\boldsymbol{A})\|_{L^\infty} < d$. (Note that we impose the strict inequality condition here.) If $d = 0$, then such an $\boldsymbol{A}$ does not exist. Hence we set $\rho_{peri}(0) := 0$. (If $d > 0$, then the product connection $\boldsymbol{A}$ is a periodic ASD connection satisfying $\|F(\boldsymbol{A})\|_{L^\infty} = 0 < d$.) Obviously we have $\rho_{peri}(d) \leqslant \rho(d)$. Our second main result is the following estimates on the "local mean dimensions".

Theorem 1.2. *For any* $[\boldsymbol{A}] \in \mathcal{M}_d$,

$$\dim_{[\boldsymbol{A}]}(\mathcal{M}_d : \mathbb{R}) \leqslant 8\rho(\boldsymbol{A}).$$

Moreover, if $\boldsymbol{A}$ *is a periodic ASD connection satisfying* $\|F(\boldsymbol{A})\|_{L^\infty} < d$*, then*

$$\dim_{[\boldsymbol{A}]}(\mathcal{M}_d : \mathbb{R}) = 8\rho(\boldsymbol{A}).$$

Therefore,

$$8\rho_{peri}(d) \leqslant \dim_{loc}(\mathcal{M}_d : \mathbb{R}) \leqslant 8\rho(d).$$

Here $\dim_{[\boldsymbol{A}]}(\mathcal{M}_d : \mathbb{R})$ is the "local mean dimension" of $\mathcal{M}_d$ at $[\boldsymbol{A}]$, and $\dim_{loc}(\mathcal{M}_d : \mathbb{R}) := \sup_{[\boldsymbol{A}] \in \mathcal{M}_d} \dim_{[\boldsymbol{A}]}(\mathcal{M}_d : \mathbb{R})$ is the "local mean dimension" of $\mathcal{M}_d$. These notions will be defined in Section 2.2.

Note that

$$\lim_{d \to +\infty} \rho_{peri}(d) = +\infty.$$

This obviously follows from the fact that for any integer $n \geqslant 0$ there exists an ASD connection on $S^3 \times (\mathbb{R}/\mathbb{Z})$ whose second Chern number is equal to n. This is a special case of the famous theorem of Taubes [18]. (Note that the intersection form of $S^3 \times S^1$ is zero.) We have $\dim(\mathcal{M}_d : \mathbb{R}) \geqslant \dim_{loc}(\mathcal{M}_d : \mathbb{R})$ (see (5) in Section 2.2). Hence the statement that $\dim(\mathcal{M}_d : \mathbb{R}) \to +\infty$ $(d \to +\infty)$ in Theorem 1.1 follows from the inequality $\dim_{loc}(\mathcal{M}_d : \mathbb{R}) \geqslant 8\rho_{peri}(d)$ in Theorem 1.2.

Remark 1.3. All principal $SU(2)$-bundles over $S^3 \times \mathbb{R}$ are gauge equivalent to the product bundle $\boldsymbol{E}$. Hence the moduli space $\mathcal{M}_d$ is equal to the space of all gauge equivalence classes $[E, A]$ such that E is a principal $SU(2)$-bundle over X, and that A is an ASD connection on E satisfying $|F(A)| \leqslant d$. We have $[E_1, A_1] = [E_2, A_2]$ if and only if there exists a bundle map $g : E_1 \to E_2$ satisfying $g(A_1) = A_2$. In this description, the topology of $\mathcal{M}_d$ is described as follows. A sequence $[E_n, A_n]$ $(n \geqslant 1)$ in $\mathcal{M}_d$ converges to $[E, A]$ if and only if there exist gauge transformations $g_n : E_n \to E$ $(n \gg 1)$ such that $g_n(A_n)$ converges to A as $n \to \infty$ in $\mathcal{C}^\infty$ over every compact subset in X.

Remark 1.4. An ASD connection satisfying the condition (1) is a Yang–Mills analogue of a "Brody curve" (cf. Brody [3]) in the entire holomorphic curve theory (Nevanlinna theory). It is widely known that there exist several similarities between the Yang–Mills gauge theory and the theory of (pseudo-)holomorphic curves (e.g. Donaldson invariant vs. Gromov–Witten invariant). On the holomorphic curve side, several researchers in the Nevanlinna theory have systematically studied the value distributions of holomorphic curves (of infinite energy) from the complex plane $\mathbb{C}$. They have found several deep structures of such infinite energy holomorphic curves. Therefore the authors hope that infinite energy ASD connections also have deep structures.

The rough ideas of the proofs of the main theorems are as follows. (For more about the outline of the proofs, see Section 3.) The upper bounds on the (local) mean dimension are proved by using the Runge-type approximation of ASD connections (originally due to Donaldson [5]). This "instanton approximation" technique gives a method to approximate infinite energy ASD connections by finite energy ones (instantons). Then we can construct "finite dimensional approximations" of $\mathcal{M}_d$ by moduli spaces of instantons. This gives an upper bound on $\dim(\mathcal{M}_d : \mathbb{R})$. The lower bound on the local mean dimension is proved by constructing an infinite dimensional deformation theory of periodic ASD connections. This method is a Yang–Mills analogue of the deformation theory of "elliptic Brody curves" developed in Tsukamoto [23].

A big technical difficulty in the study of $\mathcal{M}_d$ comes from the point that ASD equation is not elliptic. When we study the Yang–Mills theory over compact manifolds, this point can be easily overcome by using the Coulomb gauge. But in our situation (perhaps) there is no such good way to recover the ellipticity. So we will consider some "partial gauge fixings" in this paper. In the proof of the upper bound, we will consider the Coulomb gauge over S^3 instead of $S^3 \times \mathbb{R}$ (see Propositions 7.1 and 7.2). In the proof of the lower bound, we will consider the Coulomb gauge over $S^3 \times \mathbb{R}$, but it is less powerful and more technical than the usual Coulomb gauges over compact manifolds (see Proposition 9.6).

Organization of the paper: In Section 2 we review the definition of mean dimension and define local mean dimension. In Section 3 we explain the outline of the proofs of Theorems 1.1 and 1.2. Sections 4, 5 and 7 are preparations for the proof of the upper bounds on the (local) mean dimension. Section 6 is a preparation for both proofs of the upper and lower bounds. In Section 8 we prove the upper bounds. Section 9 is a preparation for the proof of the lower bound. In Section 10 we develop the deformation theory of periodic ASD connections and prove the lower bound on the local mean dimension. In Appendix A we prepare some basic results on the Green kernel of $\Delta + a$ $(a > 0)$.

2. Mean dimension and local mean dimension

2.1. Review of mean dimension

We review the definitions and basic properties of mean dimension in this subsection. For the detail, see Gromov [14] and Lindenstrauss and Weiss [16]. For some related works, see Lindenstrauss [15] and Gournay [10–13].

Let (X, d) be a compact metric space, Y be a topological space, and $f : X \to Y$ be a continuous map. For $\varepsilon > 0$, f is called an ε-embedding if we have $\operatorname{Diam} f^{-1}(y) \leqslant \varepsilon$ for all $y \in Y$. We define $\operatorname{Widim}_\varepsilon(X, d)$ as the minimum integer $n \geqslant 0$ such that there exist a polyhedron P of dimension n and an ε-embedding $f : X \to P$. We have

$$\lim_{\varepsilon \to 0} \operatorname{Widim}_\varepsilon(X, d) = \dim X,$$

where $\dim X$ denotes the topological covering dimension of X. For example, consider $[0, 1] \times [0, \varepsilon]$ with the Euclidean distance. Then the natural projection $\pi : [0, 1] \times [0, \varepsilon] \to [0, 1]$ is an ε-embedding. Hence $\operatorname{Widim}_\varepsilon([0, 1] \times [0, \varepsilon], \text{Euclidean}) \leqslant 1$. The following is given in Gromov [14, p. 333]. (For the detailed proof, see also Gournay [12, Lemma 2.5] and Tsukamoto [23, Appendix].)

Lemma 2.1. *Let $(V, \|\cdot\|)$ be a finite dimensional normed linear space over $\mathbb{R}$. Let $B_r(V)$ be the closed ball of radius $r > 0$ in V. Then*

$$\operatorname{Widim}_\varepsilon\big(B_r(V), \|\cdot\|\big) = \dim V \quad (\varepsilon < r).$$

$\operatorname{Widim}_\varepsilon(X, d)$ satisfies the following subadditivity. (The proof is obvious.)

Lemma 2.2. *For compact metric spaces (X, d_X), (Y, d_Y), we set $(X, d_X) \times (Y, d_Y) := (X \times Y, d_{X\times Y})$ with $d_{X\times Y}((x_1, y_1), (x_2, y_2)) := \max(d_X(x_1, x_2), d_Y(y_1, y_2))$. Then we have*

$$\operatorname{Widim}_\varepsilon\big((X, d_X) \times (Y, d_Y)\big) \leqslant \operatorname{Widim}_\varepsilon(X, d_X) + \operatorname{Widim}_\varepsilon(Y, d_Y).$$

The following will be used in Section 8.1

Lemma 2.3. *Let* (X, d) *be a compact metric space and suppose* $X = X_1 \cup X_2$ *with closed sets* X_1 *and* X_2*. Then*

$$\mathrm{Widim}_\varepsilon(X, d) \leqslant \mathrm{Widim}_\varepsilon(X_1, d) + \mathrm{Widim}_\varepsilon(X_2, d) + 1.$$

In general, if $X = X_1 \cup X_2 \cup \cdots \cup X_n$ *(*X_i*: closed), then*

$$\mathrm{Widim}_\varepsilon(X, d) \leqslant \sum_{i=1}^{n} \mathrm{Widim}_\varepsilon(X_i, d) + n - 1.$$

Proof. There exist a finite polyhedron P_i $(i = 1, 2)$ with $\dim P_i = \mathrm{Widim}_\varepsilon(X_i, d)$ and an ε-embedding $f_i : (X_i, d) \to P_i$. Let $P_1 * P_2 = \{tx \oplus (1-t)y \mid x \in X_1,\ y \in X_2,\ 0 \leqslant t \leqslant 1\}$ be the join of P_1 and P_2. ($P_1 * P_2 = [0,1] \times P_1 \times P_2/\sim$, where $(0, x, y) \sim (0, x', y)$ for any $x, x' \in X$ and $(1, x, y) \sim (1, x, y')$ for any $y, y' \in Y$. $tx \oplus (1-t)y$ is the equivalence class of (t, x, y).) $P_1 * P_2$ is a finite polyhedron of dimension $\mathrm{Widim}_\varepsilon(X_1, d) + \mathrm{Widim}_\varepsilon(X_2, d) + 1$. Since a finite polyhedron is ANR, there exists an open set $U_i \supset X_i$ over which the map f_i continuously extends. Let ρ be a cut-off function such that $0 \leqslant \rho \leqslant 1$, $\mathrm{supp}\,\rho \subset U_1$ and $\rho(x) = 1$ if and only if $x \in X_1$. Then $\mathrm{supp}(1-\rho) = \overline{X \setminus X_1} \subset X_2 \subset U_2$. We define a continuous map $F : X \to P_1 * P_2$ by setting $F(x) := \rho(x) f_1(x) \oplus (1-\rho(x)) f_2(x)$. F becomes an ε-embedding; Suppose $F(x) = F(y)$. If $\rho(x) = \rho(y) = 1$, then $x, y \in X_1$ and $f_1(x) = f_1(y)$. Then $d(x, y) \leqslant \varepsilon$. If $\rho(x) = \rho(y) < 1$, then $x, y \in X_2$ and $f_2(x) = f_2(y)$. Then $d(x, y) \leqslant \varepsilon$. Thus $\mathrm{Widim}_\varepsilon(X, d) \leqslant \dim P_1 * P_2 = \mathrm{Widim}_\varepsilon(X_1, d) + \mathrm{Widim}_\varepsilon(X_2, d) + 1$. □

Let Γ be a locally compact Hausdorff unimodular group with a bi-invariant Haar measure $|\cdot|$. We suppose that Γ is endowed with a left-invariant proper distance. (Properness means that every bounded closed set is compact.) In Section 2 we always assume that Γ satisfies these conditions. When Γ is discrete, we always assume that the Haar measure $|\cdot|$ is the counting measure. (That is, $|\Omega|$ is equal to the cardinality of Ω.)

Let $\Omega \subset \Gamma$ be a subset and $r > 0$. The r-boundary $\partial_r \Omega$ is the set of points $\gamma \in \Gamma$ such that the closed r-ball $B_r(\gamma)$ centered at γ has non-empty intersection with both Ω and $\Gamma \setminus \Omega$. A sequence of bounded Borel sets $\{\Omega_n\}_{n \geqslant 1}$ in Γ is called amenable (or Følner) if for any $r > 0$ the following is satisfied:

$$\lim_{n \to \infty} |\partial_r \Omega_n| / |\Omega_n| = 0.$$

Γ is called amenable group if it admits an amenable sequence.

Example 2.4. $\Gamma = \mathbb{Z}$ with the counting measure $|\cdot|$ and the standard distance $|x - y|$. Then the sequence of sets $\{0, 1, 2, \ldots, n\}$ $(n \geqslant 1)$ is amenable. The sequence of sets $\{-n, -n+1, \ldots, -1, 0, 1, \ldots, n-1, n\}$ $(n \geqslant 1)$ is also amenable.

Example 2.5. $\Gamma = \mathbb{R}$ with the Lebesgue measure $|\cdot|$ and the standard distance $|x - y|$. In this paper we always assume that $\mathbb{R}$ has these standard measure and distance. Then the sequence of sets $\{x \in \mathbb{R} \mid 0 \leqslant x \leqslant n\}$ $(n \geqslant 1)$ is amenable. The sequence of sets $\{x \in \mathbb{R} \mid -n \leqslant x \leqslant n\}$ $(n \geqslant 1)$ is also amenable.

We need the following "Ornstein–Weiss Lemma" ([14, pp. 336–338] and [16, Appendix]).

Lemma 2.6. *Suppose Γ is amenable. Let $h : \{\text{bounded sets in } \Gamma\} \to \mathbb{R}_{\geqslant 0}$ be a map satisfying the following conditions.*

(i) *If $\Omega_1 \subset \Omega_2$, then $h(\Omega_1) \leqslant h(\Omega_2)$.*
(ii) $h(\Omega_1 \cup \Omega_2) \leqslant h(\Omega_1) + h(\Omega_2)$.
(iii) *For any $\gamma \in \Gamma$ and any bounded set $\Omega \subset \Gamma$, $h(\gamma\Omega) = h(\Omega)$. Here $\gamma\Omega := \{\gamma x \in \Gamma \mid x \in \Omega\}$.*

Then for any amenable sequence $\{\Omega_n\}_{n\geqslant 1}$ in Γ, the limit $\lim_{n\to\infty} h(\Omega_n)/|\Omega_n|$ always exists and is independent of the choice of an amenable sequence $\{\Omega_n\}_{n\geqslant 1}$.

Let (X, d) be a compact metric space with a continuous action of Γ. We suppose that the action is a right-action. For a subset $\Omega \subset \Gamma$, we define a new distance $d_\Omega(\cdot,\cdot)$ on X by

$$d_\Omega(x, y) := \sup_{\gamma\in\Omega} d(x.\gamma, y.\gamma) \quad (x, y \in X).$$

Lemma 2.7. *The map $\Omega \mapsto \mathrm{Widim}_\varepsilon(X, d_\Omega)$ satisfies the conditions* (i), (ii), (iii) *in Lemma* 2.6.

Proof. If $\Omega_1 \subset \Omega_2$, then the identity map $(X, d_{\Omega_1}) \to (X, d_{\Omega_2})$ is distance non-decreasing. Hence $\mathrm{Widim}_\varepsilon(X, d_{\Omega_1}) \leqslant \mathrm{Widim}_\varepsilon(X, d_{\Omega_2})$. The map $(X, d_{\Omega_1\cup\Omega_2}) \to (X, d_{\Omega_1}) \times (X, d_{\Omega_2})$, $x \to (x, x)$, is distance preserving. Hence, by using Lemma 2.2, $\mathrm{Widim}_\varepsilon(X, d_{\Omega_1\cup\Omega_2}) \leqslant \mathrm{Widim}_\varepsilon(X, d_{\Omega_1}) + \mathrm{Widim}_\varepsilon(X, d_{\Omega_2})$. The map $(X, d_{\gamma\Omega}) \to (X, d_\Omega)$, $x \mapsto x.\gamma$, is an isometry. Hence $\mathrm{Widim}_\varepsilon(X, d_{\gamma\Omega}) = \mathrm{Widim}_\varepsilon(X, d_\Omega)$. □

Suppose that Γ is an amenable group and that an amenable sequence $\{\Omega_n\}_{n\geqslant 1}$ is given. For $\varepsilon > 0$, we set

$$\mathrm{Widim}_\varepsilon(X : \Gamma) := \lim_{n\to\infty} \frac{1}{|\Omega_n|} \mathrm{Widim}_\varepsilon(X, d_{\Omega_n}).$$

This limit exists and is independent of the choice of an amenable sequence $\{\Omega_n\}_{n\geqslant 1}$. The value of $\mathrm{Widim}_\varepsilon(X : \Gamma)$ depends on the distance d. Hence, strictly speaking, we should use the notation $\mathrm{Widim}_\varepsilon((X, d) : \Gamma)$. But we use the above notation for simplicity. We define $\dim(X : \Gamma)$ (the mean dimension of (X, Γ)) by

$$\dim(X : \Gamma) := \lim_{\varepsilon\to 0} \mathrm{Widim}_\varepsilon(X : \Gamma).$$

This becomes a topological invariant, i.e., the value of $\dim(X : \Gamma)$ does not depend on the choice of a distance d on X compatible with the topology of X.

Example 2.8. Let Γ be a finitely generated (discrete) amenable group. Let $B \subset \mathbb{R}^N$ be the closed ball. Γ acts on B^Γ by the shift. Then

$$\dim\bigl(B^\Gamma : \Gamma\bigr) = N.$$

For the proof, see Lindenstrauss and Weiss [16, Propositions 3.1, 3.3] or Tsukamoto [22, Example 9.6].

2.2. *Local mean dimension*

Let (X, d) be a compact metric space. The usual topological dimension $\dim X$ is a "local notion" as follows: For each point $p \in X$, we define the "local dimension" $\dim_p X$ at p by $\dim_p X := \lim_{r\to 0} \dim B_r(p)$. (Here $B_r(p)$ is the closed r-ball centered at p.) Then we have $\dim X = \sup_{p\in X} \dim_p X$. The authors don't know whether a similar description of the mean dimension is possible or not. Instead, in this subsection we will introduce a new notion "local mean dimension" (cf. [14, p. 406, the difficulty 1]).

Suppose that an amenable group Γ continuously acts on X from the right. ((X, d) is a compact metric space.) Let $Y \subset X$ be a closed subset. Then the map $\Omega \mapsto \sup_{\gamma\in\Gamma} \mathrm{Widim}_\varepsilon(Y, d_{\gamma\Omega})$ satisfies the conditions in Lemma 2.6. Hence we can set

$$\mathrm{Widim}_\varepsilon(Y \subset X : \Gamma) := \lim_{n\to\infty}\left(\frac{1}{|\Omega_n|}\sup_{\gamma\in\Gamma} \mathrm{Widim}_\varepsilon(Y, d_{\gamma\Omega_n})\right),$$

where $\{\Omega_n\}_{n\geqslant 1}$ is an amenable sequence. We define

$$\dim(Y \subset X : \Gamma) := \lim_{\varepsilon\to 0} \mathrm{Widim}_\varepsilon(Y \subset X : \Gamma).$$

This does not depend on the choice of a distance on X compatible with the topology of X. If Y_1 and Y_2 are closed subsets in X with $Y_1 \subset Y_2$, then

$$\dim(Y_1 \subset X : \Gamma) \leqslant \dim(Y_2 \subset X : \Gamma).$$

If $Y \subset X$ is a Γ-invariant closed subset, then $\mathrm{Widim}_\varepsilon(Y, d_{\gamma\Omega_n}) = \mathrm{Widim}_\varepsilon(Y, d_{\Omega_n})$ because $(Y, d_{\gamma\Omega_n}) \to (Y, d_{\Omega_n})$, $x \to x.\gamma$, is an isometry. Hence

$$\dim(Y \subset X : \Gamma) = \dim(Y : \Gamma),$$

where the right-hand side is the ordinary mean dimension of (Y, Γ). In particular, $\dim(X \subset X : \Gamma) = \dim(X : \Gamma)$, and hence for any closed subset $Y \subset X$ (not necessarily Γ-invariant)

$$\dim(Y \subset X : \Gamma) \leqslant \dim(X \subset X : \Gamma) = \dim(X : \Gamma).$$

Let X_1 and X_2 be compact metric spaces with continuous Γ-actions. Let $Y_1 \subset X_1$ and $Y_2 \subset X_2$ be closed subsets. If there exists a Γ-equivariant topological embedding $f : X_1 \to X_2$ satisfying $f(Y_1) \subset Y_2$, then

$$\dim(Y_1 \subset X_1 : \Gamma) \leqslant \dim(Y_2 \subset X_2 : \Gamma). \tag{4}$$

For each point $p \in X$ and $r > 0$ we define $B_r(p)_\Gamma$ (or $B_r(p; X)_\Gamma$) as the closed r-ball centered at p with respect to the distance $d_\Gamma(\cdot,\cdot)$:

$$B_r(p)_\Gamma := \left\{x \in X \,\middle|\, d_\Gamma(x, p) \leqslant r\right\}.$$

Note that $d_\Gamma(x,p) \leqslant r \Leftrightarrow d(x.\gamma, p.\gamma) \leqslant r$ for all $\gamma \in \Gamma$. $B_r(p)_\Gamma$ is a closed set in X. We define the local mean dimension of X at p by

$$\dim_p(X:\Gamma) := \lim_{r\to 0} \dim\bigl(B_r(p)_\Gamma \subset X:\Gamma\bigr).$$

This is independent of the choice of a distance compatible with the topology of X. We define the local mean dimension of X by

$$\dim_{loc}(X:\Gamma) := \sup_{p\in X} \dim_p(X:\Gamma).$$

Obviously we have

$$\dim_{loc}(X:\Gamma) \leqslant \dim(X:\Gamma). \tag{5}$$

We will use the following formula in Section 8.2. Since $(B_r(p)_\Gamma).\gamma = B_r(p.\gamma)_\Gamma$, we have

$$\mathrm{Widim}_\varepsilon\bigl(B_r(p)_\Gamma, d_{\gamma\Omega}\bigr) = \mathrm{Widim}_\varepsilon\bigl(\bigl(B_r(p)_\Gamma\bigr).\gamma, d_\Omega\bigr) = \mathrm{Widim}_\varepsilon\bigl(B_r(p.\gamma)_\Gamma, d_\Omega\bigr),$$

and hence

$$\mathrm{Widim}_\varepsilon\bigl(B_r(p)_\Gamma \subset X:\Gamma\bigr) = \lim_{n\to\infty}\left(\frac{1}{|\Omega_n|}\sup_{\gamma\in\Gamma} \mathrm{Widim}_\varepsilon\bigl(B_r(p.\gamma)_\Gamma, d_{\Omega_n}\bigr)\right). \tag{6}$$

Let X, Y be compact metric spaces with continuous Γ-actions. If there exists a Γ-equivariant topological embedding $f: X \to Y$, then, from (4), for all $p \in X$

$$\dim_p(X:\Gamma) \leqslant \dim_{f(p)}(Y:\Gamma).$$

Example 2.9. Let Γ be a finitely generated discrete amenable group, and $B \subset \mathbb{R}^N$ be the closed ball centered at the origin. Then we have

$$\dim_{\mathbf{0}}\bigl(B^\Gamma:\Gamma\bigr) = \dim_{loc}\bigl(B^\Gamma:\Gamma\bigr) = \dim\bigl(B^\Gamma:\Gamma\bigr) = N,$$

where $\mathbf{0} = (x_\gamma)_{\gamma\in\Gamma}$ with $x_\gamma = 0$ for all $\gamma \in \Gamma$.

Proof. Fix a distance on B^Γ. Then it is easy to see that for any $r > 0$ there exists $s > 0$ such that $B_s^\Gamma \subset B_r(\mathbf{0})_\Gamma$, where B_s is the s-ball in $\mathbb{R}^N$. Then

$$N = \dim\bigl(B_s^\Gamma:\Gamma\bigr) \leqslant \dim\bigl(B_r(\mathbf{0})_\Gamma \subset B^\Gamma:\Gamma\bigr) \leqslant \dim\bigl(B^\Gamma:\Gamma\bigr) = N.$$

Hence $\dim_{\mathbf{0}}(B^\Gamma:\Gamma) = N$. □

Remark 2.10. We have so far supposed that Γ has a bi-invariant Haar measure and a proper left-invariant distance. The values of mean dimension and local mean dimension depend on the choice of a Haar measure. But they are independent of the choice of a proper left-invariant distance on Γ. (We need the *existence* of a proper left-invariant distance on Γ for defining the notion "amenable sequence". But this notion is independent of the choice of a proper left-invariant distance on Γ.)

2.3. The case of $\Gamma = \mathbb{R}$

Let $\Gamma = \mathbb{R}$ with the Lebesgue measure and the standard distance. Suppose that $\mathbb{R}$ continuously acts on a compact metric space (X, d). For $T > 0$, consider the discrete subgroup $T\mathbb{Z} := \{Tn \in \mathbb{R} \mid n \in \mathbb{Z}\}$ in $\mathbb{R}$. $T\mathbb{Z}$ also acts on X. We want to compare the mean dimensions of $(X, \mathbb{R})$ and $(X, T\mathbb{Z})$. Here $T\mathbb{Z}$ is equipped with the counting measure.

Proposition 2.11.

$$\dim(X : T\mathbb{Z}) = T \dim(X : \mathbb{R}).$$

This result is given in [14, *p.* 329] *and* [16, *Proposition* 2.7]. *For any point $p \in X$,*

$$\dim_p(X : T\mathbb{Z}) = T \dim_p(X : \mathbb{R}).$$

In particular, $\dim_{loc}(X : T\mathbb{Z}) = T \dim_{loc}(X : \mathbb{R})$.

Proof. Set $\Omega_n := \{\gamma \in \mathbb{R} \mid 0 \leqslant \gamma < Tn\}$ and $\Omega'_n := \Omega_n \cap T\mathbb{Z}$. $\{\Omega_n\}_{n\geqslant 1}$ is an amenable sequence for $\mathbb{R}$, and $\{\Omega'_n\}_{n\geqslant 1}$ is an amenable sequence for $T\mathbb{Z}$. Let $Y \subset X$ be a closed subset. For $\gamma \in T\mathbb{Z}$, $d_{\gamma+\Omega'_n}(\cdot,\cdot) \leqslant d_{\gamma+\Omega_n}(\cdot,\cdot)$. Hence, for any $\varepsilon > 0$, $\mathrm{Widim}_\varepsilon(Y, d_{\gamma+\Omega'_n}) \leqslant \mathrm{Widim}_\varepsilon(Y, d_{\gamma+\Omega_n})$. Therefore

$$\dim(Y \subset X : T\mathbb{Z}) \leqslant T \dim(Y \subset X : \mathbb{R}).$$

For any $\varepsilon > 0$ there exists $\delta > 0$ such that if $d(x, y) \leqslant \delta$ then $d_{[0,2T)}(x, y) \leqslant \varepsilon$. Let $a \in \mathbb{R}$ and set $k := [a]$ (the maximum integer $\leqslant a$). If $d_{kT+\Omega'_n}(x, y) \leqslant \delta$, then $d_{aT+\Omega_n}(x, y) \leqslant \varepsilon$. Hence $\mathrm{Widim}_\varepsilon(Y, d_{aT+\Omega_n}) \leqslant \mathrm{Widim}_\delta(Y, d_{kT+\Omega'_n})$. This implies

$$\sup_{\gamma\in\mathbb{R}} \mathrm{Widim}_\varepsilon(Y, d_{\gamma+\Omega_n}) \leqslant \sup_{\gamma\in T\mathbb{Z}} \mathrm{Widim}_\delta(Y, d_{\gamma+\Omega'_n}).$$

Therefore $T \dim(Y \subset X : \mathbb{R}) \leqslant \dim(Y \subset X : T\mathbb{Z})$. Thus

$$T \dim(Y \subset X : \mathbb{R}) = \dim(Y \subset X : T\mathbb{Z}). \tag{7}$$

In particular, if $Y = X$, then $\dim(X : T\mathbb{Z}) = T \dim(X : \mathbb{R})$.

For any $r > 0$ there exists $r' > 0$ such that if $d(x, y) \leqslant r'$ then $d_{[0,T)}(x, y) \leqslant r$. Then if $d_{T\mathbb{Z}}(x, y) \leqslant r'$, we have $d_{\mathbb{R}}(x, y) \leqslant r$. Hence $B_{r'}(p)_{T\mathbb{Z}} \subset B_r(p)_{\mathbb{R}} \subset B_r(p)_{T\mathbb{Z}}$. Therefore, by using the above (7),

$$\begin{aligned}\dim\bigl(B_{r'}(p)_{T\mathbb{Z}} \subset X : T\mathbb{Z}\bigr) &= T \dim\bigl(B_{r'}(p)_{T\mathbb{Z}} \subset X : \mathbb{R}\bigr) \leqslant T \dim\bigl(B_r(p)_{\mathbb{R}} \subset X : \mathbb{R}\bigr)\\ &\leqslant T \dim\bigl(B_r(p)_{T\mathbb{Z}} \subset X : \mathbb{R}\bigr) = \dim\bigl(B_r(p)_{T\mathbb{Z}} \subset X : T\mathbb{Z}\bigr).\end{aligned}$$

Thus $\dim_p(X : T\mathbb{Z}) = T \dim_p(X : \mathbb{R})$. □

3. Outline of the proofs of the main theorems

The ideas of the proofs of Theorems 1.1 and 1.2 are simple. But the completion of the proofs needs lengthy technical arguments. So we want to describe the outline of the proofs in this section. Here we don't pursue the accuracy of the arguments for simplicity of the explanation. Some of the arguments will be replaced with different ones in the later sections.

First we explain how to get the upper bound on the mean dimension of $\mathcal{M}_d$. We define a distance on $\mathcal{M}_d$ by setting

$$\mathrm{dist}\big([\boldsymbol{A}],[\boldsymbol{B}]\big) := \inf_{g:\boldsymbol{E}\to\boldsymbol{E}}\left\{\sum_{n\geqslant 1} 2^{-n}\frac{\|g(\boldsymbol{A})-\boldsymbol{B}\|_{L^\infty(|t|\leqslant n)}}{1+\|g(\boldsymbol{A})-\boldsymbol{B}\|_{L^\infty(|t|\leqslant n)}}\right\},$$

where g runs over all gauge transformations of $\boldsymbol{E}$, and $|t|\leqslant n$ means the region $\{(\theta,t)\in S^3\times\mathbb{R} \mid |t|\leqslant n\}$. For $R=1,2,3,\dots$, we define $\Omega_R\subset\mathbb{R}$ by $\Omega_R := \{s\in\mathbb{R}\mid -R\leqslant s\leqslant R\}$. $\{\Omega_R\}_{R\geqslant 1}$ is an amenable sequence in $\mathbb{R}$.

Let $\varepsilon>0$ be a positive number, and define a positive integer $L=L(\varepsilon)$ so that

$$\sum_{n>L} 2^{-n} < \varepsilon/2. \tag{8}$$

Let $D=D(\varepsilon)$ be a large positive number which depends on ε but is independent of R, and set $T := R+L+D$. (D is chosen so that the condition (9) below is satisfied. Here we don't explain how to define D precisely.)

For $c\geqslant 0$ we define $M(c)$ as the space of the gauge equivalence classes $[A]$ where A is an ASD connection on $\boldsymbol{E}$ satisfying

$$\frac{1}{8\pi^2}\int_X |F_A|^2\,d\mathrm{vol}\leqslant c.$$

The index theorem gives the estimate:

$$\dim M(c)\leqslant 8c.$$

We want to construct an ε-embedding from $(\mathcal{M}_d,\mathrm{dist}_{\Omega_R})$ to $M(c)$ for an appropriate $c\geqslant 0$.

Let $\boldsymbol{A}$ be an ASD connection on $\boldsymbol{E}$ with $[\boldsymbol{A}]\in\mathcal{M}_d$. We "cut-off" $\boldsymbol{A}$ over the region $T<|t|<T+1$ and construct a new connection $\boldsymbol{A}'$ satisfying the following conditions. $\boldsymbol{A}'$ is a (not necessarily ASD) connection on $\boldsymbol{E}$ satisfying $\boldsymbol{A}'|_{|t|\leqslant T}=\boldsymbol{A}|_{|t|\leqslant T}$, $F(\boldsymbol{A}')=0$ over $|t|\geqslant T+1$, and

$$\frac{1}{8\pi^2}\int_X tr\big(F(\boldsymbol{A}')^2\big)\leqslant\frac{1}{8\pi^2}\int_{|t|\leqslant T}\big|F(\boldsymbol{A})\big|^2\,d\mathrm{vol}+\mathrm{const}\leqslant\frac{2T\,d^2\mathrm{vol}(S^3)}{8\pi^2}+\mathrm{const},$$

where const is a positive constant independent of ε and R. Next we "perturb" $\boldsymbol{A}'$ and construct an ASD connection $\boldsymbol{A}''$ on $\boldsymbol{E}$ satisfying

$$\left|A - A''\right| = \left|A' - A''\right| \leqslant \varepsilon/4 \quad \big(|t| \leqslant T - D = R + L\big), \tag{9}$$
$$\frac{1}{8\pi^2}\int_X \left|F\big(A''\big)\right|^2 d\mathrm{vol} = \frac{1}{8\pi^2}\int_X tr\big(F\big(A'\big)^2\big) \leqslant \frac{2T\,d^2\mathrm{vol}(S^3)}{8\pi^2} + \mathrm{const}.$$

Then we can define the map

$$\mathcal{M}_d \to M\left(\frac{2T\,d^2\mathrm{vol}(S^3)}{8\pi^2} + \mathrm{const}\right), \quad [A] \mapsto \big[A''\big].$$

The conditions (8) and (9) imply that this map is an ε-embedding with respect to the distance dist_{Ω_R}. Hence

$$\mathrm{Widim}_\varepsilon(\mathcal{M}_d, \mathrm{dist}_{\Omega_R}) \leqslant \dim M\left(\frac{2T\,d^2\mathrm{vol}(S^3)}{8\pi^2} + \mathrm{const}\right) \leqslant \frac{2T\,d^2\mathrm{vol}(S^3)}{\pi^2} + 8\cdot\mathrm{const}.$$

(Caution! This estimate will *not* be proved in this paper. The above argument contains a gap.) Recall $T = R + L + D$. Since L, D and const are independent of R, we get

$$\mathrm{Widim}_\varepsilon(\mathcal{M}_d : \mathbb{R}) = \lim_{R\to\infty} \frac{\mathrm{Widim}_\varepsilon(\mathcal{M}_d, \mathrm{dist}_{\Omega_R})}{2R} \leqslant \frac{d^2\mathrm{vol}(S^3)}{\pi^2}.$$

Hence we get

$$\dim(\mathcal{M}_d : \mathbb{R}) \leqslant \frac{d^2\mathrm{vol}(S^3)}{\pi^2} < +\infty. \tag{10}$$

This is the outline of the proof of the upper bound on the mean dimension. (The upper bound on the local mean dimension can be proved by investigating the above procedure more precisely.) Strictly speaking, the above argument contains a gap. Actually we have not so far succeeded to prove the estimate $\dim(\mathcal{M}_d : \mathbb{R}) \leqslant d^2\mathrm{vol}(S^3)/\pi^2$. In this paper we prove only $\dim(\mathcal{M}_d : \mathbb{R}) < +\infty$. A problem occurs in the cut-off construction. Indeed (we think that) there exists no canonical way to cut-off connections compatible with the gauge symmetry. Therefore we cannot define a suitable cut-off construction all over $\mathcal{M}_d$. Instead we will decompose $\mathcal{M}_d$ as $\mathcal{M}_d = \bigcup_{0\leqslant i,j\leqslant N} \mathcal{M}_{d,T}(i,j)$ (N is independent of ε and R) and define a cut-off construction for each piece $\mathcal{M}_{d,T}(i,j)$ independently. Then we will get an upper bound worse than (10) (cf. Lemma 2.3). We study the cut-off construction (the procedure $[A] \mapsto [A']$) in Section 7. In Sections 4 and 5 we study the perturbation procedure ($A' \mapsto A''$). The upper bounds on the (local) mean dimension are proved in Section 8.

Next we explain how to prove the lower bound on the local mean dimension. Let $T > 0$, $\underline{E}$ be a principal $SU(2)$-bundle over $S^3 \times (\mathbb{R}/T\mathbb{Z})$, and $\underline{A}$ be a non-flat ASD connection on $\underline{E}$ satisfying $|F(\underline{A})| < d$.

Let $\pi : S^3 \times \mathbb{R} \to S^3 \times (\mathbb{R}/T\mathbb{Z})$ be the natural projection, and set $E := \pi^*(\underline{E})$ and $A := \pi^*(\underline{A})$. We define the infinite dimensional Banach space H^1_A by

$$H^1_A := \left\{a \in \Omega^1(\mathrm{ad}\,E) \;\middle|\; \big(d_A^* + d_A^+\big)a = 0,\ \|a\|_{L^\infty} < \infty\right\}.$$

There exists a natural $T\mathbb{Z}$-action on H^1_A. Let $r > 0$ be a sufficiently small number. For each $a \in H^1_A$ with $\|a\|_{L^\infty} \leqslant r$ we can construct $\tilde{a} \in \Omega^1(\operatorname{ad} E)$ (a small perturbation of a) satisfying $F^+(A+\tilde{a}) = 0$ and $|F(A+\tilde{a})| \leqslant d$. If $a = 0$, then $\tilde{a} = 0$.

For $n \geqslant 1$, let $\pi_n : S^3 \times (\mathbb{R}/nT\mathbb{Z}) \to S^3 \times (\mathbb{R}/T\mathbb{Z})$ be the natural projection, and set $E_n := \pi_n^*(\underline{\mathrm{E}})$ and $A_n := \pi_n^*(\underline{\mathrm{A}})$. We define $H^1_{A_n}$ as the space of $a \in \Omega(\operatorname{ad} E_n)$ satisfying $(d^*_{A_n} + d^+_{A_n})a = 0$. We can identify $H^1_{A_n}$ with the subspace of H^1_A consisting of $nT\mathbb{Z}$-invariant elements. The index theorem gives

$$\dim H^1_{A_n} = 8nc_2(\underline{\mathrm{E}}).$$

We define the map from $B_r(H^1_A)$ (the r-ball of H^1_A centered at the origin) to $\mathcal{M}_d$ by

$$B_r\big(H^1_A\big) \to \mathcal{M}_d, \quad a \mapsto [E, A+\tilde{a}].$$

(Cf. the description of $\mathcal{M}_d$ in Remark 1.3.) This map becomes a $T\mathbb{Z}$-equivariant topological embedding for $r \ll 1$. (Here $B_r(H^1_A)$ is endowed with the following topology. A sequence $\{a_n\}_{n\geqslant 1}$ in $B_r(H^1_A)$ converges to a in $B_r(H^1_A)$ if and only if a_n uniformly converges to a over every compact subset.) Then we have

$$\dim_{[E,A]}(\mathcal{M}_d : T\mathbb{Z}) \geqslant \dim_0\big(B_r\big(H^1_A\big) : T\mathbb{Z}\big).$$

The right-hand side is the local mean dimension of $B_r(H^1_A)$ at the origin. We can prove that $\dim_0(B_r(H^1_A) : T\mathbb{Z})$ can be estimated from below by "the growth of periodic points":

$$\dim_0\big(B_r\big(H^1_A\big) : T\mathbb{Z}\big) \geqslant \lim_{n\to\infty} \dim H^1_{A_n}/n = 8c_2(\underline{\mathrm{E}}).$$

(This is not difficult to prove. This is just an application of Lemma 2.1.) Therefore

$$\dim_{[E,A]}(\mathcal{M}_d : \mathbb{R}) = \dim_{[E,A]}(\mathcal{M}_d : T\mathbb{Z})/T \geqslant 8c_2(\underline{\mathrm{E}})/T = 8\rho(A).$$

This is the outline of the proof of the lower bound.

4. Perturbation

In this section we construct the method of constructing ASD connections from "approximately ASD" connections over $X = S^3 \times \mathbb{R}$. We basically follow the argument of Donaldson [5]. (For a related work on "instanton approximation", see Matsuo [17].) As we promised in the introduction, the variable t means the variable of the $\mathbb{R}$-factor of $S^3 \times \mathbb{R}$.

4.1. Construction of the perturbation

Let T be a positive number, and d, d' be two non-negative real numbers. Set $\varepsilon_0 = 1/(1000)$. (The value $1/(1000)$ itself has no meaning. The point is that it is an explicit number which satisfies (14) below.) Let E be a principal $SU(2)$-bundle over X, and A be a connection on E satisfying the following conditions (i), (ii), (iii).

(i) $F_A = 0$ over $|t| > T + 1$.
(ii) F_A^+ is supported in $\{(\theta, t) \in S^3 \times \mathbb{R} \mid T < |t| < T + 1\}$, and $\|F_A^+\|_{\mathrm{T}} \leqslant \varepsilon_0$. Here $\|\cdot\|_{\mathrm{T}}$ is the "Taubes norm" defined below ((17) and (18)). ("T" of the norm $\|\cdot\|_{\mathrm{T}}$ comes from "Taubes", and it has no relation with the above positive number T. Cf. Taubes [19].)
(iii) $|F_A| \leqslant d$ on $|t| \leqslant T$ and $\|F_A^+\|_{L^\infty(X)} \leqslant d'$. (The condition (iii) is not used in Sections 4.1, 4.2, 4.3. It will be used in Section 4.4.)

Let $\Omega^+(\mathrm{ad}\, E)$ be the set of smooth self-dual 2-forms valued in $\mathrm{ad}\, E$ (not necessarily compactly supported). The first main purpose of this section is to solve the equation $F^+(A + d_A^*\phi) = 0$ for $\phi \in \Omega^+(\mathrm{ad}\, E)$. We have $F^+(A + d_A^*\phi) = F_A^+ + d_A^+ d_A^*\phi + (d_A^*\phi \wedge d_A^*\phi)^+$. The Weitzenböck formula gives [8, Chapter 6]

$$d_A^+ d_A^*\phi = \frac{1}{2}\nabla_A^*\nabla_A\phi + \left(\frac{S}{6} - W^+\right)\phi + F_A^+ \cdot \phi, \tag{11}$$

where S is the scalar curvature of X and W^+ is the self-dual part of the Weyl curvature. Since X is conformally flat, we have $W^+ = 0$. The scalar curvature S is a positive constant. Then the equation $F^+(A + d_A^*\phi) = 0$ becomes

$$\left(\nabla_A^*\nabla_A + S/3\right)\phi + 2F_A^+ \cdot \phi + 2\left(d_A^*\phi \wedge d_A^*\phi\right)^+ = -2F_A^+. \tag{12}$$

Set $c_0 = 10$. Then

$$\left|F_A^+ \cdot \phi\right| \leqslant c_0\left|F_A^+\right| \cdot |\phi|, \qquad \left|\left(d_A^*\phi_1 \wedge d_A^*\phi_2\right)^+\right| \leqslant c_0|\nabla_A\phi_1| \cdot |\nabla_A\phi_2|. \tag{13}$$

(These are not best possible.[1]) The positive constant $\varepsilon_0 = 1/1000$ in the above satisfies

$$50c_0\varepsilon_0 < 1. \tag{14}$$

Let $\Delta = \nabla^*\nabla$ be the Laplacian on functions over X, and $g(x, y)$ be the Green kernel of $\Delta + S/3$. We prepare basic facts on $g(x, y)$ in Appendix A. Here we state some of them without the proofs. For the proofs, see Appendix A. $g(x, y)$ satisfies

$$(\Delta_y + S/3)g(x, y) = \delta_x(y).$$

This equation means that, for any compactly supported smooth function φ,

$$\varphi(x) = \int_X g(x, y)(\Delta_y + S/3)\varphi(y)\, d\mathrm{vol}(y),$$

where $d\mathrm{vol}(y)$ denotes the volume form of X. $g(x, y)$ is smooth outside the diagonal and it has a singularity of order $1/d(x, y)^2$ along the diagonal:

$$\mathrm{const}_1/d(x, y)^2 \leqslant g(x, y) \leqslant \mathrm{const}_2/d(x, y)^2 \quad \left(d(x, y) \leqslant \mathrm{const}_3\right), \tag{15}$$

[1] Strictly speaking, the choice of c_0 depends on the convention of the metric (inner product) on $su(2)$. Our convention is: $\langle A, B\rangle = -tr(AB)$ for $A, B \in su(2)$.

where $d(x, y)$ is the distance on X, and const_1, const_2, const_3 are positive constants. $g(x, y) > 0$ for $x \neq y$ (Lemma A.1), and it has an exponential decay (Lemma A.2):

$$0 < g(x, y) < \mathrm{const}_4 \cdot e^{-\sqrt{S/3}d(x,y)} \quad \big(d(x, y) \geqslant 1\big). \tag{16}$$

Since $S^3 \times \mathbb{R} = SU(2) \times \mathbb{R}$ is a Lie group and its Riemannian metric is two-sided invariant, we have $g(zx, zy) = g(xz, yz) = g(x, y)$. In particular, for $x = (\theta_1, t_1)$ and $y = (\theta_2, t_2)$, we have $g((\theta_1, t_1 - t_0), (\theta_2, t_2 - t_0)) = g((\theta_1, t_1), (\theta_2, t_2))$ ($t_0 \in \mathbb{R}$). That is, $g(x, y)$ is invariant under the translation $t \mapsto t - t_0$.

For $\phi \in \Omega^+(\mathrm{ad}\, E)$, we define the pointwise Taubes norm $|\phi|_{\mathrm{T}}(x)$ by setting

$$|\phi|_{\mathrm{T}}(x) := \int_X g(x, y)\big|\phi(y)\big|\, d\mathrm{vol}(y) \quad (x \in X). \tag{17}$$

(Recall $g(x, y) > 0$ for $x \neq y$.) This may be infinity. We define the Taubes norm $\|\phi\|_{\mathrm{T}}$ by

$$\|\phi\|_{\mathrm{T}} := \sup_{x\in X} |\phi|_{\mathrm{T}}(x). \tag{18}$$

Set

$$K := \int_X g(x, y)\, d\mathrm{vol}(y) \quad \text{(this is independent of } x \in X).$$

(This is finite by (15) and (16).) We have

$$\|\phi\|_{\mathrm{T}} \leqslant K\|\phi\|_{L^\infty}.$$

We define $\Omega^+(\mathrm{ad}\, E)_0$ as the set of $\phi \in \Omega^+(\mathrm{ad}\, E)$ which vanish at infinity: $\lim_{x\to\infty} |\phi(x)| = 0$. (Here $x = (\theta, t) \to \infty$ means $|t| \to +\infty$.) If $\phi \in \Omega^+(\mathrm{ad}\, E)_0$, then $\|\phi\|_{\mathrm{T}} < \infty$ and $\lim_{x\to\infty} |\phi|_{\mathrm{T}}(x) = 0$. (See the proof of Proposition A.7.)

Let $\eta \in \Omega^+(\mathrm{ad}\, E)_0$. There uniquely exists $\phi \in \Omega^+(\mathrm{ad}\, E)_0$ satisfying $(\nabla_A^*\nabla_A + S/3)\phi = \eta$. (See Proposition A.7.) We set $(\nabla_A^*\nabla_A + S/3)^{-1}\eta := \phi$. This satisfies

$$\big|\phi(x)\big| \leqslant |\eta|_{\mathrm{T}}(x), \quad \text{and hence} \quad \|\phi\|_{L^\infty} \leqslant \|\eta\|_{\mathrm{T}}. \tag{19}$$

Lemma 4.1. $\lim_{x\to\infty} |\nabla_A\phi(x)| = 0$.

Proof. From the condition (i) in the beginning of this section, A is flat over $|t| > T+1$. Therefore there exists a bundle map $g : E|_{|t|>T+1} \to X_{|t|>T+1} \times SU(2)$ such that $g(A)$ is the product connection. Here $X_{|t|>T+1} = \{(\theta, t) \in S^3 \times \mathbb{R} \mid |t| > T + 1\}$ and $E|_{|t|>T+1}$ is the restriction of E to $X_{|t|>T+1}$. We sometimes use similar notations in this paper. Set $\phi' := g(\phi)$ and $\eta' := g(\eta)$. They satisfy $(\nabla^*\nabla + S/3)\phi' = \eta'$. (Here ∇ is defined by the product connection on $X|_{|t|>T+1} \times SU(2)$ and the Levi-Civita connection.)

For $|t| > T + 2$, we set $B_t := S^3 \times (t - 1, t + 1)$. From the elliptic estimates, for any $\theta \in S^3$,

$$\big|\nabla\phi'(\theta, t)\big| \leqslant C\big(\big\|\phi'\big\|_{L^\infty(B_t)} + \big\|\eta'\big\|_{L^\infty(B_t)}\big),$$

where C is a constant independent of t. This means

$$\left|\nabla_A\phi(\theta,t)\right| \leqslant C\left(\|\phi\|_{L^\infty(B_t)} + \|\eta\|_{L^\infty(B_t)}\right).$$

The right-hand side goes to 0 as $|t|$ goes to infinity. □

The following lemma shows a power of the Taubes norm. (Here $\eta \in \Omega^+(\mathrm{ad}\,E)_0$ and $\phi = (\nabla_A^*\nabla_A + S/3)^{-1}\eta \in \Omega^+(\mathrm{ad}\,E)_0$.)

Lemma 4.2.

$$\left||\nabla_A\phi|^2\right|_{\mathrm{T}}(x) := \int_X g(x,y)\left|\nabla_A\phi(y)\right|^2 d\mathrm{vol}(y) \leqslant \|\eta\|_{\mathrm{T}}|\eta|_{\mathrm{T}}(x).$$

In particular, $\||\nabla_A\phi|^2\|_{\mathrm{T}} := \sup_{x\in X}||\nabla_A\phi|^2|_{\mathrm{T}}(x) \leqslant \|\eta\|_{\mathrm{T}}^2$ *and* $\|(d_A^*\phi \wedge d_A^*\phi)^+\|_{\mathrm{T}} \leqslant c_0\|\eta\|_{\mathrm{T}}^2$.

Proof. $\nabla|\phi|^2 = 2(\nabla_A\phi, \phi)$ vanishes at infinity (Lemma 4.1).

$$(\Delta + 2S/3)|\phi|^2 = 2\left(\nabla_A^*\nabla_A\phi + (S/3)\phi, \phi\right) - 2|\nabla_A\phi|^2 = 2(\eta,\phi) - 2|\nabla_A\phi|^2.$$

In particular, $(\Delta + S/3)|\phi|^2$ vanishes at infinity (Lemma 4.1). Hence $|\phi|^2$, $\nabla|\phi|^2$, $(\Delta + S/3)|\phi|^2$ vanish at infinity (in particular, they are contained in L^∞). Then we can apply Lemma A.3 in Appendix A to $|\phi|^2$ and get

$$\int_X g(x,y)(\Delta_y + S/3)\left|\phi(y)\right|^2 d\mathrm{vol}(y) = \left|\phi(x)\right|^2.$$

We have

$$\begin{aligned}|\nabla_A\phi|^2 &= (\eta,\phi) - \frac{1}{2}(\Delta + S/3)|\phi|^2 - \frac{S}{6}|\phi|^2\\ &\leqslant (\eta,\phi) - \frac{1}{2}(\Delta + S/3)|\phi|^2.\end{aligned}$$

Therefore

$$\begin{aligned}\int_X g(x,y)\left|\nabla_A\phi(y)\right|^2 d\mathrm{vol}(y) &\leqslant \int_X g(x,y)\big(\eta(y),\phi(y)\big)\,d\mathrm{vol}(y) - \frac{1}{2}\left|\phi(x)\right|^2\\ &\leqslant \int_X g(x,y)\big(\eta(y),\phi(y)\big)\,d\mathrm{vol}(y)\\ &\leqslant \|\phi\|_{L^\infty}\int_X g(x,y)\left|\eta(y)\right| d\mathrm{vol}(y) \leqslant \|\eta\|_{\mathrm{T}}|\eta|_{\mathrm{T}}(x).\end{aligned}$$

In the last line we have used (19). □

For $\eta_1, \eta_2 \in \Omega^+(\mathrm{ad}\,E)_0$, set $\phi_i := (\nabla_A^*\nabla_A + S/3)^{-1}\eta_i \in \Omega^+(\mathrm{ad}\,E)_0$ $(i = 1, 2)$ and

$$\beta(\eta_1, \eta_2) := \left(d_A^*\phi_1 \wedge d_A^*\phi_2\right)^+ + \left(d_A^*\phi_2 \wedge d_A^*\phi_1\right)^+. \tag{20}$$

β is symmetric and $|\beta(\eta_1, \eta_2)| \leqslant 2c_0|\nabla_A\phi_1| \cdot |\nabla_A\phi_2|$. In particular, $\beta(\eta_1, \eta_2) \in \Omega^+(\mathrm{ad}\,E)_0$ (Lemma 4.1).

Lemma 4.3. $\|\beta(\eta_1, \eta_2)\|_{\mathrm{T}} \leqslant 4c_0\|\eta_1\|_{\mathrm{T}}\|\eta_2\|_{\mathrm{T}}$.

Proof. From Lemma 4.2, $\|\beta(\eta, \eta)\|_{\mathrm{T}} \leqslant 2c_0\|\eta\|_{\mathrm{T}}^2$. Suppose $\|\eta_1\|_{\mathrm{T}} = \|\eta_2\|_{\mathrm{T}} = 1$. Since $4\beta(\eta_1, \eta_2) = \beta(\eta_1 + \eta_2, \eta_1 + \eta_2) - \beta(\eta_1 - \eta_2, \eta_1 - \eta_2)$,

$$4\left\|\beta(\eta_1, \eta_2)\right\|_{\mathrm{T}} \leqslant 2c_0\|\eta_1 + \eta_2\|_{\mathrm{T}}^2 + 2c_0\|\eta_1 - \eta_2\|_{\mathrm{T}}^2 \leqslant 16c_0.$$

Hence $\|\beta(\eta_1, \eta_2)\|_{\mathrm{T}} \leqslant 4c_0$. The general case follows from this. □

For $\eta \in \Omega^+(\mathrm{ad}\,E)_0$, we set $\phi := (\nabla_A^*\nabla_A + S/3)^{-1}\eta \in \Omega^+(\mathrm{ad}\,E)_0$ and define

$$\Phi(\eta) := -2F_A^+ \cdot \phi - \beta(\eta, \eta) - 2F_A^+ \in \Omega^+(\mathrm{ad}\,E)_0.$$

If η satisfies $\eta = \Phi(\eta)$, then ϕ satisfies the ASD equation (12).

Lemma 4.4. *For $\eta_1, \eta_2 \in \Omega^+(\mathrm{ad}\,E)_0$,*

$$\left\|\Phi(\eta_1) - \Phi(\eta_2)\right\|_{\mathrm{T}} \leqslant 2c_0\left(\left\|F_A^+\right\|_{\mathrm{T}} + 2\|\eta_1 + \eta_2\|_{\mathrm{T}}\right)\|\eta_1 - \eta_2\|_{\mathrm{T}}.$$

Proof.

$$\Phi(\eta_1) - \Phi(\eta_2) = -2F_A^+ \cdot (\phi_1 - \phi_2) + \beta(\eta_1 + \eta_2, \eta_2 - \eta_1).$$

From Lemma 4.3 and $\|\phi_1 - \phi_2\|_{L^\infty} \leqslant \|\eta_1 - \eta_2\|_{\mathrm{T}}$ (see (19)),

$$\begin{aligned}\left\|\Phi(\eta_1) - \Phi(\eta_2)\right\|_{\mathrm{T}} &\leqslant 2c_0\left\|F_A^+\right\|_{\mathrm{T}}\|\phi_1 - \phi_2\|_{L^\infty} + 4c_0\|\eta_1 + \eta_2\|_{\mathrm{T}}\|\eta_1 - \eta_2\|_{\mathrm{T}}, \\ &\leqslant 2c_0\left(\left\|F_A^+\right\|_{\mathrm{T}} + 2\|\eta_1 + \eta_2\|_{\mathrm{T}}\right)\|\eta_1 - \eta_2\|_{\mathrm{T}}.\end{aligned}$$ □

Proposition 4.5. *The sequence $\{\eta_n\}_{n\geqslant 0}$ in $\Omega^+(\mathrm{ad}\,E)_0$ defined by*

$$\eta_0 = 0, \qquad \eta_{n+1} = \Phi(\eta_n),$$

becomes a Cauchy sequence with respect to the Taubes norm $\|\cdot\|_{\mathrm{T}}$ and satisfies

$$\|\eta_n\|_{\mathrm{T}} \leqslant 3\varepsilon_0,$$

for all $n \geqslant 0$.

Proof. Set $B := \{\eta \in \Omega^+(\operatorname{ad} E)_0 \mid \|\eta\|_{\mathrm{T}} \leqslant 3\varepsilon_0\}$. For $\eta \in B$ (recall: $\|F_A^+\|_{\mathrm{T}} \leqslant \varepsilon_0$),

$$\begin{aligned}\|\Phi(\eta)\|_{\mathrm{T}} &\leqslant 2c_0\|F_A^+\|_{\mathrm{T}}\|\phi\|_{L^\infty} + 2c_0\|\eta\|_{\mathrm{T}}^2 + 2\|F_A^+\|_{\mathrm{T}}\\ &\leqslant 2c_0\varepsilon_0\|\eta\|_{\mathrm{T}} + 2c_0\|\eta\|_{\mathrm{T}}^2 + 2\varepsilon_0\\ &\leqslant (24c_0\varepsilon_0 + 2)\varepsilon_0 \leqslant 3\varepsilon_0.\end{aligned}$$

Here we have used (14). Hence $\Phi(\eta) \in B$. Lemma 4.4 implies (for $\eta_1, \eta_2 \in B$)

$$\|\Phi(\eta_1) - \Phi(\eta_2)\|_{\mathrm{T}} \leqslant 2c_0\left(\|F_A^+\|_{\mathrm{T}} + 2\|\eta_1 + \eta_2\|_{\mathrm{T}}\right)\|\eta_1 - \eta_2\|_{\mathrm{T}} \leqslant 26c_0\varepsilon_0\|\eta_1 - \eta_2\|_{\mathrm{T}}.$$

$26c_0\varepsilon_0 < 1$ by (14). Hence $\Phi : B \to B$ becomes a contraction map with respect to the norm $\|\cdot\|_{\mathrm{T}}$. Thus $\eta_{n+1} = \Phi(\eta_n)$ $(\eta_0 = 0)$ becomes a Cauchy sequence. □

The sequence $\phi_n \in \Omega^+(\operatorname{ad} E)_0$ $(n \geqslant 0)$ defined by $\phi_n := (\nabla_A^*\nabla_A + S/3)^{-1}\eta_n$ satisfies $\|\phi_n - \phi_m\|_{L^\infty} \leqslant \|\eta_n - \eta_m\|_{\mathrm{T}}$. Hence it becomes a Cauchy sequence in $L^\infty(\Lambda^+(\operatorname{ad} E))$. Therefore ϕ_n converges to some ϕ_A in $L^\infty(\Lambda^+(\operatorname{ad} E))$. ϕ_A is continuous since every ϕ_n is continuous. Indeed we will see later that ϕ_A is smooth and satisfies the ASD equation $F^+(A + d_A^*\phi_A) = 0$.

We have $\eta_{n+1} = \Phi(\eta_n) = -2F_A^+ \cdot \phi_n - 2(d_A^*\phi_n \wedge d_A^*\phi_n)^+ - 2F_A^+$.

$$\begin{aligned}\left|2F_A^+ \cdot \phi_n\right|_{\mathrm{T}}(x) &\leqslant 2c_0\int g(x,y)\left|F_A^+(y)\right|\left|\phi_n(y)\right| d\mathrm{vol}(y)\\ &\leqslant 2c_0\left|F_A^+\right|_{\mathrm{T}}(x)\|\phi_n\|_{L^\infty} \leqslant 2c_0\left|F_A^+\right|_{\mathrm{T}}(x)\|\eta_n\|_{\mathrm{T}},\\ \left|2\left(d_A^*\phi_n \wedge d_A^*\phi_n\right)^+\right|_{\mathrm{T}}(x) &\leqslant 2c_0\|\eta_n\|_{\mathrm{T}}|\eta_n|_{\mathrm{T}}(x) \quad \text{(Lemma 4.2)}.\end{aligned}$$

Hence

$$|\eta_{n+1}|_{\mathrm{T}}(x) \leqslant 2c_0\|\eta_n\|_{\mathrm{T}}\left|F_A^+\right|_{\mathrm{T}}(x) + 2c_0\|\eta_n\|_{\mathrm{T}}|\eta_n|_{\mathrm{T}}(x) + 2\left|F_A^+\right|_{\mathrm{T}}(x).$$

Since $\|\eta_n\|_{\mathrm{T}} \leqslant 3\varepsilon_0$,

$$|\eta_{n+1}|_{\mathrm{T}}(x) \leqslant 6c_0\varepsilon_0|\eta_n|_{\mathrm{T}}(x) + (6c_0\varepsilon_0 + 2)\left|F_A^+\right|_{\mathrm{T}}(x).$$

By (14),

$$|\eta_n|_{\mathrm{T}}(x) \leqslant \frac{(6c_0\varepsilon_0 + 2)|F_A^+|_{\mathrm{T}}(x)}{1 - 6c_0\varepsilon_0} \leqslant 3\left|F_A^+\right|_{\mathrm{T}}(x).$$

Recall that F_A^+ is supported in $\{T < |t| < T + 1\}$ and that $g(x,y) > 0$ for $x \neq y$. Set

$$\delta(x) := \int_{T<|t|<T+1} g(x,y)\, d\mathrm{vol}(y) \quad (x \in X).$$

Then $|F_A^+|_{\mathrm{T}}(x) \leqslant \delta(x)\|F_A^+\|_{L^\infty}$. Note that $\delta(x)$ vanishes at infinity because $g(x,y) \leqslant \text{const} \cdot e^{-\sqrt{S/3}d(x,y)}$ for $d(x,y) \geqslant 1$. (See (16).) We get the following decay estimate.

Proposition 4.6. $|\phi_n(x)| \leqslant |\eta_n|_{\mathrm{T}}(x) \leqslant 3\delta(x)\|F_A^+\|_{L^\infty}$. *Hence* $|\phi_A(x)| \leqslant 3\delta(x)\|F_A^+\|_{L^\infty}$. *In particular,* ϕ_A *vanishes at infinity.*

4.2. Regularity and the behavior at the end

From the definition of ϕ_n, we have

$$\big(\nabla_A^*\nabla_A + S/3\big)\phi_{n+1} = \eta_{n+1} = -2F_A^+ \cdot \phi_n - 2\big(d_A^*\phi_n \wedge d_A^*\phi_n\big)^+ - 2F_A^+. \tag{21}$$

Lemma 4.7. $\sup_{n\geqslant 1}\|\nabla_A\phi_n\|_{L^\infty} < +\infty$.

Proof. We use the rescaling argument of Donaldson [5, Section 2.4]. Recall that ϕ_n are uniformly bounded and uniformly go to zero at infinity (Proposition 4.6). Moreover $\|\nabla_A\phi_n\|_{L^\infty} < \infty$ for each $n \geqslant 1$ by Lemma 4.1. Suppose $\sup_{n\geqslant 1}\|\nabla_A\phi_n\|_{L^\infty} = +\infty$. Then there exists a sequence $n_1 < n_2 < n_3 < \cdots$ such that $R_k := \|\nabla_A\phi_{n_k}\|_{L^\infty}$ go to infinity and $R_k \geqslant \max_{1\leqslant n\leqslant n_k}\|\nabla_A\phi_n\|_{L^\infty}$. Since $|\nabla_A\phi_n|$ vanishes at infinity (see Lemma 4.1), we can take $x_k \in X$ satisfying $R_k = |\nabla_A\phi_{n_k}(x_k)|$. From Eq. (21), $|\nabla_A^*\nabla_A\phi_{n_k}| \leqslant \mathrm{const}_A \cdot R_k^2$. Here "$\mathrm{const}_A$" means a positive constant depending on A (but independent of $k \geqslant 1$). Let $r_0 > 0$ be a positive number less than the injectivity radius of X. We consider the geodesic coordinate centered at x_k for each $k \geqslant 1$, and we take a bundle trivialization of E over each geodesic ball $B(x_k, r_0)$ by the exponential gauge centered at x_k. Then we can consider ϕ_{n_k} as a vector-valued function in the ball $B(x_k, r_0)$. Under this setting, ϕ_{n_k} satisfies

$$\left|\sum_{i,j} g_{(k)}^{ij}\partial_i\partial_j\phi_{n_k}\right| \leqslant \mathrm{const}_A \cdot R_k^2 \quad \text{on } B(x_k, r_0), \tag{22}$$

where $(g_{(k)}^{ij}) = (g_{(k),ij})^{-1}$ and $g_{(k),ij}$ is the Riemannian metric tensor in the geodesic coordinate centered at x_k. (Indeed $S^3 \times \mathbb{R} = SU(2) \times \mathbb{R}$ is a Lie group. Hence we can take the geodesic coordinates so that $g_{(k),ij}$ are independent of k.) Set $\tilde\phi_k(x) := \phi_{n_k}(x/R_k)$. $\tilde\phi_k(x)$ is a vector-valued function defined over the r_0R_k-ball in $\mathbb{R}^4$ centered at the origin. $\tilde\phi_k$ ($k \geqslant 1$) satisfy $|\nabla\tilde\phi_k(0)| = 1$, and they are uniformly bounded. From (22), they satisfy

$$\left|\sum_{i,j} \tilde g_{(k)}^{ij}\partial_i\partial_j\tilde\phi_k\right| \leqslant \mathrm{const}_A,$$

where $\tilde g_{(k)}^{ij}(x) = g_{(k)}^{ij}(x/R_k)$. $\{\tilde g_{(k)}^{ij}\}_{k\geqslant 1}$ converges to δ^{ij} (the Kronecker delta) as $k \to +\infty$ in the $\mathcal{C}^\infty$-topology over compact subsets in $\mathbb{R}^4$. Hence there exists a subsequence $\{\tilde\phi_{k_l}\}_{l\geqslant 1}$ which converges to some $\tilde\phi$ in the $\mathcal{C}^1$-topology over compact subsets in $\mathbb{R}^4$. Since $|\nabla\tilde\phi_k(0)| = 1$, we have $|\nabla\tilde\phi(0)| = 1$.

If $\{x_{k_l}\}_{l\geqslant 1}$ is a bounded sequence, then $\{\tilde\phi_{k_l}\}$ has a subsequence which converges to a constant function uniformly over every compact subset because ϕ_n converges to ϕ_A (a continuous section) in the $\mathcal{C}^0$-topology ($= L^\infty$-topology) and $R_k \to \infty$. But this contradicts the above $|\nabla\tilde\phi(0)| = 1$. Hence $\{x_{k_l}\}$ is an unbounded sequence. Since ϕ_n uniformly go to zero at infinity, $\{\tilde\phi_{k_l}\}$ has a subsequence which converges to 0 uniformly over every compact subset. Then this also contradicts $|\nabla\tilde\phi(0)| = 1$. □

From Lemma 4.7 and Eq. (21), the elliptic estimates show that ϕ_n converges to ϕ_A in the $\mathcal{C}^\infty$-topology over every compact subset in X. In particular, ϕ_A is smooth. (Indeed $\phi_A \in \Omega^+(\operatorname{ad} E)_0$ from Proposition 4.6.) From Eq. (21),

$$\left(\nabla_A^*\nabla_A + S/3\right)\phi_A = -2F_A^+ \cdot \phi_A - 2\left(d_A^*\phi_A \wedge d_A^*\phi_A\right)^+ - 2F_A^+. \tag{23}$$

This implies that $A + d_A^*\phi_A$ is an ASD connection.

Lemma 4.1 shows $\lim_{x\to\infty} |\nabla_A\phi_n(x)| = 0$ for each n. Indeed we can prove a stronger result:

Lemma 4.8. *For each $\varepsilon > 0$, there exists a compact set $K \subset X$ such that for all n*

$$\left|\nabla_A\phi_n(x)\right| \leqslant \varepsilon \quad (x \in X \setminus K).$$

Therefore, $\lim_{x\to\infty} |\nabla_A\phi_A(x)| = 0$.

Proof. Suppose the statement is false. Then there are $\delta > 0$, a sequence $n_1 < n_2 < n_3 < \cdots$, and a sequence of points $x_1, x_2, x_3, \ldots$ in X which goes to infinity such that

$$\left|\nabla_A\phi_{n_k}(x_k)\right| \geqslant \delta \quad (k = 1, 2, 3, \ldots).$$

Let $x_k = (\theta_k, t_k) \in S^3 \times \mathbb{R} = X$. $|t_k|$ goes to infinity. We can suppose $|t_k| > T + 2$.

Since A is flat in $|t| > T + 1$, there exists a bundle trivialization $g : E|_{|t|>T+1} \to X_{|t|>T+1} \times SU(2)$ such that $g(A)$ is equal to the product connection. (Here $X_{|t|>T+1} = \{(\theta, t) \in S^3 \times \mathbb{R} \mid |t| > T + 1\}$.) Set $\phi_n' := g(\phi_n)$. We have

$$\left(\nabla^*\nabla + S/3\right)\phi_n' = -2\left(d^*\phi_{n-1}' \wedge d^*\phi_{n-1}'\right)^+ \quad \left(|t| > T + 1\right),$$

where ∇ is defined by using the product connection on $X_{|t|>T+1} \times SU(2)$. From this equation and Lemma 4.7,

$$\left|\left(\nabla^*\nabla + S/3\right)\phi_n'\right| \leqslant \text{const} \quad \left(|t| > T + 1\right),$$

where const is independent of n. We define $\varphi_k \in \Gamma(S^3 \times (-1, 1), \Lambda^+ \otimes su(2))$ by $\varphi_k(\theta, t) := \phi_{n_k}'(\theta, t_k + t)$. We have $|(\nabla^*\nabla + S/3)\varphi_k| \leqslant \text{const}$. Since $|\phi_n'(x)| \leqslant 3\delta(x)\|F_A^+\|_{L^\infty}$ and $|t_k| \to +\infty$, the sequence φ_k converges to 0 in $L^\infty(S^3 \times (-1, 1))$. Using the elliptic estimate, we get $\varphi_k \to 0$ in $\mathcal{C}^1(S^3 \times [-1/2, 1/2])$. On the other hand, $|\nabla\varphi_k(\theta_k, 0)| = |\nabla_A\phi_{n_k}(\theta_k, t_k)| \geqslant \delta > 0$. This is a contradiction. □

Set

$$\eta_A := \left(\nabla_A^*\nabla_A + S/3\right)\phi_A = -2F_A^+ \cdot \phi_A - 2\left(d_A^*\phi_A \wedge d_A^*\phi_A\right)^+ - 2F_A^+. \tag{24}$$

This is contained in $\Omega^+(\operatorname{ad} E)_0$ (Lemma 4.8). The sequence η_n defined in Proposition 4.5 satisfies

$$\eta_{n+1} = -2F_A^+ \cdot \phi_n - 2\left(d_A^*\phi_n \wedge d_A^*\phi_n\right)^+ - 2F_A^+.$$

Corollary 4.9. *The sequence η_n converges to η_A in L^∞. In particular, $\|\eta_n - \eta_A\|_{\mathrm{T}} \to 0$ as $n \to \infty$. Hence $\|\eta_A\|_{\mathrm{T}} \leqslant 3\varepsilon_0$.* (*Proposition* 4.5.)

Proof.

$$\eta_{n+1} - \eta_A = -2F_A^+ \cdot (\phi_n - \phi_A) + 2\big\{d_A^*(\phi_A - \phi_n) \wedge d_A^*\phi_A + d_A^*\phi_n \wedge d_A^*(\phi_A - \phi_n)\big\}^+.$$

Hence

$$|\eta_{n+1} - \eta_A| \leqslant 2c_0 \big\| F_A^+ \big\|_{L^\infty} \|\phi_n - \phi_A\|_{L^\infty} + 2c_0\big(|\nabla_A\phi_n| + |\nabla_A\phi_A|\big)|\nabla_A\phi_A - \nabla_A\phi_n|.$$

$\phi_n \to \phi_A$ in $L^\infty(X)$ and in $\mathcal{C}^\infty$ over every compact subset. Moreover $|\nabla_A\phi_n|$ are uniformly bounded and uniformly go to zero at infinity (Lemmas 4.7 and 4.8). Then the above inequality implies that $\|\eta_{n+1} - \eta_A\|_{L^\infty}$ goes to 0. $\square$

Lemma 4.10. $\|d_A d_A^*\phi_A\|_{L^\infty} < \infty$.

Proof. It is enough to prove $|d_A d_A^*\phi_A(\theta, t)| \leqslant \mathrm{const}$ for $|t| > T + 2$. Take a trivialization g of E over $|t| > T + 1$ such that $g(A)$ is the product connection, and set $\phi' := g(\phi_A)$. This satisfies

$$\big(\nabla^*\nabla + S/3\big)\phi' = -2\big(d^*\phi' \wedge d^*\phi'\big)^+ \quad \big(|t| > T + 1\big).$$

Since $|\phi'|$ and $|\nabla\phi'|$ go to zero at infinity (Proposition 4.6 and Lemma 4.8), this shows (by using the elliptic estimates) that $|dd^*\phi'|$ is bounded. $\square$

Lemma 4.11.

$$\frac{1}{8\pi^2}\int_X \big|F\big(A + d_A^*\phi_A\big)\big|^2 \, d\mathrm{vol} = \frac{1}{8\pi^2}\int_X tr\big(F_A^2\big).$$

Recall that A is flat over $|t| > T + 1$. Hence the right-hand side is finite. (*Indeed it is a non-negative integer by the Chern–Weil theory.*)

Proof. Set $a := d_A^*\phi_A$ and $cs_A(a) := \frac{1}{8\pi^2} tr(2a \wedge F_A + a \wedge d_A a + \frac{2}{3}a^3)$. We have $\frac{1}{8\pi^2} tr(F(A + a)^2) - \frac{1}{8\pi^2} tr(F(A)^2) = d\, cs_A(a)$. Since $A + a$ is ASD, we have $|F(A + a)|^2 \, d\mathrm{vol} = tr(F(A + a)^2)$ and

$$\frac{1}{8\pi^2}\int_{|t|\leqslant R} tr\big(F(A + a)^2\big) - \frac{1}{8\pi^2}\int_{|t|\leqslant R} tr\big(F(A)^2\big) = \int_{t=R} cs_A(a) - \int_{t=-R} cs_A(a).$$

From Lemma 4.8, $|a| = |d_A^*\phi_A|$ goes to zero at infinity. From Lemma 4.10, $|d_A a| = |d_A d_A^*\phi_A|$ is bounded. F_A vanishes over $|t| > T + 1$. Hence $|cs_A(a)|$ goes to zero at infinity. Thus the right-hand side of the above equation goes to zero as $R \to \infty$. $\square$

4.3. Conclusion of the construction

The following is the conclusion of Sections 4.1 and 4.2. This will be used in Sections 5 and 8. (Notice that we have not so far used the condition (iii) in the beginning of Section 4.1.)

Proposition 4.12. *Let $T > 0$. Let E be a principal $SU(2)$-bundle over X, and A be a connection on E satisfying $F_A = 0$ $(|t| > T + 1)$, $\operatorname{supp} F_A^+ \subset \{T < |t| < T + 1\}$ and $\|F_A^+\|_{\mathrm{T}} \leqslant \varepsilon_0 = 1/1000$. Then we can construct $\phi_A \in \Omega^+(\operatorname{ad} E)_0$ satisfying the following conditions.*

(a) $A + d_A^*\phi_A$ *is an ASD connection*: $F^+(A + d_A^*\phi_A) = 0$.

(b)

$$\frac{1}{8\pi^2}\int_X \left|F\left(A + d_A^*\phi_A\right)\right|^2 d\mathrm{vol} = \frac{1}{8\pi^2}\int_X tr\left(F_A^2\right).$$

(c) $|\phi_A(x)| \leqslant 3\delta(x)\|F_A^+\|_{L^\infty}$, *where* $\delta(x) = \int_{T<|t|<T+1} g(x,y)\,d\mathrm{vol}(y)$.

(d) $\eta_A := (\nabla_A^*\nabla_A + S/3)\phi_A$ *is contained in* $\Omega^+(\operatorname{ad} E)_0$ *and* $\|\eta_A\|_{\mathrm{T}} \leqslant 3\varepsilon_0$.

Moreover this construction $(E, A) \mapsto \phi_A$ is gauge equivariant, i.e., if F is another principal $SU(2)$-bundle over X admitting a bundle map $g : E \to F$, then $\phi_{g(A)} = g(\phi_A)$.

Proof. The conditions (a), (b), (c), (d) have been already proved. The gauge equivariance is obvious by the construction of ϕ_A in Section 4.1. □

4.4. Interior estimate

In the proof of the upper bound on the mean dimension, we need to use an "interior estimate" of ϕ_A (Lemma 4.14 below), which we investigate in this subsection. We use the argument of Donaldson [5, pp. 189–190]. Recall that $|F_A| \leqslant d$ on $|t| \leqslant T$ and $\|F_A^+\|_{L^\infty(X)} \leqslant d'$ by the condition (iii) in the beginning of Section 4.1. We fix $r_0 > 0$ so that r_0 is less than the injectivity radius of $S^3 \times \mathbb{R}$ (cf. the proof of Lemma 4.7).

Lemma 4.13. *For any $\varepsilon > 0$, there exists a constant $\delta_0 = \delta_0(d, \varepsilon) > 0$ depending only on d and ε such that the following statement holds. For any $\phi \in \Omega^+(\operatorname{ad} E)$ and any closed r_0-ball B contained in $S^3 \times [-T + 1, T - 1]$, if ϕ satisfies*

$$\left(\nabla_A^*\nabla_A + S/3\right)\phi = -2\left(d_A^*\phi \wedge d_A^*\phi\right)^+ \quad \text{over } B \quad \text{and} \quad \|\phi\|_{L^\infty(B)} \leqslant \delta_0, \tag{25}$$

then we have

$$\sup_{x\in B}\left|\nabla_A\phi(x)\right| d(x, \partial B) \leqslant \varepsilon.$$

Here $d(x, \partial B)$ is the distance between x and ∂B. (If $T < 1$, then $S^3 \times [-T + 1, T - 1]$ is empty, and the above statement has no meaning.)

Proof. Suppose ϕ satisfies

$$\sup_{x\in B}\left|\nabla_A\phi(x)\right| d(x,\partial B) > \varepsilon,$$

and the supremum is attained at $x_0 \in B$ (x_0 is an inner point of B). Set $R := |\nabla_A\phi(x_0)|$ and $r_0' := d(x_0, \partial B)/2$. We have $2r_0'R > \varepsilon$. Let B' be the closed r_0'-ball centered at x_0. We have $|\nabla_A\phi| \leqslant 2R$ on B'. We consider the geodesic coordinate over B' centered at x_0, and we trivialize the bundle E over B' by the exponential gauge centered at x_0. Since A is ASD and $|F_A| \leqslant d$ over $-T \leqslant t \leqslant T$, the $\mathcal{C}^1$-norm of the connection matrix of A in the exponential gauge over B' is bounded by a constant depending only on d. From Eq. (25) and $|\nabla_A\phi| \leqslant 2R$ on B',

$$\left|\sum g^{ij}\partial_i\partial_j\phi\right| \leqslant \mathrm{const}_{d,\varepsilon}\cdot R^2 \quad \text{over } B',$$

where $(g^{ij}) = (g_{ij})^{-1}$ and g_{ij} is the Riemannian metric tensor in the geodesic coordinate over B'. Here we consider ϕ as a vector-valued function over B'. Set $\tilde\phi(x) := \phi(x/R)$. Since $2r_0'R > \varepsilon$, $\tilde\phi$ is defined over the $\varepsilon/2$-ball $B(\varepsilon/2)$ centered at the origin in $\mathbb{R}^4$, and it satisfies

$$\left|\sum \tilde g^{ij}\partial_i\partial_i\tilde\phi\right| \leqslant \mathrm{const}_{d,\varepsilon} \quad \text{over } B(\varepsilon/2).$$

Here $\tilde g^{ij}(x) := g^{ij}(x/R)$. The eigenvalues of the matrix $(\tilde g^{ij})$ are bounded from below by a positive constant depending only on the geometry of X, and the $\mathcal{C}^1$-norm of $\tilde g^{ij}$ is bounded from above by a constant depending only on ε and the geometry of X. (Note that $R > \varepsilon/(2r_0') \geqslant \varepsilon/(2r_0)$.) Then by using the elliptic estimate [9, Theorem 9.11] and the Sobolev embedding $L^8_2(B(\varepsilon/4)) \hookrightarrow \mathcal{C}^{1,1/2}(B(\varepsilon/4))$ (the Hölder space), we get

$$\|\tilde\phi\|_{\mathcal{C}^{1,1/2}(B(\varepsilon/4))} \leqslant \mathrm{const}_\varepsilon \cdot \|\tilde\phi\|_{L^8_2(B(\varepsilon/4))} \leqslant C = C(d,\varepsilon).$$

Hence $|\nabla\tilde\phi(x) - \nabla\tilde\phi(0)| \leqslant C|x|^{1/2}$ on $B(\varepsilon/4)$. Set $u := \nabla\tilde\phi(0)$. From the definition, we have $|u| = 1$.

$$\tilde\phi(tu) - \tilde\phi(0) = t\int_0^1 \nabla\tilde\phi(tsu)\cdot u\,ds = t + t\int_0^1 \bigl(\nabla\tilde\phi(tsu) - u\bigr)\cdot u\,ds.$$

Hence

$$\left|\tilde\phi(tu) - \tilde\phi(0)\right| \geqslant t - t\int_0^1 C|tsu|^{1/2}\,ds = t - 2Ct^{3/2}/3.$$

We can suppose $C \geqslant 2/\sqrt{\varepsilon}$. Then $u/C^2 \in B(\varepsilon/4)$ and

$$\left|\tilde\phi\bigl(u/C^2\bigr) - \tilde\phi(0)\right| \geqslant 1/\bigl(3C^2\bigr).$$

If $|\phi| \leqslant \delta_0 < 1/(6C^2)$, then this inequality becomes a contradiction. □

The following will be used in Section 8.

Lemma 4.14. *For any $\varepsilon > 0$ there exists a positive number $D = D(d, d', \varepsilon)$ such that*

$$\left\| d_A^* \phi_A \right\|_{L^\infty(S^3 \times [-T+D, T-D])} \leqslant \varepsilon.$$

(If $D > T$, then $S^3 \times [-T + D, T - D]$ is the empty set.) Here the important point is that D is independent of T.

Proof. Note that $|d_A^* \phi_A| \leqslant \sqrt{3/2} |\nabla_A \phi_A|$. We have $|\phi_A(x)| \leqslant 3d' \delta(x)$ by Proposition 4.12(c) (or Proposition 4.6) and

$$\delta(x) = \int_{T < |t| < T+1} g(x, y)\, d\mathrm{vol}(y).$$

Set $D' := D - r_0$. (We choose D so that $D' \geqslant 1$.) Since $g(x, y) \leqslant \mathrm{const} \cdot e^{-\sqrt{S/3} d(x,y)}$ for $d(x, y) \geqslant 1$, we have

$$\delta(x) \leqslant C \cdot e^{-\sqrt{S/3} D'} \quad \text{for } x \in S^3 \times \left[-T + D', T - D'\right].$$

We choose $D = D(d, d', \varepsilon) \geqslant r_0 + 1$ so that

$$3d' C e^{-\sqrt{S/3} D'} \leqslant \delta_0\left(d, r_0 \varepsilon \sqrt{2/3}\right).$$

Here $\delta_0(d, r_0 \varepsilon \sqrt{2/3})$ is the positive constant introduced in Lemma 4.13. Note that this condition is independent of T. Then ϕ_A satisfies, for $x \in S^3 \times [-T + D', T - D']$,

$$\left|\phi_A(x)\right| \leqslant \delta_0\left(d, r_0 \varepsilon \sqrt{2/3}\right).$$

ϕ_A satisfies $(\nabla_A^* \nabla_A + S/3)\phi_A = -2(d_A^* \phi_A \wedge d_A^* \phi_A)^+$ over $|t| \leqslant T$. Then Lemma 4.13 implies

$$\left|\nabla_A \phi_A(x)\right| \leqslant \varepsilon \sqrt{2/3} \quad \text{for } x \in S^3 \times [-T + D, T - D].$$

(Note that, for $x \in S^3 \times [-T + D, T - D]$, we have $B(x, r_0) \subset S^3 \times [-T + D', T - D']$ and hence $|\phi_A| \leqslant \delta_0(d, r_0 \varepsilon \sqrt{2/3})$ over $B(x, r_0)$.) Then, for $x \in S^3 \times [-T + D, T - D]$,

$$\left|d_A^* \phi_A(x)\right| \leqslant \sqrt{3/2} \left|\nabla_A \phi_A(x)\right| \leqslant \varepsilon. \qquad \square$$

5. Continuity of the perturbation

The purpose of this section is to show the continuity of the perturbation construction in Section 4. The conclusion of Section 5 is Proposition 5.6. As in Section 4, $X = S^3 \times \mathbb{R}$, $T > 0$ is a positive number, and $E \to X$ is a principal $SU(2)$-bundle. Let ρ be a flat connection on $E|_{|t|>T+1}$. ($E|_{|t|>T+1}$ is the restriction of E to $X_{|t|>T+1} = \{(\theta, t) \in S^3 \times \mathbb{R} \mid |t| > T + 1\}$.) We define $\mathcal{A}'$ as the set of connections A on E satisfying the following (i), (ii), (iii).

(i) $A|_{|t|>T+1} = \rho$, i.e., A coincides with ρ over $|t| > T+1$.
(ii) F_A^+ is supported in $\{(\theta,t) \in S^3 \times \mathbb{R} \mid T < |t| < T+1\}$.
(iii) $\|F_A^+\|_{\mathrm{T}} \leqslant \varepsilon_0 = 1/1000$.

By Proposition 4.12, for each $A \in \mathcal{A}'$, we have $\phi_A \in \Omega^+(\operatorname{ad} E)_0$ and $\eta_A = (\nabla_A^*\nabla_A + S/3)\phi_A \in \Omega^+(\operatorname{ad} E)_0$ satisfying

$$\eta_A = -2F_A^+ \cdot \phi_A - 2\big(d_A^*\phi_A \wedge d_A^*\phi_A\big)^+ - 2F_A^+, \qquad \|\eta_A\|_{\mathrm{T}} \leqslant 3\varepsilon_0. \tag{26}$$

The first equation in the above is equivalent to the ASD equation $F^+(A + d_A^*\phi_A) = 0$. Since $\phi_A = (\nabla_A^*\nabla_A + S/3)^{-1}\eta_A$, we have ((19) and Lemma 4.2)

$$\|\phi_A\|_{L^\infty} \leqslant \|\eta_A\|_{\mathrm{T}} \leqslant 3\varepsilon_0, \qquad \big\||\nabla_A\phi_A|^2\big\|_{\mathrm{T}} \leqslant \|\eta_A\|_{\mathrm{T}}^2 \leqslant 9\varepsilon_0^2.$$

Then (by the Cauchy–Schwartz inequality)

$$\|\nabla_A\phi_A\|_{\mathrm{T}} := \sup_{x\in X} \int_X g(x,y)\big|\nabla_A\phi_A(y)\big|\,d\mathrm{vol}(y) \leqslant 3\varepsilon_0\sqrt{K},$$

where $K = \int_X g(x,y)\,d\mathrm{vol}(y)$. (The value of K is independent of $x \in X$.)

Let $A, B \in \mathcal{A}'$. We want to estimate $\|\phi_A - \phi_B\|_{L^\infty}$. Set $a := B - A$. Since both A and B coincide with ρ (the fixed flat connection) over $|t| > T+1$, a is compactly supported. We set

$$\|a\|_{\mathcal{C}_A^1} := \|a\|_{L^\infty} + \|\nabla_A a\|_{L^\infty}.$$

We suppose

$$\|a\|_{\mathcal{C}_A^1} \leqslant 1.$$

Lemma 5.1. $\|\phi_A - \phi_B\|_{L^\infty} \leqslant \|\eta_A - \eta_B\|_{\mathrm{T}} + \mathrm{const}\|a\|_{\mathcal{C}_A^1}$, *where* const *is a universal constant independent of* A, B.

Proof. We have $\eta_A = (\nabla_A^*\nabla_A + S/3)\phi_A$ and

$$\eta_B = \big(\nabla_B^*\nabla_B + S/3\big)\phi_B = \big(\nabla_A^*\nabla_A + S/3\big)\phi_B + \big(\nabla_A^* a\big) * \phi_B + a * \nabla_B\phi_B + a * a * \phi_B,$$

where $*$ are algebraic multiplications. Then

$$\begin{aligned}
\|\phi_A - \phi_B\|_{L^\infty} &\leqslant \big\|\big(\nabla_A^*\nabla_A + S/3\big)(\phi_A - \phi_B)\big\|_{\mathrm{T}} \\
&\leqslant \|\eta_A - \eta_B\|_{\mathrm{T}} + \mathrm{const}\big(\|\nabla_A a\|_{L^\infty}\|\phi_B\|_{\mathrm{T}} + \|a\|_{L^\infty}\|\nabla_B\phi_B\|_{\mathrm{T}} + \|a\|_{L^\infty}^2\|\phi_B\|_{\mathrm{T}}\big) \\
&\leqslant \|\eta_A - \eta_B\|_{\mathrm{T}} + \mathrm{const}\|a\|_{\mathcal{C}_A^1}. \qquad \square
\end{aligned}$$

Lemma 5.2.

$$\left\|\left(d_A^*\phi_A\wedge d_A^*\phi_A\right)^+-\left(d_B^*\phi_B\wedge d_B^*\phi_B\right)^+\right\|_{\mathrm{T}}\leqslant\left(\frac{1}{4}+\mathrm{const}\|a\|_{\mathcal{C}_A^1}\right)\|\eta_A-\eta_B\|_{\mathrm{T}}+\mathrm{const}\|a\|_{\mathcal{C}_A^1}.$$

Proof.

$$\begin{aligned}&\left(d_A^*\phi_A\wedge d_A^*\phi_A\right)^+-\left(d_B^*\phi_B\wedge d_B^*\phi_B\right)^+\\&\quad=\underbrace{\left(d_A^*\phi_A\wedge d_A^*\phi_A\right)^+-\left(d_A^*\phi_B\wedge d_A^*\phi_B\right)^+}_{(I)}+\underbrace{\left(d_A^*\phi_B\wedge d_A^*\phi_B\right)^+-\left(d_B^*\phi_B\wedge d_B^*\phi_B\right)^+}_{(II)}.\end{aligned}$$

We first estimate the term (II). Since $B=A+a$,

$$\begin{aligned}&\left(d_B^*\phi_B\wedge d_B^*\phi_B\right)^+-\left(d_A^*\phi_B\wedge d_A^*\phi_B\right)^+\\&\quad=\left(d_A^*\phi_B\wedge(a*\phi_B)\right)^++\left((a*\phi_B)\wedge d_A^*\phi_B\right)^++\left((a*\phi_B)\wedge(a*\phi_B)\right)^+,\\&\quad\left\|(II)\right\|_{\mathrm{T}}\leqslant\mathrm{const}\|\nabla_A\phi_B\|_{\mathrm{T}}\|a\|_{L^\infty}\|\phi_B\|_{L^\infty}+\mathrm{const}\|a\|_{L^\infty}^2\|\phi_B\|_{L^\infty}^2\\&\qquad\qquad\leqslant\mathrm{const}\cdot\|\nabla_A\phi_B\|_{\mathrm{T}}\|a\|_{L^\infty}+\mathrm{const}\cdot\|a\|_{L^\infty}.\end{aligned}$$

We have

$$\|\nabla_A\phi_B\|_{\mathrm{T}}=\|\nabla_B\phi_B+a*\phi_B\|_{\mathrm{T}}\leqslant\|\nabla_B\phi_B\|_{\mathrm{T}}+\mathrm{const}\|a\|_{L^\infty}\|\phi_B\|_{L^\infty}\leqslant\mathrm{const}.$$

Hence $\|(II)\|_{\mathrm{T}}\leqslant\mathrm{const}\|a\|_{L^\infty}$.

Next we estimate the term (I). For $\eta_1,\eta_2\in\Omega^+(\mathrm{ad}\,E)_0$, set $\phi_i:=(\nabla_A^*\nabla_A+S/3)^{-1}\eta_i\in\Omega^+(\mathrm{ad}\,E)_0$ $(i=1,2)$, and define (see (20))

$$\beta_A(\eta_1,\eta_2):=\left(d_A^*\phi_1\wedge d_A^*\phi_2\right)^++\left(d_A^*\phi_2\wedge d_A^*\phi_1\right)^+.$$

Set $\eta'_B:=(\nabla_A^*\nabla_A+S/3)\phi_B=\eta_B+(\nabla_A^*a)*\phi_B+a*\nabla_B\phi_B+a*a*\phi_B$. Then $(d_A^*\phi_B\wedge d_A^*\phi_B)^+=\beta_A(\eta'_B,\eta'_B)/2$ and $(I)=(\beta_A(\eta_A,\eta_A)-\beta_A(\eta'_B,\eta'_B))/2=\beta_A(\eta_A+\eta'_B,\eta_A-\eta'_B)/2$. From Lemma 4.3,

$$\left\|(I)\right\|_{\mathrm{T}}\leqslant 2c_0\left\|\eta_A+\eta'_B\right\|_{\mathrm{T}}\left\|\eta_A-\eta'_B\right\|_{\mathrm{T}}.$$

$\|\eta_A+\eta'_B\|_{\mathrm{T}}\leqslant\|\eta_A+\eta_B\|_{\mathrm{T}}+\|\eta'_B-\eta_B\|_{\mathrm{T}}\leqslant 6\varepsilon_0+\mathrm{const}\|a\|_{\mathcal{C}_A^1}$, and $\|\eta_A-\eta'_B\|_{\mathrm{T}}\leqslant\|\eta_A-\eta_B\|_{\mathrm{T}}+\mathrm{const}\|a\|_{\mathcal{C}_A^1}$. From (14), we have $12c_0\varepsilon_0\leqslant 1/4$. Then

$$\left\|(I)\right\|_{\mathrm{T}}\leqslant\left(\frac{1}{4}+\mathrm{const}\|a\|_{\mathcal{C}_A^1}\right)\|\eta_A-\eta_B\|_{\mathrm{T}}+\mathrm{const}\|a\|_{\mathcal{C}_A^1}.\qquad\square$$

We have $F_B^+=F_A^++d_A^+a+(a\wedge a)^+$. Recall that we have supposed $\|a\|_{\mathcal{C}_A^1}\leqslant 1$. Hence

$$\left|F_B^+-F_A^+\right|\leqslant\mathrm{const}\|a\|_{\mathcal{C}_A^1}.$$

Proposition 5.3. *There exists* $\delta > 0$ *such that if* $\|a\|_{\mathcal{C}^1_A} \leqslant \delta$ *then*

$$\|\eta_A - \eta_B\|_{\mathrm{T}} \leqslant \mathrm{const}\|a\|_{\mathcal{C}^1_A}.$$

Proof. From (26),

$$\eta_A - \eta_B = 2\big(F_B^+ - F_A^+\big)\cdot\phi_B + 2F_A^+\cdot(\phi_B - \phi_A) + 2\big(\big(d_B^*\phi_B \wedge d_B^*\phi_B\big)^+ - \big(d_A^*\phi_A \wedge d_A^*\phi_A\big)^+\big) + 2\big(F_B^+ - F_A^+\big).$$

Using $\|\phi_B\|_{L^\infty} \leqslant 3\varepsilon_0$, $\|F_A^+\|_{\mathrm{T}} \leqslant \varepsilon_0$ and Lemma 5.2,

$$\|\eta_A - \eta_B\|_{\mathrm{T}} \leqslant \mathrm{const}\|a\|_{\mathcal{C}^1_A} + 2c_0\varepsilon_0\|\phi_A - \phi_B\|_{L^\infty} + \left(\frac{1}{2} + \mathrm{const}\|a\|_{\mathcal{C}^1_A}\right)\|\eta_A - \eta_B\|_{\mathrm{T}}.$$

Using Lemma 5.1,

$$\|\eta_A - \eta_B\|_{\mathrm{T}} \leqslant \mathrm{const}\|a\|_{\mathcal{C}^1_A} + \left(\frac{1}{2} + \mathrm{const}\|a\|_{\mathcal{C}^1_A} + 2c_0\varepsilon_0\right)\|\eta_A - \eta_B\|_{\mathrm{T}}.$$

From (14), we can choose $\delta > 0$ so that if $\|a\|_{\mathcal{C}^1_A} \leqslant \delta$ then

$$\left(\frac{1}{2} + \mathrm{const}\|a\|_{\mathcal{C}^1_A} + 2c_0\varepsilon_0\right) \leqslant 3/4.$$

Then we get

$$\|\eta_A - \eta_B\|_{\mathrm{T}} \leqslant \mathrm{const}\|a\|_{\mathcal{C}^1_A} + (3/4)\|\eta_A - \eta_B\|_{\mathrm{T}}.$$

Then $\|\eta_A - \eta_B\|_{\mathrm{T}} \leqslant \mathrm{const}\|a\|_{\mathcal{C}^1_A}$. □

From Lemma 5.1, we get (under the condition $\|a\|_{\mathcal{C}^1_A} \leqslant \delta$)

$$\|\phi_A - \phi_B\|_{L^\infty} \leqslant \|\eta_A - \eta_B\|_{\mathrm{T}} + \mathrm{const}\|a\|_{\mathcal{C}^1_A} \leqslant \mathrm{const}\|a\|_{\mathcal{C}^1_A}.$$

Therefore we get the following.

Corollary 5.4. *The map*

$$\big(\mathcal{A}', \mathcal{C}^1\text{-}topology\big) \to \big(\Omega^+(\mathrm{ad}\,E)_0, \|\cdot\|_{L^\infty}\big), \quad A \mapsto \phi_A,$$

is continuous.

Let A_n $(n \geqslant 1)$ be a sequence in $\mathcal{A}'$ which converges to $A \in \mathcal{A}'$ in the $\mathcal{C}^1$-topology: $\|A_n - A\|_{\mathcal{C}^1_A} \to 0$ $(n \to \infty)$. By Corollary 5.4, we get $\|\phi_{A_n} - \phi_A\|_{L^\infty} \to 0$. Set $a_n := A_n - A$.

Lemma 5.5. $\sup_{n\geqslant 1}\|\nabla_{A_n}\phi_{A_n}\|_{L^\infty}<\infty$. (*Equivalently,* $\sup_{n\geqslant 1}\|\nabla_A\phi_{A_n}\|_{L^\infty}<\infty$.)

Proof. Note that $|\nabla_{A_n}\phi_{A_n}|$ vanishes at infinity (see Lemma 4.8). Hence we can take a point $x_n\in S^3\times\mathbb{R}$ satisfying $|\nabla_{A_n}\phi_{A_n}(x_n)|=\|\nabla_{A_n}\phi_{A_n}\|_{L^\infty}$. $\|\phi_{A_n}-\phi_A\|_{L^\infty}\to 0$ $(n\to\infty)$, and ϕ_{A_n} uniformly go to zero at infinity (see Proposition 4.12(c) or Proposition 4.6). Then the rescaling argument as in the proof of Lemma 4.7 shows the above statement. □

Since $(\nabla_{A_n}^*\nabla_{A_n}+S/3)\phi_{A_n}=-2F_{A_n}^+\cdot\phi_{A_n}-2(d_{A_n}^*\phi_{A_n}\wedge d_{A_n}^*\phi_{A_n})^+-2F_{A_n}^+$,

$$\sup_{n\geqslant 1}\left\|\nabla_{A_n}^*\nabla_{A_n}\phi_{A_n}\right\|_{L^\infty}<\infty.$$

We have $\nabla_{A_n}^*\nabla_{A_n}\phi_{A_n}=\nabla_A^*\nabla_A\phi_{A_n}+(\nabla_A^*a_n)*\phi_{A_n}+a_n*\nabla_{A_n}\phi_{A_n}+a_n*a_n*\phi_{A_n}$. Hence

$$\sup_{n\geqslant 1}\left\|\nabla_A^*\nabla_A\phi_{A_n}\right\|_{L^\infty}<\infty.$$

By the elliptic estimate, we conclude that ϕ_{A_n} converges to ϕ_A in $\mathcal{C}^1$ over every compact subset. Then we get the following conclusion. This will be used in Section 8.

Proposition 5.6. *Let $\{A_n\}_{n\geqslant 1}$ be a sequence in $\mathcal{A}'$ which converges to $A\in\mathcal{A}'$ in the $\mathcal{C}^1$-topology. Then ϕ_{A_n} converges to ϕ_A in the $\mathcal{C}^1$-topology over every compact subset in X. Therefore $d_{A_n}^*\phi_{A_n}$ converges to $d_A^*\phi_A$ in the $\mathcal{C}^0$-topology over every compact subset in X. Moreover, for any $n\geqslant 1$,*

$$\int_X\left|F\left(A_n+d_{A_n}^*\phi_{A_n}\right)\right|^2 d\mathrm{vol}=\int_X\left|F\left(A+d_A^*\phi_A\right)\right|^2 d\mathrm{vol}.$$

(*This means that no energy is lost at the end.*)

Proof. The last statement follows from Proposition 4.12(b) (or Lemma 4.11) and the fact that for any A and B in $\mathcal{A}'$ we have

$$\int_X tr\left(F_A^2\right)=\int_X tr\left(F_B^2\right).$$

This is because $tr\,F_B^2-tr\,F_A^2=d(tr(2a\wedge F_A+a\wedge d_Aa+\frac{2}{3}a^3))$ $(a=B-A)$, and both A and B coincide with the fixed flat connection ρ over $|t|>T+1$. □

6. "Non-flat" implies "irreducible"

This section is short. But the results in this section are crucial for both proofs of the upper and lower bounds on the mean dimension. Note that the following trivial fact: if a smooth function u on $\mathbb{R}$ is bounded and convex $(u''\geqslant 0)$ then u is a constant function.

Lemma 6.1. *If a smooth function f on $S^3 \times \mathbb{R}$ is bounded, non-negative and sub-harmonic ($\Delta f \leqslant 0$),*[2] *then f is a constant function.*

Proof. We have $\Delta = -\partial^2/\partial t^2 + \Delta_{S^3}$ where t is the coordinate of the $\mathbb{R}$-factor of $S^3 \times \mathbb{R}$ and Δ_{S^3} is the Laplacian of S^3. We have

$$\frac{\partial^2}{\partial t^2} f^2 = 2\left(\frac{\partial f}{\partial t}\right)^2 + 2f\Delta_{S^3} f - 2f\Delta f.$$

Then we have

$$\frac{1}{2}\frac{\partial^2}{\partial t^2}\int_{S^3\times\{t\}} f^2\, d\mathrm{vol} = \int_{S^3\times\{t\}} \left(\left|\frac{\partial f}{\partial t}\right|^2 + |\nabla_{S^3} f|^2 + f(-\Delta f)\right) d\mathrm{vol} \geqslant 0.$$

Here we have used $f \geqslant 0$ and $\Delta f \leqslant 0$. This shows that $u(t) = \int_{S^3\times\{t\}} f^2$ is a bounded convex function. Hence it is a constant function. In particular $u'' \equiv 0$. Then the above formula implies $\partial f/\partial t \equiv \nabla_{S^3} f \equiv 0$. This means that f is a constant function. □

Lemma 6.2. *If A is a $U(1)$-ASD connection on $S^3 \times \mathbb{R}$ satisfying $\|F_A\|_{L^\infty} < \infty$, then A is flat.*

Proof. We have $F_A \in \sqrt{-1}\Omega^-$. The Weitzenböck formula (cf. (11)) gives $(\nabla^*\nabla + S/3)F_A = 2d^- d^* F_A = 0$. We have

$$\Delta|F_A|^2 = -2|\nabla F_A|^2 + 2(F_A, \nabla^*\nabla F_A) = -2|\nabla F_A|^2 - (2S/3)|F_A|^2 \leqslant 0.$$

This shows that $|F_A|^2$ is a non-negative, bounded, subharmonic function. Hence it is a constant function. In particular $\Delta|F_A|^2 \equiv 0$. Then the above formula implies $F_A \equiv 0$. □

An $SU(2)$-connection A is said to be reducible if there is a gauge transformation $g \neq \pm 1$ satisfying $g(A) = A$. A is said to be irreducible if it is not reducible.

Corollary 6.3. *If A is a non-flat $SU(2)$-ASD connection on $S^3 \times \mathbb{R}$ satisfying $\|F_A\|_{L^\infty} < \infty$, then A is irreducible.*

This corollary will be used in the proof of the lower bound on the mean dimension. The following proposition will be used in the cut-off construction in Section 7.

Proposition 6.4. *Let A be a non-flat $SU(2)$-ASD connection on $S^3 \times \mathbb{R}$ satisfying $\|F_A\|_{L^\infty} < +\infty$. The restriction of A to $S^3 \times \{0\}$ is irreducible.*

[2] Our convention of the sign of the Laplacian is geometric; we have $\Delta = -\partial^2/\partial x_1^2 - \partial^2/\partial x_2^2 - \partial^2/\partial x_3^2 - \partial^2/\partial x_4^2$ on $\mathbb{R}^4$.

Proof. This follows from the above Corollary 6.3 and the result of Taubes [20, Theorem 5]. Here we give a brief proof for readers' convenience. Note that the Riemannian metric on $S^3 \times \mathbb{R}$ is real analytic. (Later we will use Cauchy–Kovalevskaya's theorem. Hence the real analyticity of all data is essential.) Suppose $A|_{S^3\times\{0\}}$ is reducible. Fix $p \in S^3$ and take a small open neighborhood $\Omega \subset S^3$ of p. Let $\varepsilon > 0$ be a small positive number. By using the Uhlenbeck gauge [24, Corollary 1.4], we can suppose that A is represented by a real analytic connection matrix over $\Omega \times (-\varepsilon, \varepsilon)$. Moreover, by using the (real analytic) temporal gauge (see Donaldson [6, Chapter 2]), we can assume that the (real analytic) connection matrix of A over $\Omega \times (-\varepsilon, \varepsilon)$ is dt-part free and satisfies

$$\frac{\partial}{\partial t} A(t) = *_3 F\big(A(t)\big)_3,$$

where $A(t) := A|_{\Omega\times\{t\}}$ and $F(A(t))_3$ is the curvature of $A(t)$ as a connection over the 3-manifold $\Omega \times \{t\}$. $*_3$ is the Hodge star on $\Omega \times \{t\}$.

Since $A(0)$ is reducible, there exists a real analytic gauge transformation u ($\neq \pm 1$) over Ω satisfying $u(A(0)) = A(0)$. Set $B := u(A)$ over $\Omega \times (-\varepsilon, \varepsilon)$. B is real analytic and satisfies

$$\frac{\partial}{\partial t} B(t) = *_3 F\big(B(t)\big)_3.$$

A and B are both real analytic and satisfy the same real analytic equation of the normal form with the same real analytic initial value $A(0) = B(0)$. Therefore Cauchy–Kovalevskaya's theorem implies $A = B = u(A)$. This means that A is reducible over an open set $\Omega \times (-\varepsilon, \varepsilon) \subset S^3 \times \mathbb{R}$. Then the unique continuation principle (see Donaldson and Kronheimer [7, Lemma 4.3.21]) implies that A is reducible all over $S^3 \times \mathbb{R}$. But this contradicts Corollary 6.3. □

7. Cut-off constructions

As we explained in Section 3, we need to define a 'cut-off' of $[\boldsymbol{A}] \in \mathcal{M}_d$. Section 7.1 is a preparation to define a cut-off construction, and we define it in Section 7.2.

Let $\delta_1 > 0$. We define $\delta_1' = \delta_1'(\delta_1)$ by

$$\delta_1' := \sup_{x\in S^3\times\mathbb{R}} \Bigg(\int\limits_{S^3\times(-\delta_1,\delta_1)} g(x,y)\, d\mathrm{vol}(y) \Bigg).$$

Since we have $g(x,y) \leqslant \mathrm{const}/d(x,y)^2$ (see (15) and (16)),

$$\int\limits_{d(x,y)\leqslant(\delta_1)^{1/4}} g(x,y)\, d\mathrm{vol}(y) \leqslant \mathrm{const} \int\limits_0^{(\delta_1)^{1/4}} r\, dr = \mathrm{const}' \sqrt{\delta_1},$$

$$\int\limits_{\{d(x,y)\geqslant(\delta_1)^{1/4}\}\cap S^3\times(-\delta_1,\delta_1)} g(x,y)\, d\mathrm{vol}(y) \leqslant \mathrm{const}\cdot \delta_1 \frac{1}{\sqrt{\delta_1}} = \mathrm{const}\sqrt{\delta_1}.$$

Hence $\delta_1' \leqslant \text{const}\sqrt{\delta_1}$ (this calculation is due to [5, pp. 190–191]). In particular, we have $\delta_1' \to 0$ as $\delta_1 \to 0$. For $d \geqslant 0$, we choose $\delta_1 = \delta_1(d)$ so that $0 < \delta_1 < 1$ and $\delta_1' = \delta_1'(\delta_1(d))$ satisfies

$$\left(5 + 7d + d^2\right)\delta_1' \leqslant \varepsilon_0/4 = 1/(4000). \tag{27}$$

The reason of this choice will be revealed in Proposition 7.6.

7.1. Gauge fixing on S^3 and gluing instantons

Let $F := S^3 \times SU(2)$ be the product principal $SU(2)$-bundle over S^3. Let $\mathcal{A}_{S^3}$ be the space of connections on F, and $\mathcal{G}$ be the gauge transformation group of F. $\mathcal{A}_{S^3}$ and $\mathcal{G}$ are equipped with the $\mathcal{C}^\infty$-topology. Set $\mathcal{B}_{S^3} := \mathcal{A}_{S^3}/\mathcal{G}$ (with the quotient topology), and let $\pi : \mathcal{A}_{S^3} \to \mathcal{B}_{S^3}$ be the natural projection. Note that the gauge transformations ± 1 trivially act on $\mathcal{A}_{S^3}$.

Proposition 7.1. *Let $d \geqslant 0$, and $A \in \mathcal{A}_{S^3}$ be an irreducible connection. There exist a closed neighborhood U_A of $[A]$ in $\mathcal{B}_{S^3}$ and a continuous map $\Phi_A : \pi^{-1}(U_A) \to \mathcal{G}/\{\pm 1\}$ such that, for any $B \in \pi^{-1}(U_A)$, $[g] := \Phi_A(B)$ satisfies the following.*

(i) *$g(B) = A + a$ with $\|a\|_{L^\infty} \leqslant \delta_1 = \delta_1(d)$. ($\delta_1$ is the positive constant chosen in the above* (27).)
(ii) *For any gauge transformation h of F, we have $\Phi_A(h(B)) = [gh^{-1}]$.*

Proof. Let $\varepsilon > 0$ be sufficiently small, and we take a closed neighborhood U_A of $[A]$ in $\mathcal{B}_{S^3}$ such that

$$U_A \subset \left\{[B] \in \mathcal{B}_{S^3} \,\middle|\, \exists g\colon \text{gauge transformation of } F \text{ s.t. } \left\|g(B) - A\right\|_{L^4_1} < \varepsilon\right\}.$$

The usual Coulomb gauge construction shows that, for each $B \in \pi^{-1}(U_A)$, there uniquely exists $[g] \in \mathcal{G}/\{\pm 1\}$ such that $g(B) = A + a$ with $d_A^* a = 0$ and $\|a\|_{L^4_1} \leqslant \text{const} \cdot \varepsilon$. Since $L^4_1(S^3) \hookrightarrow \mathcal{C}^0(S^3)$, we have $\|a\|_{L^\infty} \leqslant \text{const} \cdot \varepsilon \leqslant \delta_1$ for sufficiently small ε. We define $\Phi_A(B) := [g]$. Then the condition (i) is obviously satisfied, and the condition (ii) follows from the uniqueness of $[g]$. □

Proposition 7.2. *Let $d \geqslant 0$, and Θ be the product connection on $F = S^3 \times SU(2)$. There exist a closed neighborhood U_Θ of $[\Theta]$ in $\mathcal{B}_{S^3}$ and a continuous map $\Phi_\Theta : \pi^{-1}(U_\Theta) \to \mathcal{G}$ such that, for any $A \in \pi^{-1}(U_\Theta)$, $g := \Phi_\Theta(A)$ satisfies the following.*

(i) *$g(A) = \Theta + a$ with $\|a\|_{L^\infty} \leqslant \delta_1 = \delta_1(d)$.*
(ii) *For any gauge transformation h of F, there exists a constant gauge transformation h' of F (i.e. $h'(\Theta) = \Theta$) such that $\Phi_\Theta(h(A)) = h'gh^{-1}$.*

Proof. Fix a point $\theta_0 \in S^3$. Let $\varepsilon > 0$ be sufficiently small, and we take a closed neighborhood U_Θ of $[\Theta]$ in $\mathcal{B}_{S^3}$ such that

$$U_\Theta \subset \left\{[A] \in \mathcal{B}_{S^3} \,\middle|\, \exists g\colon \text{gauge transformation of } F \text{ s.t. } \left\|g(A) - \Theta\right\|_{L^4_1} < \varepsilon\right\}.$$

For any $A \in \pi^{-1}(U_\Theta)$, there uniquely exists a gauge transformation g with $g(\theta_0) = 1$ such that $g(A) = \Theta + a$ with $d_\Theta^* a = 0$ and $\|a\|_{L_1^4} \leqslant \text{const} \cdot \varepsilon\ (\leqslant \delta_1)$. We set $\Phi_\Theta(A) := g$. The condition (i) is obvious, and condition (ii) follows from $\Phi_\Theta(h(A)) = h(\theta_0)gh^{-1}$. (Here $h(\theta_0)$ is a constant gauge transformation. Note that $(h(\theta_0)gh^{-1})(\theta_0) = 1$.) □

Recall the settings in Section 1. Let $d \geqslant 0$. The moduli space $\mathcal{M}_d$ is the space of all gauge equivalence classes $[\boldsymbol{A}]$ where $\boldsymbol{A}$ is an ASD connection on $\boldsymbol{E} := X \times SU(2)$ satisfying $|F(\boldsymbol{A})| \leqslant d$.

We define $K_d \subset \mathcal{B}_{S^3}$ by

$$K_d := \{[\boldsymbol{A}|_{S^3\times\{0\}}] \in \mathcal{B}_{S^3} \mid [\boldsymbol{A}] \in \mathcal{M}_d\},$$

where we identify $\boldsymbol{E}|_{S^3\times\{0\}}$ with F. From the Uhlenbeck compactness [24,25], $\mathcal{M}_d$ is compact, and hence K_d is also compact. Proposition 6.4 implies that, for any $[\boldsymbol{A}] \in \mathcal{M}_d$, $\boldsymbol{A}|_{S^3\times\{0\}}$ is irreducible or a flat connection. (The important point is that $\boldsymbol{A}|_{S^3\times\{0\}}$ never be a non-flat reducible connection.)

Set $A_0 := \Theta$ (the product connection on F). There exist irreducible connections $A_1, A_2, \ldots, A_N$ ($N = N(d)$) on F such that $K_d \subset \mathrm{Int}(U_{A_0}) \cup \mathrm{Int}(U_{A_1}) \cup \cdots \cup \mathrm{Int}(U_{A_N})$ and $[A_i] \in K_d$ ($0 \leqslant i \leqslant N$). Here $\mathrm{Int}(U_{A_i})$ is the interior of the closed set U_{A_i} introduced in Propositions 7.1 and 7.2. Note that we can naturally identify K_d with the space $\{[\boldsymbol{A}|_{S^3\times\{T\}}] \in \mathcal{B}_{S^3} \mid [\boldsymbol{A}] \in \mathcal{M}_d\}$ for any real number T because $\mathcal{M}_d$ admits the natural $\mathbb{R}$-action.

For the statement of the next proposition, we introduce a new notation. We define $F \times \mathbb{R}$ as the pull-back of F by the natural projection $X = S^3 \times \mathbb{R} \to S^3$. So $F \times \mathbb{R}$ is a principal $SU(2)$-bundle over X. Of course, we can naturally identify $F \times \mathbb{R}$ with $\boldsymbol{E}$, but here we use this notation for the later convenience. We define $\hat{A}_0$ as the pull-back of Θ by the projection $X = S^3 \times \mathbb{R} \to S^3$. (Hence $\hat{A}_0$ is the product connection on $F \times \mathbb{R}$ under the natural identification $F \times \mathbb{R} = \boldsymbol{E}$.)

Proposition 7.3. *For each $i = 1, 2, \ldots, N$ there exists a connection $\hat{A}_i$ on $F \times \mathbb{R}$ satisfying the following.* (*Recall* $0 < \delta_1 < 1$.)

- (i) $\hat{A}_i = A_i$ *over* $S^3 \times [-\delta_1, \delta_1]$. *Here* A_i (*a connection on* $F \times \mathbb{R}$) *means the pull-back of* A_i (*a connection on* F) *by the natural projection* $X \to S^3$.
- (ii) $F(\hat{A}_i)$ *is supported in* $S^3 \times (-1, 1)$.
- (iii) $\|F^+(\hat{A}_i)|_{\delta_1<|t|<1}\|_{\mathrm{T}} \leqslant \varepsilon_0/4 = 1/(4000)$, *where* $F^+(\hat{A}_i)|_{\delta_1<|t|<1} = F^+(\hat{A}_i) \times 1_{\delta_1<|t|<1}$ *and* $1_{\delta_1<|t|<1}$ *is the characteristic function of the set* $\{(\theta, t) \in S^3 \times \mathbb{R} \mid \delta_1 < |t| < 1\}$.

Proof. By using a cut-off function, we can construct a connection A_i' on $F \times \mathbb{R}$ such that $A_i' = A_i$ over $S^3 \times [-\delta_1, \delta_1]$ and $\operatorname{supp} F(A_i') \subset S^3 \times (-1, 1)$. We can reduce the self-dual part of $F(A_i')$ by "gluing instantons" to A_i' over $\delta_1 < |t| < 1$. This technique is essentially well known for the specialists in the gauge theory. For the detail, see Donaldson [5, pp. 190–199].

By the argument of [5, pp. 196–198], we get the following situation. For any $\varepsilon > 0$, there exists a connection $\hat{A}_i$ satisfying the following. $\hat{A}_i = A_i' = A_i$ over $|t| \leqslant \delta_1$, and $\operatorname{supp} F(\hat{A}_i) \subset S^3 \times (-1, 1)$. Moreover $F^+(\hat{A}_i) = F_1^+ + F_2^+$ over $\delta_1 < |t| < 1$ such that $|F_1^+| \leqslant \varepsilon$ and

$$|F_2^+| \leqslant \text{const}, \qquad \mathrm{vol}(\operatorname{supp}(F_2^+)) \leqslant \varepsilon,$$

where const is a positive constant depending only on A_i' and independent of ε. If we take ε sufficiently small, then

$$\left\| F^+(\hat{A}_i)|_{\delta_1<|t|<1} \right\|_{\mathrm{T}} \leqslant \varepsilon_0/4. \qquad \square$$

7.2. Cut-off construction

Let T be a positive real number. We define a closed subset $\mathcal{M}_{d,T}(i,j) \subset \mathcal{M}_d$ $(0 \leqslant i, j \leqslant N = N(d))$ as the set of $[\boldsymbol{A}] \in \mathcal{M}_d$ satisfying $[\boldsymbol{A}|_{S^3\times\{T\}}] \in U_{A_i}$ and $[\boldsymbol{A}|_{S^3\times\{-T\}}] \in U_{A_j}$. Here we naturally identify $E_T := \boldsymbol{E}|_{S^3\times\{T\}}$ and $E_{-T} := \boldsymbol{E}|_{S^3\times\{-T\}}$ with F, and A_i $(0 \leqslant i \leqslant N)$ are the connections on F introduced in the previous subsection. We have

$$\mathcal{M}_d = \bigcup_{0\leqslant i,j\leqslant N} \mathcal{M}_{d,T}(i,j). \tag{28}$$

Of course, this decomposition depends on the parameter $T > 0$. The important point is that N is independent of T. We will define a cut-off construction for each piece $\mathcal{M}_{d,T}(i,j)$.

Let $\boldsymbol{A}$ be an ASD connection on $\boldsymbol{E}$ satisfying $[\boldsymbol{A}] \in \mathcal{M}_{d,T}(i,j)$. Let $u_+ : \boldsymbol{E}|_{t\geqslant T} \to E_T \times [T,+\infty)$ be the temporal gauge of $\boldsymbol{A}$ with $u_+ = \mathrm{id}$ on $\boldsymbol{E}|_{S^3\times\{T\}} = E_T$. (See Donaldson [6, Chapter 2].) Here $\boldsymbol{E}|_{t\geqslant T}$ is the restriction of $\boldsymbol{E}$ to $S^3 \times [T,+\infty)$, and $E_T \times [T,+\infty)$ is the pull-back of E_T by the projection $S^3 \times [T,\infty) \to S^3 \times \{T\}$. We will repeatedly use these kinds of notations. In the same way, let $u_- : \boldsymbol{E}|_{t\leqslant -T} \to E_{-T} \times (-\infty,-T]$ be the temporal gauge of $\boldsymbol{A}$ with $u_- = \mathrm{id}$ on $\boldsymbol{E}|_{S^3\times\{-T\}} = E_{-T}$. We define $A(t)$ $(|t| \geqslant T)$ by setting $A(t) := u_+(\boldsymbol{A})$ for $t \geqslant T$ and $A(t) := u_-(\boldsymbol{A})$ for $t \leqslant -T$. $A(t)$ becomes dt-part free. Since $\boldsymbol{A}$ is ASD, we have

$$\frac{\partial A(t)}{\partial t} = *_3 F\bigl(A(t)\bigr)_3, \tag{29}$$

where $*_3$ is the Hodge star on $S^3 \times \{t\}$ and $F(A(t))_3$ is the curvature of $A(t)$ as a connection on the 3-manifold $S^3 \times \{t\}$.

We have $[A(T)] \in U_{A_i}$ and $[A(-T)] \in U_{A_j}$. By using Propositions 7.1 and 7.2, we set $[g_+] := \Phi_{A_i}(A(T))$ if $i > 0$ and $g_+ := \Phi_\Theta(A(T))$ if $i = 0$. (If $i > 0$, the gauge transformation g_+ is not uniquely determined because there exists the ambiguity coming from ± 1. For this point, see Lemma 7.4 and its proof.) In the same way we set $[g_-] := \Phi_{A_j}(A(-T))$ if $j > 0$ and $g_- := \Phi_\Theta(A(-T))$ if $j = 0$. We consider g_+ (resp. g_-) as the gauge transformation of E_T (resp. E_{-T}). They satisfy

$$\left\| g_+\bigl(A(T)\bigr) - A_i \right\|_{L^\infty} \leqslant \delta_1, \qquad \left\| g_-\bigl(A(-T)\bigr) - A_j \right\|_{L^\infty} \leqslant \delta_1. \tag{30}$$

We define a principal $SU(2)$-bundle $\boldsymbol{E}'$ over X by

$$\boldsymbol{E}' := \boldsymbol{E}|_{|t|<T+\delta_1/4} \sqcup E_T \times (T,+\infty) \sqcup E_{-T} \times (-\infty,-T)/\sim,$$

where the identification $\sim$ is given as follows. $\boldsymbol{E}|_{|t|<T+\delta_1/4}$ is identified with $E_T \times (T,+\infty)$ over the region $T < t < T + \delta_1/4$ by the map $g_+ \circ u_+ : \boldsymbol{E}|_{T<t<T+\delta_1/4} \to E_T \times (T, T+\delta_1/4)$. Here we consider g_+ as a gauge transformation of $E_T \times (T, T+\delta_1/4)$ by $g_+ : E_T \times (T, T+\delta_1/4) \to E_T \times (T, T+\delta_1/4)$, $(p,t) \mapsto (g_+(p), t)$. Similarly, we identify $\boldsymbol{E}|_{|t|<T+\delta_1/4}$ with $E_{-T} \times$

$(-\infty, -T)$ over the region $-T - \delta/4 < t < -T$ by the map $g_- \circ u_- : \boldsymbol{E}|_{-T-\delta_1/4<t<-T} \to E_{-T} \times (-T - \delta_1/4, -T)$.

Let $\rho(t)$ be a smooth function on $\mathbb{R}$ such that $0 \leqslant \rho \leqslant 1$, $\rho = 0$ ($|t| \leqslant \delta_1/4$), $\rho = 1$ ($|t| \geqslant 3\delta_1/4$) and

$$\left|\rho'\right| \leqslant 4/\delta_1.$$

We define a (not necessarily ASD) connection $\boldsymbol{A}'$ on $\boldsymbol{E}'$ as follows. Over the region $|t| < T + \delta_1/4$ where $\boldsymbol{E}'$ is equal to $\boldsymbol{E}$, we set

$$\boldsymbol{A}' := \boldsymbol{A} \quad \text{on } \boldsymbol{E}|_{|t|<T+\delta_1/4}. \tag{31}$$

Over the region $t > T$, we set

$$\boldsymbol{A}' := \bigl(1 - \rho(t - T)\bigr) g_+\bigl(A(t)\bigr) + \rho(t - T)\hat{A}_{i,T} \quad \text{on } E_T \times (T, +\infty), \tag{32}$$

where $\hat{A}_{i,T}$ is the pull-back of the connection $\hat{A}_i$ introduced in the previous subsection (see Proposition 7.3) by the map $t \mapsto t - T$. So, in particular, $\hat{A}_{i,T} = A_i$ over $T - \delta_1 \leqslant t \leqslant T + \delta_1$ and $F(\hat{A}_{i,T}) = 0$ over $t \geqslant T + 1$. (32) is compatible with (31) over $T < t < T + \delta_1/4$ where $\rho(t - T) = 0$. In the same way, over the region $t < -T$, we set

$$\boldsymbol{A}' := \bigl(1 - \rho(t + T)\bigr) g_-\bigl(A(t)\bigr) + \rho(t + T)\hat{A}_{j,-T} \quad \text{on } E_{-T} \times (-\infty, -T).$$

We have $F(\boldsymbol{A}') = 0$ ($|t| \geqslant T + 1$). Then we have constructed $(\boldsymbol{E}', \boldsymbol{A}')$ from $\boldsymbol{A}$ with $[\boldsymbol{A}] \in \mathcal{M}_{d,T}(i, j)$.

Lemma 7.4. *The gauge equivalence class of $(\boldsymbol{E}', \boldsymbol{A}')$ depends only on the gauge equivalence class of $\boldsymbol{A}$.*

Proof. Suppose $[\boldsymbol{A}] \in \mathcal{M}_{d,T}(0, 1)$. Other cases can be proved in the same way. Let $h : \boldsymbol{E} \to \boldsymbol{E}$ be a gauge transformation and set $\boldsymbol{B} := h(\boldsymbol{A})$. Let $(\boldsymbol{E}'_{\boldsymbol{A}}, \boldsymbol{A}')$ and $(\boldsymbol{E}'_{\boldsymbol{B}}, \boldsymbol{B}')$ be the bundles and connections constructed by the above cut-off procedure form $\boldsymbol{A}$ and $\boldsymbol{B}$, respectively. Let $u_{\pm,\boldsymbol{A}}$ and $u_{\pm,\boldsymbol{B}}$ be the temporal gauges of $\boldsymbol{A}$ and $\boldsymbol{B}$ over $t \geqslant T$ or $t \leqslant -T$. We have $u_{\pm,\boldsymbol{B}} = h_{\pm T} \circ u_{\pm,\boldsymbol{A}} \circ h^{-1}$ where $h_{\pm T} := h|_{t=\pm T}$ on $E_{\pm T}$. Set $g_{+,\boldsymbol{A}} := \Phi_\Theta(A(T))$ and $g_{+,\boldsymbol{B}} := \Phi_\Theta(B(T)) = \Phi_\Theta(h_T(A(T)))$. From Proposition 7.2(ii), we have $g_{+,\boldsymbol{B}} = h' g_{+,\boldsymbol{A}} h_T^{-1}$, where h' is a constant gauge transformation of E_T ($h'(\Theta) = \Theta$). Set $[g_{-,\boldsymbol{A}}] := \Phi_{A_1}(A(-T))$ and $[g_{-,\boldsymbol{B}}] := \Phi_{A_1}(B(-T)) = \Phi_{A_1}(h_{-T}(A(-T)))$. We have $g_{-,\boldsymbol{B}} = \pm g_{-,\boldsymbol{A}} h_{-T}^{-1}$. We define a gauge transformation $g : \boldsymbol{E}'_{\boldsymbol{A}} \to \boldsymbol{E}'_{\boldsymbol{B}}$ by the following way: Over the region $|t| < T + \delta_1/4$, we set $g := h$ on $\boldsymbol{E}|_{|t|<T+\delta_1/4}$. Over the region $t > T$, we set $g := h'$ on $E_T \times (T, +\infty)$. Over the region $t < -T$, we set $g := \pm 1$ on $E_{-T} \times (-\infty, -T)$. We have $g(\boldsymbol{A}') = \boldsymbol{B}'$. Indeed, over the region $t > T$,

$$h'\bigl((1 - \rho) g_{+,\boldsymbol{A}}\bigl(A(t)\bigr) + \rho\Theta\bigr) = (1 - \rho) g_{+,\boldsymbol{B}}\bigl(B(t)\bigr) + \rho\Theta \quad \bigl(\rho = \rho(t - T)\bigr),$$

because $h' g_{+,\boldsymbol{A}} u_{+,\boldsymbol{A}} = g_{+,\boldsymbol{B}} u_{+,\boldsymbol{B}} h$ and $h'(\Theta) = \Theta$. $\square$

Lemma 7.5.

$$\left|F^+(\boldsymbol{A}')\right| \leqslant 5 + 7d + d^2 \quad \text{on } T \leqslant |t| \leqslant T + \delta_1.$$

Proof. We consider the case $T < t \leqslant T + \delta_1$ where $\hat{A}_{i,T} = A_i$. We have $\boldsymbol{A}' = (1-\rho)g_+(A(t)) + \rho A_i$, $\rho = \rho(t-T)$. Set $a := A_i - g_+(A(t))$. Then $\boldsymbol{A}' = g_+(A(t)) + \rho a$. We have

$$F^+(\boldsymbol{A}') = (\rho' dt \wedge a)^+ + \frac{\rho}{2}\big(F(A_i) + *_3 F(A_i) \wedge dt\big) + (\rho^2 - \rho)(a \wedge a)^+.$$

We have $|F(A_i)| \leqslant d$ and $|\rho'| \leqslant 4/\delta_1$. From (30), $|A_i - g_+(A(T))| \leqslant \delta_1$. From the ASD equation (29) and $|F(\boldsymbol{A})| \leqslant d$, $|A(t) - A(T)| \leqslant d|t-T| \leqslant d\delta_1$. Hence

$$|a| \leqslant \left|A_i - g_+\big(A(T)\big)\right| + \left|g_+\big(A(T)\big) - g_+\big(A(t)\big)\right| \leqslant (1+d)\delta_1 \quad (T \leqslant t \leqslant T + \delta_1). \tag{33}$$

Therefore, for $T \leqslant t \leqslant T + \delta_1$,

$$\left|F^+(\boldsymbol{A}')\right| \leqslant 4(1+d) + d + (1+d)^2 = 5 + 7d + d^2. \qquad \square$$

Proposition 7.6. *$F(\boldsymbol{A}') = 0$ over $|t| \geqslant T+1$, and $F^+(\boldsymbol{A}')$ is supported in $\{T < |t| < T+1\}$. We have $|F(\boldsymbol{A}')| \leqslant d$ over $|t| \leqslant T$, and*

$$\left\|F^+(\boldsymbol{A}')\right\|_{L^\infty} \leqslant d', \qquad \left\|F^+(\boldsymbol{A}')\right\|_{\mathrm{T}} \leqslant \varepsilon_0 = 1/(1000), \tag{34}$$

where $d' = d'(d)$ is a positive constant depending only on d. Moreover

$$\frac{1}{8\pi^2}\int_X tr\big(F(\boldsymbol{A}')^2\big) \leqslant \frac{1}{8\pi^2}\int_{|t|\leqslant T} \left|F(\boldsymbol{A})\right|^2 d\mathrm{vol} + C_1(d) \leqslant \frac{2T\, d^2 \mathrm{vol}(S^3)}{8\pi^2} + C_1(d).$$

Here $C_1(d)$ depends only on d.

Proof. The statements about the supports of $F(\boldsymbol{A}')$ and $F^+(\boldsymbol{A}')$ are obvious by the construction. Since $\boldsymbol{A}' = \boldsymbol{A}$ over $|t| \leqslant T$, $|F(\boldsymbol{A}')| \leqslant d$ over $|t| \leqslant T$. We have $\boldsymbol{A}' = \hat{A}_{i,T}$ for $t \geqslant T + \delta_1$ and $\boldsymbol{A}' = \hat{A}_{j,-T}$ for $t \leqslant -T - \delta_1$. Hence (from Lemma 7.5)

$$\left\|F^+(\boldsymbol{A}')\right\|_{L^\infty} \leqslant d' := \max\big(5 + 7d + d^2, \left\|F^+(\hat{A}_1)\right\|_{L^\infty}, \left\|F^+(\hat{A}_2)\right\|_{L^\infty}, \dots, \left\|F^+(\hat{A}_N)\right\|_{L^\infty}\big).$$

By using Lemma 7.5, (27) and Proposition 7.3(iii) (note that $g(x,y)$ is invariant under the translations $t \mapsto t - T$ and $t \mapsto t + T$),

$$\left\|F^+(\boldsymbol{A}')\right\|_{\mathrm{T}} \leqslant 2\big(5 + 7d + d^2\big)\delta_1' + \varepsilon_0/2 \leqslant \varepsilon_0.$$

We have $\boldsymbol{A}' = \boldsymbol{A}$ over $|t| \leqslant T$ and

$$F(\boldsymbol{A}') = (1-\rho)g_+ \circ u_+\big(F(\boldsymbol{A})\big) + \rho F(A_i) + \rho' dt \wedge a + (\rho^2 - \rho)a^2,$$

over $T < t < T + \delta_1$. Hence $|F(A')| \leqslant \text{const}_d$ over $T < |t| < T + \delta_1$ by using (33). Then the last statement can be easily proved. □

7.3. Continuity of the cut-off

Fix $0 \leqslant i, j \leqslant N$. Let $[\boldsymbol{A}_n]$ $(n \geqslant 1)$ be a sequence in $\mathcal{M}_{d,T}(i, j)$ converging to $[\boldsymbol{A}] \in \mathcal{M}_{d,T}(i, j)$ in the $\mathcal{C}^\infty$-topology over every compact subset in X. Let $[\boldsymbol{E}'_n, \boldsymbol{A}'_n]$ (resp. $[\boldsymbol{E}', \boldsymbol{A}']$) be the gauge equivalence classes of the connections constructed by cutting off $[\boldsymbol{A}_n]$ (resp. $[\boldsymbol{A}]$) as in Section 7.2.

Lemma 7.7. *There are gauge transformations $h_n : \boldsymbol{E}'_n \to \boldsymbol{E}'$ $(n \gg 1)$ such that $h_n(\boldsymbol{A}'_n) = \boldsymbol{A}'$ for $|t| \geqslant T + 1$ and $h_n(\boldsymbol{A}'_n)$ converges to $\boldsymbol{A}'$ in the $\mathcal{C}^\infty$-topology over X.* (*Indeed, we will need only $\mathcal{C}^1$-convergence in the later argument.*)

Proof. We can suppose that $\boldsymbol{A}_n$ converges to $\boldsymbol{A}$ in the $\mathcal{C}^\infty$-topology over $|t| \leqslant T + 2$. Let $u_{+,n} : \boldsymbol{E}|_{t \geqslant T} \to E_T \times [T, +\infty)$ (resp. u_+) be the temporal gauge of $\boldsymbol{A}_n$ (resp. $\boldsymbol{A}$), and set $A_n(t) := u_{+,n}(\boldsymbol{A}_n)$ and $A(t) := u_+(\boldsymbol{A})$ for $t \geqslant T$. We set $[g_{+,n}] := \Phi_{A_i}(A_n(T))$ and $[g_+] := \Phi_{A_i}(A(T))$ if $i > 0$, and we set $g_{+,n} := \Phi_\Theta(A_n(T))$ and $g_+ := \Phi_\Theta(A(T))$ if $i = 0$.

$u_{+,n}$ converges to u_+ in the $\mathcal{C}^\infty$-topology over $T \leqslant t \leqslant T + 1$, and we can suppose that $g_{+,n}$ converges to g_+ in the $\mathcal{C}^\infty$-topology. Hence there are $\chi_n \in \Gamma(S^3 \times [T, T+1], \text{ad}\, E_T \times [T, T+1])$ $(n \gg 1)$ satisfying $g_+ \circ u_+ = e^{\chi_n} g_{+,n} \circ u_{+,n}$. $\chi_n \to 0$ in the $\mathcal{C}^\infty$-topology over $T \leqslant t \leqslant T + 1$. Let φ be a smooth function on X such that $0 \leqslant \varphi \leqslant 1$, $\varphi = 1$ over $t \leqslant T + \delta_1$ and $\varphi = 0$ over $t \geqslant T + 1$. We define $h_n : \boldsymbol{E}'_n \to \boldsymbol{E}'$ $(n \gg 1)$ as follows.

- (i) In the case of $|t| < T + \delta_1/4$, we set $h_n := \text{id} : \boldsymbol{E} \to \boldsymbol{E}$.
- (ii) In the case of $t > T$, we set $h_n := e^{\varphi \chi_n} : E_T \times (T, +\infty) \to E_T \times (T, +\infty)$. This is compatible with the case (i).
- (iii) In the case of $t < -T$, we define $h_n : E_{-T} \times (-\infty, -T) \to E_{-T} \times (-\infty, -T)$ in the same way as in the above (ii).

Then we can easily check that these h_n satisfy the required properties. □

8. Proofs of the upper bounds

8.1. Proof of $\dim(\mathcal{M}_d : \mathbb{R}) < \infty$

As in Section 1, $\boldsymbol{E} = X \times SU(2)$ and $\mathcal{M}_d$ $(d \geqslant 0)$ is the space of all gauge equivalence classes $[\boldsymbol{A}]$ where $\boldsymbol{A}$ is an ASD connection on $\boldsymbol{E}$ satisfying $\|F(\boldsymbol{A})\|_{L^\infty} \leqslant d$. We define a distance on $\mathcal{M}_d$ as follows. For $[\boldsymbol{A}], [\boldsymbol{B}] \in \mathcal{M}_d$, we set

$$\text{dist}\big([\boldsymbol{A}], [\boldsymbol{B}]\big) := \inf_{g:\boldsymbol{E}\to\boldsymbol{E}} \left\{ \sum_{n\geqslant 1} 2^{-n} \frac{\|g(\boldsymbol{A}) - \boldsymbol{B}\|_{L^\infty(|t|\leqslant n)}}{1 + \|g(\boldsymbol{A}) - \boldsymbol{B}\|_{L^\infty(|t|\leqslant n)}} \right\},$$

where g runs over all gauge transformations of $\boldsymbol{E}$, and $|t| \leqslant n$ means the region $\{(\theta, t) \in S^3 \times \mathbb{R} \mid |t| \leqslant n\}$. This distance is compatible with the topology of $\mathcal{M}_d$ introduced in Section 1. For

$R = 1, 2, 3, \dots$, we define an amenable sequence $\Omega_R \subset \mathbb{R}$ by $\Omega_R = \{s \in \mathbb{R} \mid -R \leqslant s \leqslant R\}$. We define $\mathrm{dist}_{\Omega_R}([\boldsymbol{A}], [\boldsymbol{B}])$ as in Section 2.1, i.e.,

$$\mathrm{dist}_{\Omega_R}\big([\boldsymbol{A}], [\boldsymbol{B}]\big) := \sup_{s \in \Omega_R} \mathrm{dist}\big(\big[s^*\boldsymbol{A}\big], \big[s^*\boldsymbol{B}\big]\big),$$

where $s^*\boldsymbol{A}$ is the pull-back of $\boldsymbol{A}$ by $s : \boldsymbol{E} \to \boldsymbol{E}$.

Let $\varepsilon > 0$. We take a positive integer $L = L(\varepsilon)$ so that

$$\sum_{n>L} 2^{-n} < \varepsilon/2. \tag{35}$$

We define $D = D(d, d', \varepsilon/4)$ as the positive number introduced in Lemma 4.14, where $d' = d'(d)$ is the positive constant introduced in Proposition 7.6. We set $T = T(R, d, \varepsilon) = R + L + D > 0$.

We have the decomposition $\mathcal{M}_d = \bigcup_{0 \leqslant i, j \leqslant N} \mathcal{M}_{d,T}(i, j)$ $(N = N(d))$ as in Section 7.2. $\mathcal{M}_{d,T}(i, j)$ is the space of $[\boldsymbol{A}] \in \mathcal{M}_d$ satisfying $[\boldsymbol{A}|_{S^3 \times \{T\}}] \in U_{A_i}$ and $[\boldsymbol{A}|_{S^3 \times \{-T\}}] \in U_{A_j}$. Fix $0 \leqslant i, j \leqslant N$. Let $\boldsymbol{A}$ be an ASD connection on $\boldsymbol{E}$ satisfying $[\boldsymbol{A}] \in \mathcal{M}_{d,T}(i, j)$. By the cut-off construction in Section 7.2, we have constructed $(\boldsymbol{E}', \boldsymbol{A}')$ satisfying the following conditions (see Proposition 7.6). $\boldsymbol{E}'$ is a principal $SU(2)$-bundle over X, and $\boldsymbol{A}'$ is a connection on $\boldsymbol{E}'$ such that $F(\boldsymbol{A}') = 0$ for $|t| \geqslant T + 1$, $F^+(\boldsymbol{A}')$ is supported in $\{T < |t| < T + 1\}$, and that

$$\big\|F^+\big(\boldsymbol{A}'\big)\big\|_{\mathrm{T}} \leqslant \varepsilon_0, \qquad \big\|F^+\big(\boldsymbol{A}'\big)\big\|_{L^\infty} \leqslant d', \qquad \big\|F\big(\boldsymbol{A}'\big)\big\|_{L^\infty(|t| \leqslant T)} \leqslant d.$$

We can identify $\boldsymbol{E}'$ with $\boldsymbol{E}$ over $|t| < T + \delta_1/4$ by the definition, and

$$\boldsymbol{A}'|_{|t|<T+\delta_1/4} = \boldsymbol{A}|_{|t|<T+\delta_1/4}. \tag{36}$$

$(\boldsymbol{E}', \boldsymbol{A}')$ satisfies the conditions (i), (ii), (iii) in the beginning of Section 4.1. Therefore, by using the perturbation argument in Section 4 (see Proposition 4.12), we can construct the ASD connection $\boldsymbol{A}'' := \boldsymbol{A}' + d^*_{\boldsymbol{A}'}\phi_{\boldsymbol{A}'}$ on $\boldsymbol{E}'$. By Lemma 4.14,

$$\big|\boldsymbol{A} - \boldsymbol{A}''\big| = \big|\boldsymbol{A}' - \boldsymbol{A}''\big| \leqslant \varepsilon/4 \quad \big(|t| \leqslant T - D = R + L\big). \tag{37}$$

From Propositions 7.6 and 4.12(b),

$$\begin{aligned} \frac{1}{8\pi^2} \int_X \big|F\big(\boldsymbol{A}''\big)\big|^2 d\mathrm{vol} &= \frac{1}{8\pi^2} \int_X tr\big(F\big(\boldsymbol{A}'\big)^2\big) \\ &\leqslant \frac{1}{8\pi^2} \int_{|t| \leqslant T} \big|F(\boldsymbol{A})\big|^2 d\mathrm{vol} + C_1(d) \leqslant \frac{2T\, d^2 \mathrm{vol}(S^3)}{8\pi^2} + C_1(d), \end{aligned} \tag{38}$$

where $C_1(d)$ is a positive constant depending only on d. Since the cut-off and perturbation constructions respect the gauge symmetry (see Proposition 4.12 and Lemma 7.4), the gauge equivalence class $[\boldsymbol{E}', \boldsymbol{A}'']$ depends only on the gauge equivalence class $[\boldsymbol{A}]$. We set $F_{i,j}([\boldsymbol{A}]) := [\boldsymbol{E}', \boldsymbol{A}'']$.

For $c \geqslant 0$, we define $M(c)$ as the space of all gauge equivalence classes $[E, A]$ satisfying the following. E is a principal $SU(2)$-bundle over X, and A is an ASD connection on E satisfying

$$\frac{1}{8\pi^2}\int_X |F_A|^2\,d\mathrm{vol} \leqslant c.$$

The topology of $M(c)$ is defined as follows. A sequence $[E_n, A_n] \in M(c)$ $(n \geqslant 1)$ converges to $[E, A] \in M(c)$ if the following two conditions are satisfied:

(i) $\int_X |F(A_n)|^2\,d\mathrm{vol} = \int_X |F(A)|^2\,d\mathrm{vol}$ for $n \gg 1$.
(ii) There are gauge transformations $g_n : E_n \to E$ $(n \gg 1)$ such that for any compact set $K \subset X$ we have $\|g_n(A_n) - A\|_{\mathcal{C}^0(K)} \to 0$.

Using the index theorem, we have

$$\dim M(c) \leqslant 8c. \tag{39}$$

Here $\dim M(c)$ denotes the topological covering dimension of $M(c)$. By (38), we get the map

$$F_{i,j} : \mathcal{M}_{d,T}(i,j) \to M\left(\frac{2T\,d^2\mathrm{vol}(S^3)}{8\pi^2} + C_1(d)\right), \quad [\boldsymbol{A}] \mapsto \left[\boldsymbol{E}', \boldsymbol{A}''\right].$$

Lemma 8.1. *For $[\boldsymbol{A}_1]$ and $[\boldsymbol{A}_2]$ in $\mathcal{M}_{d,T}(i,j)$, if $F_{i,j}([\boldsymbol{A}_1]) = F_{i,j}([\boldsymbol{A}_2])$, then*

$$\mathrm{dist}_{\Omega_R}\big([\boldsymbol{A}_1], [\boldsymbol{A}_2]\big) < \varepsilon.$$

Proof. From (36) and (37), there exists a gauge transformation g of $\boldsymbol{E}$ defined over $|t| < T + \delta_1/4$ such that $|g(\boldsymbol{A}_1) - \boldsymbol{A}_2| \leqslant \varepsilon/2$ over $|t| \leqslant R + L$. There exists a gauge transformation $\tilde{g}$ of $\boldsymbol{E}$ defined all over X satisfying $\tilde{g} = g$ on $|t| \leqslant T = R + L + D$. Then we have $|\tilde{g}(\boldsymbol{A}_1) - \boldsymbol{A}_2| \leqslant \varepsilon/2$ on $|t| \leqslant R + L$. For $s \in \Omega_R$ (i.e. $|s| \leqslant R$), by using (35),

$$\begin{aligned}\mathrm{dist}\big(\big[s^*\boldsymbol{A}_1\big], \big[s^*\boldsymbol{A}_2\big]\big) &\leqslant \sum_{n\geqslant 1} 2^{-n}\frac{\|\tilde{g}(\boldsymbol{A}_1) - \boldsymbol{A}_2\|_{L^\infty(|t-s|\leqslant n)}}{1 + \|\tilde{g}(\boldsymbol{A}_1) - \boldsymbol{A}_2\|_{L^\infty(|t-s|\leqslant n)}} \\ &\leqslant \sum_{n=1}^{L} 2^{-n}(\varepsilon/2) + \sum_{n>L} 2^{-n} < \varepsilon/2 + \varepsilon/2 = \varepsilon. \qquad \Box\end{aligned}$$

Lemma 8.2. *The map $F_{i,j} : \mathcal{M}_{d,T}(i,j) \to M(\frac{2T\,d^2\mathrm{vol}(S^3)}{8\pi^2} + C_1(d))$ is continuous.*

Proof. Let $[\boldsymbol{A}_n] \in \mathcal{M}_{d,T}(i,j)$ be a sequence converging to $[\boldsymbol{A}] \in \mathcal{M}_{d,T}(i,j)$. From Lemma 7.7, there are gauge transformations $h_n : \boldsymbol{E}'_n \to \boldsymbol{E}'$ $(n \gg 1)$ such that $h_n(\boldsymbol{A}'_n) = \boldsymbol{A}'$ over $|t| \geqslant T + 1$ and that $h_n(\boldsymbol{A}'_n)$ converges to $\boldsymbol{A}'$ in the $\mathcal{C}^\infty$-topology over X. Since the perturbation construction in Section 4 is gauge equivariant (Proposition 4.12), we have

$$\big(\boldsymbol{E}', h_n\big(\boldsymbol{A}'_n\big) + d^*_{h_n(\boldsymbol{A}'_n)}\phi_{h_n(\boldsymbol{A}'_n)}\big) = \big(h_n\big(\boldsymbol{E}'_n\big), h_n\big(\boldsymbol{A}''_n\big)\big).$$

From Proposition 5.6, $d^*_{h_n(\boldsymbol{A}'_n)}\phi_{h_n(\boldsymbol{A}'_n)}$ converges to $d^*_{\boldsymbol{A}'}\phi_{\boldsymbol{A}'}$ in the $\mathcal{C}^0$-topology over every compact subset in X and

$$\int_X \left|F\left(h_n\left(\boldsymbol{A}'_n\right)+d^*_{h_n(\boldsymbol{A}'_n)}\phi_{h_n(\boldsymbol{A}'_n)}\right)\right|^2 d\mathrm{vol}=\int_X \left|F\left(\boldsymbol{A}'+d^*_{\boldsymbol{A}'}\phi_{\boldsymbol{A}'}\right)\right|^2 d\mathrm{vol}\quad \text{for } n\gg 1.$$

This shows that the sequence $[\boldsymbol{E}'_n,\boldsymbol{A}''_n]=[\boldsymbol{E}',h_n(\boldsymbol{A}'_n)+d^*_{h_n(\boldsymbol{A}'_n)}\phi_{h_n(\boldsymbol{A}'_n)}]$ $(n\geqslant 1)$ converges to $[\boldsymbol{E}',\boldsymbol{A}'']=[\boldsymbol{E}',\boldsymbol{A}'+d^*_{\boldsymbol{A}'}\phi_{\boldsymbol{A}'}]$ in $M_T(\frac{2T\,d^2\mathrm{vol}(S^3)}{8\pi^2}+C_1(d))$. $\square$

From Lemmas 8.1 and 8.2, $F_{i,j}$ becomes an ε-embedding with respect to the distance dist_{Ω_R}. Hence

$$\mathrm{Widim}_\varepsilon\left(\mathcal{M}_{d,T}(i,j),\mathrm{dist}_{\Omega_R}\right)\leqslant \dim M\left(\frac{2T\,d^2\mathrm{vol}(S^3)}{8\pi^2}+C_1(d)\right).$$

Since $\mathcal{M}_d=\bigcup_{0\leqslant i,j\leqslant N}\mathcal{M}_{d,T}(i,j)$ (each $\mathcal{M}_{d,T}(i,j)$ is a closed set), by using Lemma 2.3, we get

$$\mathrm{Widim}_\varepsilon(\mathcal{M}_d,\mathrm{dist}_{\Omega_R})\leqslant (N+1)^2\dim M\left(\frac{2T\,d^2\mathrm{vol}(S^3)}{8\pi^2}+C_1(d)\right)+(N+1)^2-1.$$

From (39) and $T=R+L+D$,

$$\dim M\left(\frac{2T\,d^2\mathrm{vol}(S^3)}{8\pi^2}+C_1(d)\right)\leqslant \frac{2(R+L+D)\,d^2\mathrm{vol}(S^3)}{\pi^2}+8C_1(d).$$

Since $N=N(d)$, $L=L(\varepsilon)$, $D=D(d,d'(d),\varepsilon/4)$ are independent of R, we get

$$\mathrm{Widim}_\varepsilon(\mathcal{M}_d:\mathbb{R})=\lim_{R\to\infty}\frac{\mathrm{Widim}_\varepsilon(\mathcal{M}_d,\mathrm{dist}_{\Omega_R})}{|\Omega_R|}\leqslant\frac{(N+1)^2\,d^2\mathrm{vol}(S^3)}{\pi^2}.$$

This holds for any $\varepsilon>0$. Thus

$$\dim(\mathcal{M}_d:\mathbb{R})=\lim_{\varepsilon\to 0}\mathrm{Widim}_\varepsilon(\mathcal{M}_d:\mathbb{R})\leqslant\frac{(N+1)^2\,d^2\mathrm{vol}(S^3)}{\pi^2}<\infty.$$

8.2. Upper bound on the local mean dimension

Lemma 8.3. *There exists $r_1=r_1(d)>0$ satisfying the following. For any $[\boldsymbol{A}]\in\mathcal{M}_d$ and $s\in\mathbb{R}$, there exists an integer i $(0\leqslant i\leqslant N)$ such that if $[\boldsymbol{B}]\in\mathcal{M}_d$ satisfies* $\mathrm{dist}_{\mathbb{R}}([\boldsymbol{A}],[\boldsymbol{B}])\leqslant r_1$ *then*

$$[\boldsymbol{B}|_{S^3\times\{s\}}]\in U_{A_i}.$$

Here we identify $\boldsymbol{E}|_{S^3\times\{s\}}$ with F, and U_{A_i} is the closed set introduced in Section 7.1. *Recall* $\mathrm{dist}_{\mathbb{R}}([\boldsymbol{A}],[\boldsymbol{B}])=\sup_{s\in\mathbb{R}}\mathrm{dist}([s^*\boldsymbol{A}],[s^*\boldsymbol{B}])$.

Proof. There exists $r_1 > 0$ (the Lebesgue number) satisfying the following. For any $[\boldsymbol{A}] \in \mathcal{M}_d$, there exists $i = i([\boldsymbol{A}])$ such that if $[\boldsymbol{B}] \in \mathcal{M}_d$ satisfies $\mathrm{dist}([\boldsymbol{A}], [\boldsymbol{B}]) \leqslant r_1$ then $[\boldsymbol{B}|_{S^3\times\{0\}}] \in U_{A_i}$. If $\mathrm{dist}_{\mathbb{R}}([\boldsymbol{A}], [\boldsymbol{B}]) \leqslant r_1$, then for each $s \in \mathbb{R}$ we have $\mathrm{dist}([s^*\boldsymbol{A}], [s^*\boldsymbol{B}]) \leqslant r_1$ and hence

$$[\boldsymbol{B}|_{S^3\times\{s\}}] = \left[\left(s^*\boldsymbol{B}\right)\big|_{S^3\times\{0\}}\right] \in U_{A_i},$$

for $i = i([s^*\boldsymbol{A}])$. $\square$

Lemma 8.4. *For any $\varepsilon' > 0$, there exists $r_2 = r_2(\varepsilon') > 0$ such that if $[\boldsymbol{A}]$ and $[\boldsymbol{B}]$ in $\mathcal{M}_d$ satisfy $\mathrm{dist}_{\mathbb{R}}([\boldsymbol{A}], [\boldsymbol{B}]) \leqslant r_2$ then*

$$\left\| \left|F(\boldsymbol{A})\right|^2 - \left|F(\boldsymbol{B})\right|^2 \right\|_{L^\infty(X)} \leqslant \varepsilon'.$$

Proof. The map $\mathcal{M}_d \ni [\boldsymbol{A}] \mapsto |F(\boldsymbol{A})|^2 \in \mathcal{C}^0(S^3 \times [0,1])$ is continuous. Hence there exists $r_2 > 0$ such that if $\mathrm{dist}([\boldsymbol{A}], [\boldsymbol{B}]) \leqslant r_2$ then

$$\left\| \left|F(\boldsymbol{A})\right|^2 - \left|F(\boldsymbol{B})\right|^2 \right\|_{L^\infty(S^3\times[0,1])} \leqslant \varepsilon'.$$

Then for each $s \in \mathbb{R}$, if $\mathrm{dist}([s^*\boldsymbol{A}], [s^*\boldsymbol{B}]) \leqslant r_2$,

$$\left\| \left|F(\boldsymbol{A})\right|^2 - \left|F(\boldsymbol{B})\right|^2 \right\|_{L^\infty(S^3\times[s,s+1])} \leqslant \varepsilon'.$$

Therefore if $\mathrm{dist}_{\mathbb{R}}([\boldsymbol{A}], [\boldsymbol{B}]) \leqslant r_2$, then $\||F(\boldsymbol{A})|^2 - |F(\boldsymbol{B})|^2\|_{L^\infty(X)} \leqslant \varepsilon'$. $\square$

Let $[\boldsymbol{A}] \in \mathcal{M}_d$, and $\varepsilon, \varepsilon' > 0$ be arbitrary two positive numbers. There exists $T_0 = T_0([\boldsymbol{A}], \varepsilon') > 0$ such that for any $T_1 \geqslant T_0$

$$\frac{1}{8\pi^2 T_1} \sup_{t\in\mathbb{R}} \int\limits_{S^3\times[t,t+T_1]} \left|F(\boldsymbol{A})\right|^2 d\mathrm{vol} \leqslant \rho(\boldsymbol{A}) + \varepsilon'/2.$$

The important point for the later argument is the following: We can arrange T_0 so that $T_0([s^*\boldsymbol{A}], \varepsilon') = T_0([\boldsymbol{A}], \varepsilon')$ for all $s \in \mathbb{R}$. We set

$$r = r\left(d, \varepsilon'\right) = \min\left(r_1(d), r_2\left(\frac{4\pi^2\varepsilon'}{\mathrm{vol}(S^3)}\right)\right),$$

where $r_1(\cdot)$ and $r_2(\cdot)$ are the positive constants introduced in Lemmas 8.3 and 8.4. By Lemma 8.4, if $[\boldsymbol{B}] \in B_r([\boldsymbol{A}])_{\mathbb{R}}$ (the closed ball of radius r centered at $[\boldsymbol{A}]$ in $\mathcal{M}_d$ with respect to the distance $\mathrm{dist}_{\mathbb{R}}$), then for any $T_1 \geqslant T_0$

$$\frac{1}{8\pi^2 T_1} \sup_{t\in\mathbb{R}} \int\limits_{S^3\times[t,t+T_1]} \left|F(\boldsymbol{B})\right|^2 d\mathrm{vol} \leqslant \rho(\boldsymbol{A}) + \varepsilon'/2 + \varepsilon'/2 = \rho(\boldsymbol{A}) + \varepsilon'. \tag{40}$$

We define positive numbers $L = L(\varepsilon)$ and $D = D(d, d'(d), \varepsilon/4)$ as in the previous subsection. ($L = L(\varepsilon)$ is a positive integer satisfying (35), and $D = D(d, d'(d), \varepsilon/4)$ is the positive number introduced in Lemma 4.14.) Let R be an integer with $R \geqslant T_0$, and set $T := R + L + D$. By Lemma 8.3, there exist i, j $(0 \leqslant i, j \leqslant N)$ depending on $[\boldsymbol{A}]$ and T such that all $[\boldsymbol{B}] \in B_r([\boldsymbol{A}])_{\mathbb{R}}$ satisfy $[\boldsymbol{B}|_{S^3\times\{T\}}] \in U_{A_i}$ and $[\boldsymbol{B}|_{S^3\times\{-T\}}] \in U_{A_j}$. (That is, $B_r([\boldsymbol{A}])_{\mathbb{R}} \subset \mathcal{M}_{d,T}(i,j)$.)

As in the previous subsection, by using the cut-off construction and perturbation, for each $[\boldsymbol{B}] \in B_r([\boldsymbol{A}])_{\mathbb{R}}$ we can construct the ASD connection $[\boldsymbol{E}', \boldsymbol{B}'']$. By (38), (40) and $T \geqslant T_0$,

$$\frac{1}{8\pi^2}\int_X \left|F\left(\boldsymbol{B}''\right)\right|^2 d\mathrm{vol} \leqslant \frac{1}{8\pi^2}\int_{|t|\leqslant T} \left|F(\boldsymbol{B})\right|^2 d\mathrm{vol} + C_1(d) \leqslant 2T\left(\rho(\boldsymbol{A})+\varepsilon'\right) + C_1(d),$$

where $C_1(d)$ depends only on d. Therefore we get the map

$$B_r\left([\boldsymbol{A}]\right)_{\mathbb{R}} \to M\left(2T\left(\rho(\boldsymbol{A})+\varepsilon'\right)+C_1(d)\right), \quad [\boldsymbol{B}] \mapsto \left[\boldsymbol{E}', \boldsymbol{B}''\right].$$

This is an ε-embedding with respect to the distance dist_{Ω_R} by Lemmas 8.1 and 8.2. Therefore we get (by (39))

$$\mathrm{Widim}_\varepsilon\left(B_r\left([\boldsymbol{A}]\right)_{\mathbb{R}}, \mathrm{dist}_{\Omega_R}\right) \leqslant 16T\left(\rho(\boldsymbol{A})+\varepsilon'\right) + 8C_1(d),$$

for $R \geqslant T_0([\boldsymbol{A}], \varepsilon')$ and $r = r(d, \varepsilon')$. As we pointed out before, we have $T_0([s^*\boldsymbol{A}], \varepsilon') = T_0([\boldsymbol{A}], \varepsilon')$ for $s \in \mathbb{R}$. Hence for all $s \in \mathbb{R}$ and $R \geqslant T_0 = T_0([\boldsymbol{A}], \varepsilon')$, we have the same upper bound on $\mathrm{Widim}_\varepsilon(B_r([s^*\boldsymbol{A}])_{\mathbb{R}}, \mathrm{dist}_{\Omega_R})$. Then for $R \geqslant T_0$,

$$\frac{1}{|\Omega_R|}\sup_{s\in\mathbb{R}} \mathrm{Widim}_\varepsilon\left(B_r\left(\left[s^*\boldsymbol{A}\right]\right)_{\mathbb{R}}, \mathrm{dist}_{\Omega_R}\right) \leqslant \frac{16T(\rho(\boldsymbol{A})+\varepsilon')+8C_1(d)}{2R}.$$

$T = R + L + D$. Here $L = L(\varepsilon)$ and $D = D(d, d'(d), \varepsilon/4)$ are independent of R. Hence

$$\begin{aligned}\mathrm{Widim}_\varepsilon\left(B_r\left([\boldsymbol{A}]\right)_{\mathbb{R}} \subset \mathcal{M}_d : \mathbb{R}\right) &= \lim_{R\to\infty}\left(\frac{1}{|\Omega_R|}\sup_{s\in\mathbb{R}} \mathrm{Widim}_\varepsilon\left(B_r\left(\left[s^*\boldsymbol{A}\right]\right)_{\mathbb{R}}, \mathrm{dist}_{\Omega_R}\right)\right)\\ &\leqslant 8\left(\rho(\boldsymbol{A})+\varepsilon'\right).\end{aligned}$$

Here we have used (6). This holds for any $\varepsilon > 0$. (Note that $r = r(d, \varepsilon')$ is independent of ε.) Hence

$$\dim\left(B_r\left([\boldsymbol{A}]\right)_{\mathbb{R}} \subset \mathcal{M}_d : \mathbb{R}\right) = \lim_{\varepsilon\to 0} \mathrm{Widim}_\varepsilon\left(B_r\left([\boldsymbol{A}]\right)_{\mathbb{R}} \subset \mathcal{M}_d : \mathbb{R}\right) \leqslant 8\left(\rho(\boldsymbol{A})+\varepsilon'\right).$$

Since $\dim_{[\boldsymbol{A}]}(\mathcal{M}_d : \mathbb{R}) \leqslant \dim(B_r([\boldsymbol{A}])_{\mathbb{R}} \subset \mathcal{M}_d : \mathbb{R})$,

$$\dim_{[\boldsymbol{A}]}(\mathcal{M}_d : \mathbb{R}) \leqslant 8\left(\rho(\boldsymbol{A})+\varepsilon'\right).$$

This holds for any $\varepsilon' > 0$. Thus

$$\dim_{[\boldsymbol{A}]}(\mathcal{M}_d : \mathbb{R}) \leqslant 8\rho(\boldsymbol{A}).$$

Therefore we get the conclusion:

Theorem 8.5. *For any* $[\boldsymbol{A}] \in \mathcal{M}_d$,

$$\dim_{[\boldsymbol{A}]}(\mathcal{M}_d : \mathbb{R}) \leqslant 8\rho(\boldsymbol{A}).$$

9. Analytic preliminaries for the lower bound

Let $T > 0$ be a positive real number, $\underline{\mathrm{E}}$ be a principal $SU(2)$-bundle over $S^3 \times (\mathbb{R}/T\mathbb{Z})$, and $\underline{\mathrm{A}}$ be an ASD connection on $\underline{\mathrm{E}}$. Suppose $\underline{\mathrm{A}}$ is not flat. Let $\pi : S^3 \times \mathbb{R} \to S^3 \times (\mathbb{R}/T\mathbb{Z})$ be the natural projection, and $E := \pi^*\underline{\mathrm{E}}$ and $A := \pi^*\underline{\mathrm{A}}$ be the pull-backs. Obviously A is a non-flat ASD connection satisfying $\|F_A\|_{L^\infty} < \infty$. Hence it is irreducible (Corollary 6.3). Some constants introduced below (e.g. C_2, C_3, ε_1, ε_2) will depend on $(\underline{\mathrm{E}}, \underline{\mathrm{A}})$. But we consider that $(\underline{\mathrm{E}}, \underline{\mathrm{A}})$ is fixed, and hence the dependence on it will not be explicitly written.

Lemma 9.1. *There exists* $C_2 > 0$ *such that for any* $u \in \Omega^0(\mathrm{ad}\, E)$

$$\int_{S^3\times[0,T]} |u|^2 \leqslant C_2 \int_{S^3\times[0,T]} |d_A u|^2.$$

Then, from the natural T*-periodicity of* A*, for every* $n \in \mathbb{Z}$

$$\int_{S^3\times[nT,(n+1)T]} |u|^2 \leqslant C_2 \int_{S^3\times[nT,(n+1)T]} |d_A u|^2.$$

Proof. Since A is ASD and irreducible, the restriction of A to $S^3 \times (0, T)$ is also irreducible (by the unique continuation [7, Section 4.3.4]). Suppose the above statement is false, then there exist u_n $(n \geqslant 1)$ such that

$$1 = \int_{S^3\times[0,T]} |u_n|^2 > n \int_{S^3\times[0,T]} |d_A u_n|^2.$$

If we take a subsequence, then the restrictions of u_n to $S^3 \times (0, T)$ converge to some u weakly in $L^2_1(S^3 \times (0, T))$ and strongly in $L^2(S^3 \times (0, T))$. We have $\|u\|_{L^2} = 1$ (in particular $u \neq 0$) and $d_A u = 0$. This means that A is reducible over $S^3 \times (0, T)$. This is a contradiction. □

Lemma 9.2. *Let* $4 < q < \infty$*. For any* $u \in L^q_1(S^3 \times (0, T), \Lambda^0(\mathrm{ad}\, E))$,

$$\|u\|_{L^\infty(S^3\times(0,T))} \leqslant \mathrm{const}_q \|d_A u\|_{L^q(S^3\times(0,T))}.$$

Proof. Note that the Sobolev embedding $L^q_1(S^3 \times (0, T)) \hookrightarrow \mathcal{C}^0(S^3 \times [0, T])$ is a compact operator. Then this lemma can be proved in the same way as in Lemma 9.1. □

Lemma 9.3. *Let* $4 < q < \infty$*. For any gauge transformation* $g : E \to E$ *and* $n \in \mathbb{Z}$*,*

$$\min\bigl(\|g-1\|_{L^\infty(S^3\times(nT,(n+1)T))}, \|g+1\|_{L^\infty(S^3\times(nT,(n+1)T))}\bigr) \leqslant \mathrm{const}_q \|d_A g\|_{L^q(S^3\times(nT,(n+1)T))}.$$

Here const_q *is independent of* g *and* n*.*

Proof. From the T-periodicity of A, it is enough to prove the case of $n=0$. Suppose the statement is false. Then there exists a sequence of gauge transformations $\{g_n\}_{n\geqslant 1}$ satisfying

$$\min\bigl(\|g_n-1\|_{L^\infty(S^3\times(0,T))}, \|g_n+1\|_{L^\infty(S^3\times(0,T))}\bigr) > n\|d_A g_n\|_{L^q(S^3\times(0,T))}.$$

If we take a subsequence, then g_n converges to some g weakly in $L^q_1(S^3\times(0,T))$ and strongly in $\mathcal{C}^0(S^3\times[0,T])$. In particular we have $d_A g=0$. Hence $g=\pm 1$ since A is irreducible. By multiplying ± 1 to g_n, we can assume that $g=1$. Then there exists $u_n \in L^q_1(S^3\times(0,T), \Lambda^0(\mathrm{ad}\,E))$ $(n\gg 1)$ satisfying $g_n = e^{u_n}$ and $\|u_n\|_{L^\infty(S^3\times(0,T))} \leqslant \mathrm{const}\|g_n-1\|_{L^\infty(S^3\times(0,T))}$. Then, by using Lemma 9.2, we have

$$\begin{aligned}\|g_n-1\|_{L^\infty(S^3\times(0,T))} &\leqslant \mathrm{const}\|u_n\|_{L^\infty(S^3\times(0,T))}\\ &\leqslant \mathrm{const}'\|d_A u_n\|_{L^q(S^3\times(0,T))} \leqslant \mathrm{const}''\|d_A g_n\|_{L^q(S^3\times(0,T))}.\end{aligned}$$

This is a contradiction. □

Lemma 9.4. *There exists* $\varepsilon_1 > 0$ *such that, for any gauge transformation* $g : E \to E$*, if* $\|d_A g\|_{L^\infty(X)} \leqslant \varepsilon_1$ *then*

$$\min\bigl(\|g-1\|_{L^\infty(X)}, \|g+1\|_{L^\infty(X)}\bigr) \leqslant \mathrm{const}\|d_A g\|_{L^\infty(X)}.$$

Proof. From Lemma 9.3

$$\min\bigl(\|g-1\|_{L^\infty(S^3\times(nT,(n+1)T))}, \|g+1\|_{L^\infty(S^3\times(nT,(n+1)T))}\bigr) \leqslant C\|d_A g\|_{L^\infty(X)} \leqslant C\cdot\varepsilon_1.$$

Suppose $\min(\|g-1\|_{L^\infty(S^3\times(0,T))}, \|g+1\|_{L^\infty(S^3\times(0,T))}) = \|g-1\|_{L^\infty(S^3\times(0,T))}$. We want to prove that for all $n\in\mathbb{Z}$

$$\begin{aligned}&\min\bigl(\|g-1\|_{L^\infty(S^3\times(nT,(n+1)T))}, \|g+1\|_{L^\infty(S^3\times(nT,(n+1)T))}\bigr)\\ &\quad = \|g-1\|_{L^\infty(S^3\times(nT,(n+1)T))}.\end{aligned} \tag{41}$$

We have $\|g-1\|_{L^\infty(S^3\times(0,T))} \leqslant C\cdot\varepsilon_1 \ll 1$. From $|d_A g|\leqslant\varepsilon_1$, $\|g-1\|_{L^\infty(S^3\times(T,2T))} \leqslant (C+T)\varepsilon_1$, and hence $\|g+1\|_{L^\infty(S^3\times(T,2T))} \geqslant 2-(C+T)\varepsilon_1$. We choose $\varepsilon_1>0$ so that $(C+T)\varepsilon_1<1$. Then (41) holds for $n=1$. In the same way, by using induction, we can prove that (41) holds for all $n\in\mathbb{Z}$. Then Lemma 9.3 implies $\|g-1\|_{L^\infty(X)} \leqslant C\|d_A g\|_{L^\infty(X)}$. □

Let $N>0$ be a large positive integer which will be fixed later, and set $R := NT$. Let φ be a smooth function on $\mathbb{R}$ such that $0\leqslant\varphi\leqslant 1$, $\varphi=1$ on $[0,R]$, $\varphi=0$ over $t\geqslant 2R$ and $t\leqslant -R$, and

$|\varphi'|, |\varphi''| \leqslant 2/R$. Then for any $u \in \Omega^0(\mathrm{ad}\,E)$ (not necessarily compactly supported),

$$\int_{S^3\times[0,R]} |d_A u|^2 \leqslant \int_{S^3\times\mathbb{R}} \left|d_A(\varphi u)\right|^2 = \int_{S^3\times\mathbb{R}} \left(\Delta_A(\varphi u), \varphi u\right).$$

Here $\Delta_A := \nabla_A^*\nabla_A = -*d_A*d_A$ on $\Omega^0(\mathrm{ad}\,E)$. We have $\Delta_A(\varphi u) = \varphi\Delta_A u + \Delta\varphi\cdot u + *(*d\varphi\wedge d_A u - d\varphi\wedge *d_A u)$. Then $\Delta_A(\varphi u) = \Delta_A u$ over $S^3\times[0,R]$ and

$$\left|\Delta_A(\varphi u)\right| \leqslant (2/R)|u| + (4/R)|d_A u| + |\Delta_A u|.$$

Hence

$$\int_{S^3\times[0,R]} |d_A u|^2 \leqslant (2/R)\int_{t\in[-R,0]\cup[R,2R]} |u|^2 + (4/R)\int_{t\in[-R,0]\cup[R,2R]} |u||d_A u| + \int_{S^3\times[-R,2R]} |\Delta_A u||u|.$$

From Lemma 9.1,

$$\int_{t\in[-R,0]\cup[R,2R]} |u|^2 \leqslant C_2 \int_{t\in[-R,0]\cup[R,2R]} |d_A u|^2,$$

$$\int_{t\in[-R,0]\cup[R,2R]} |u||d_A u| \leqslant \sqrt{\int_{t\in[-R,0]\cup[R,2R]} |u|^2}\sqrt{\int_{t\in[-R,0]\cup[R,2R]} |d_A u|^2} \leqslant \sqrt{C_2}\int_{t\in[-R,0]\cup[R,2R]} |d_A u|^2.$$

Hence

$$\int_{S^3\times[0,R]} |d_A u|^2 \leqslant \frac{2C_2 + 4\sqrt{C_2}}{R}\int_{t\in[-R,0]\cup[R,2R]} |d_A u|^2 + \int_{S^3\times[-R,2R]} |\Delta_A u||u|.$$

For a function (or a section of some Riemannian vector bundle) f on $S^3\times\mathbb{R}$ and $p\in[1,\infty]$, we set

$$\|f\|_{\ell^\infty L^p} := \sup_{n\in\mathbb{Z}} \|f\|_{L^p(S^3\times(nR,(n+1)R))}.$$

Then the above implies

$$\int_{S^3\times[0,R]} |d_A u|^2 \leqslant \frac{4C_2 + 8\sqrt{C_2}}{R}\|d_A u\|^2_{\ell^\infty L^2} + 3\big\||\Delta_A u|\cdot|u|\big\|_{\ell^\infty L^1}.$$

In the same way, for any $n \in \mathbb{Z}$,

$$\int_{S^3\times[nR,(n+1)R]} |d_A u|^2 \leqslant \frac{4C_2 + 8\sqrt{C_2}}{R} \|d_A u\|^2_{\ell^\infty L^2} + 3\big\| |\Delta_A u| \cdot |u| \big\|_{\ell^\infty L^1}.$$

Then we have

$$\|d_A u\|^2_{\ell^\infty L^2} \leqslant \frac{4C_2 + 8\sqrt{C_2}}{R} \|d_A u\|^2_{\ell^\infty L^2} + 3\big\| |\Delta_A u| \cdot |u| \big\|_{\ell^\infty L^1}.$$

We fix $N > 0$ so that $(4C_2 + 8\sqrt{C_2})/R \leqslant 1/2$ (recall: $R = NT$). If $\|d_A u\|_{\ell^\infty L^2} < \infty$, then we get

$$\|d_A u\|^2_{\ell^\infty L^2} \leqslant 6\big\| |\Delta_A u| \cdot |u| \big\|_{\ell^\infty L^1}.$$

From Hölder's inequality and Lemma 9.1,

$$\big\| |\Delta_A u| \cdot |u| \big\|_{\ell^\infty L^1} \leqslant \|\Delta_A u\|_{\ell^\infty L^2} \|u\|_{\ell^\infty L^2} \leqslant \sqrt{C_2} \|\Delta_A u\|_{\ell^\infty L^2} \|d_A u\|_{\ell^\infty L^2}.$$

Hence $\|d_A u\|_{\ell^\infty L^2} \leqslant 6\sqrt{C_2}\|\Delta_A u\|_{\ell^\infty L^2}$, and $\|u\|_{\ell^\infty L^2} \leqslant \sqrt{C_2}\|d_A u\|_{\ell^\infty L^2} \leqslant 6C_2\|\Delta_A u\|_{\ell^\infty L^2}$. Then we get the following conclusion.

Lemma 9.5. *There exists a constant $C_3 > 0$ such that, for any $u \in \Omega^0(\operatorname{ad} E)$ with $\|d_A u\|_{\ell^\infty L^2} < \infty$, we have*

$$\|u\|_{\ell^\infty L^2} + \|d_A u\|_{\ell^\infty L^2} \leqslant C_3 \|\Delta_A u\|_{\ell^\infty L^2}.$$

The following result gives the "partial Coulomb gauge slice" in our situation.

Proposition 9.6. *There exists $\varepsilon_2 > 0$ satisfying the following. For any a and b in $\Omega^1(\operatorname{ad} E)$ satisfying $d_A^* a = d_A^* b = 0$ and $\|a\|_{L^\infty}, \|b\|_{L^\infty} \leqslant \varepsilon_2$, if there is a gauge transformation g of E satisfying $g(A + a) = A + b$ then $a = b$ and $g = \pm 1$.*

Proof. Since $g(A + a) = A + b$, we have $d_A g = ga - bg$. Then we have $|d_A g| \leqslant 2\varepsilon_2$. We choose $\varepsilon_2 > 0$ so that $2\varepsilon_2 \leqslant \varepsilon_1$. ($\varepsilon_1$ is the positive constant introduced in Lemma 9.4.) From Lemma 9.4, by multiplying ± 1 to g, we can suppose $\|g - 1\|_{L^\infty} \leqslant \text{const} \cdot \varepsilon_2 \ll 1$. Then there exists $u \in \Omega^0(\operatorname{ad} E)$ satisfying $g = e^u$ and $\|u\|_{L^\infty} \leqslant \text{const} \cdot \varepsilon_2$. We have

$$d_A e^u = d_A u + (d_A u \cdot u + u d_A u)/2! + \big(d_A u \cdot u^2 + u d_A u \cdot u + u^2 d_A u\big)/3! + \cdots.$$

Since $|u| \leqslant \text{const} \cdot \varepsilon_2 \ll 1$,

$$\big|d_A e^u\big| \geqslant |d_A u|\big(2 - e^{|u|}\big) \geqslant |d_A u|/2.$$

Hence $|d_A u| \leqslant 2|d_A g| \leqslant 4\varepsilon_2$. In particular, $\|d_A u\|_{\ell^\infty L^2} < \infty$. In the same way we get $|d_A g| \leqslant 2|d_A u|$, and hence

$$\|d_A g\|_{\ell^\infty L^2} \leqslant 2\|d_A u\|_{\ell^\infty L^2} \leqslant 2C_3\|\Delta_A u\|_{\ell^\infty L^2}. \tag{42}$$

Here we have used Lemma 9.5. Since $d_A^* a = d_A^* b = 0$ and $d_A g = ga - bg$, we have

$$\Delta_A g = -{*}d_A{*}d_A g = -{*}(d_A g \wedge {*}a + {*}b \wedge d_A g).$$

Therefore $\|\Delta_A g\|_{\ell^\infty L^2} \leqslant (\|a\|_{L^\infty} + \|b\|_{L^\infty})\|d_A g\|_{\ell^\infty L^2} < \infty$. Moreover, by using the above (42) and $\|a\|_{L^\infty}, \|b\|_{L^\infty} \leqslant \varepsilon_2$, we get

$$\|\Delta_A g\|_{\ell^\infty L^2} \leqslant 4C_3\varepsilon_2\|\Delta_A u\|_{\ell^\infty L^2}. \tag{43}$$

A direct calculation shows $|\Delta_A u^n| \leqslant n(n-1)|u|^{n-2}|d_A u|^2 + n|u|^{n-1}|\Delta_A u|$. Hence

$$\left|\Delta_A\left(e^u - u\right)\right| \leqslant e^{|u|}|d_A u|^2 + \left(e^{|u|} - 1\right)|\Delta_A u| \leqslant C\varepsilon_2\left(|d_A u| + |\Delta_A u|\right). \tag{44}$$

Here we have used $|u|, |d_A u| \leqslant \text{const} \cdot \varepsilon_2 \ll 1$. Hence $(1 - C\varepsilon_2)|\Delta_A u| \leqslant C\varepsilon_2|d_A u| + |\Delta_A g|$, and $(1 - C\varepsilon_2)\|\Delta_A u\|_{\ell^\infty L^2} \leqslant C\varepsilon_2\|d_A u\|_{\ell^\infty L^2} + \|\Delta_A g\|_{\ell^\infty L^2} < \infty$. We choose $\varepsilon_2 > 0$ so that $(1 - C\varepsilon_2) > 0$. Then $\|\Delta_A u\|_{\ell^\infty L^2} < \infty$.

The above (44) implies

$$\|\Delta_A g - \Delta_A u\|_{\ell^\infty L^2} \leqslant C\varepsilon_2\left(\|d_A u\|_{\ell^\infty L^2} + \|\Delta_A u\|_{\ell^\infty L^2}\right).$$

Using Lemma 9.5, we get

$$\|\Delta_A g - \Delta_A u\|_{\ell^\infty L^2} \leqslant C'\varepsilon_2\|\Delta_A u\|_{\ell^\infty L^2}.$$

Then the inequality (43) gives

$$(1 - 4C_3\varepsilon_2)\|\Delta_A u\|_{\ell^\infty L^2} \leqslant C'\varepsilon_2\|\Delta_A u\|_{\ell^\infty L^2}.$$

If we choose $\varepsilon_2 > 0$ so small that $(1 - 4C_3\varepsilon_2) > C'\varepsilon_2$, then this estimate gives $\Delta_A u = 0$. (Here we have used $\|\Delta_A u\|_{\ell^\infty L^2} < \infty$.) Then we get (from Lemma 9.5) $u = 0$. This shows $g = 1$ and $a = b$. $\square$

The following "L^∞-estimate" will be used in the next section. For its proof, see Proposition A.5 in Appendix A.

Proposition 9.7. *Let ξ be a $\mathcal{C}^2$-section of $\Lambda^+(\operatorname{ad} E)$ over $S^3 \times \mathbb{R}$, and set $\eta := (\nabla_A^* \nabla_A + S/3)\xi$. If $\|\xi\|_{L^\infty}, \|\eta\|_{L^\infty} < \infty$, then*

$$\|\xi\|_{L^\infty} \leqslant (24/S)\|\eta\|_{L^\infty}.$$

10. Proof of the lower bound: deformation theory

The argument in this section is a Yang–Mills analogue of the deformation theory developed in Tsukamoto [23]. Let d be a positive real number. As in Section 9, let $T > 0$ be a positive real number, $\underline{\mathrm{E}}$ be a principal $SU(2)$-bundle over $S^3 \times (\mathbb{R}/T\mathbb{Z})$, and $\underline{\mathrm{A}}$ be an ASD connection on $\underline{\mathrm{E}}$. Suppose that $\underline{\mathrm{A}}$ is not flat and

$$\left\|F(\underline{\mathrm{A}})\right\|_{L^\infty} < d. \tag{45}$$

Set $E := \pi^*\underline{\mathrm{E}}$ and $A := \pi^*\underline{\mathrm{A}}$ where $\pi : S^3 \times \mathbb{R} \to S^3 \times (\mathbb{R}/T\mathbb{Z})$ is the natural projection. Some constants introduced below depend on $(\underline{\mathrm{E}}, \underline{\mathrm{A}})$. But we don't explicitly write their dependence on it because we consider that $(\underline{\mathrm{E}}, \underline{\mathrm{A}})$ is fixed.

We define the Banach space H^1_A by setting

$$H^1_A := \left\{a \in \Omega^1(\operatorname{ad} E) \;\middle|\; \left(d_A^* + d_A^+\right)a = 0,\ \|a\|_{L^\infty} < \infty\right\}.$$

$(H^1_A, \|\cdot\|_{L^\infty})$ becomes an infinite dimensional Banach space. The additive group $T\mathbb{Z} = \{nT \in \mathbb{R} \mid n \in \mathbb{Z}\}$ acts on H^1_A as follows. From the definition of E and A, we have $(T^*E, T^*A) = (E, A)$ where $T : S^3 \times \mathbb{R} \to S^3 \times \mathbb{R}$, $(\theta, t) \mapsto (\theta, t + T)$. Hence for any $a \in H^1_A$, we have $T^*a \in H^1_A$ and $\|T^*a\|_{L^\infty} = \|a\|_{L^\infty}$.

Fix $0 < \alpha < 1$. We want to define the Hölder space $\mathcal{C}^{k,\alpha}(\Lambda^+(\operatorname{ad} E))$ for $k \geqslant 0$. Let $\{U_\lambda\}_{\lambda=1}^{\Lambda}$, $\{U'_\lambda\}_{\lambda=1}^{\Lambda}$, $\{U''_\lambda\}_{\lambda=1}^{\Lambda}$ be finite open coverings of $S^3 \times (\mathbb{R}/T\mathbb{Z})$ satisfying the following conditions.

(i) $\bar{U}_\lambda \subset U'_\lambda$ and $\bar{U}'_\lambda \subset U''_\lambda$. U_λ, U'_λ and U''_λ are connected, and their boundaries are smooth. Each U''_λ is a coordinate chart, i.e., a diffeomorphism between U''_λ and an open set in $\mathbb{R}^4$ is given for each λ.

(ii) The covering map $\pi : S^3 \times \mathbb{R} \to S^3 \times (\mathbb{R}/T\mathbb{Z})$ can be trivialized over each U''_λ, i.e., we have a disjoint union $\pi^{-1}(U''_\lambda) = \bigsqcup_{n\in\mathbb{Z}} U''_{n\lambda}$ such that $\pi : U''_{n\lambda} \to U''_\lambda$ is diffeomorphic. We set $U_{n\lambda} := U''_{n\lambda} \cap \pi^{-1}(U_\lambda)$ and $U'_{n\lambda} := U''_{n\lambda} \cap \pi^{-1}(U'_\lambda)$. We have $\pi^{-1}(U_\lambda) = \bigsqcup_{n\in\mathbb{Z}} U_{n\lambda}$ and $\pi^{-1}(U'_\lambda) = \bigsqcup_{n\in\mathbb{Z}} U'_{n\lambda}$.

(iii) A trivialization of the principal $SU(2)$-bundle $\underline{\mathrm{E}}$ over each U''_λ is given.

From the conditions (ii) and (iii), we have a coordinate system and a trivialization of E over each $U''_{n\lambda}$. Let u be a section of $\Lambda^i(\operatorname{ad} E)$ $(0 \leqslant i \leqslant 4)$ over $S^3 \times \mathbb{R}$. Then $u|_{U''_{n\lambda}}$ can be seen as a vector-valued function over $U''_{n\lambda}$. Hence we can consider the Hölder norm $\|u\|_{\mathcal{C}^{k,\alpha}(\bar{U}_{n\lambda})}$ of u as a vector-valued function over $\bar{U}_{n\lambda}$ (cf. Gilbarg and Trudinger [9, Chapter 4]). We define the Hölder norm $\|u\|_{\mathcal{C}^{k,\alpha}}$ by setting

$$\|u\|_{\mathcal{C}^{k,\alpha}} := \sup_{n\in\mathbb{Z}, 1\leqslant\lambda\leqslant\Lambda} \|u\|_{\mathcal{C}^{k,\alpha}(\bar{U}_{n\lambda})}.$$

For $a \in H^1_A$, we have $\|a\|_{\mathcal{C}^{k,\alpha}} \leqslant \mathrm{const}_k \|a\|_{L^\infty} < \infty$ for every $k = 0, 1, 2, \ldots$ by the elliptic regularity. We define the Banach space $\mathcal{C}^{k,\alpha}(\Lambda^+(\operatorname{ad} E))$ as the space of sections u of $\Lambda^+(\operatorname{ad} E)$ over $S^3 \times \mathbb{R}$ satisfying $\|u\|_{\mathcal{C}^{k,\alpha}} < \infty$.

Consider the following map:

$$\Phi : H^1_A \times \mathcal{C}^{2,\alpha}\big(\Lambda^+(\operatorname{ad} E)\big) \to \mathcal{C}^{0,\alpha}\big(\Lambda^+(\operatorname{ad} E)\big), \quad (a,\phi) \mapsto F^+\big(A + a + d_A^*\phi\big).$$

This is a smooth map between the Banach spaces. Since $F^+(A+a) = (a\wedge a)^+$,

$$F^+\big(A + a + d_A^*\phi\big) = (a\wedge a)^+ + d_A^+ d_A^*\phi + \big[a \wedge d_A^*\phi\big]^+ + \big(d_A^*\phi \wedge d_A^*\phi\big)^+. \tag{46}$$

The derivative of Φ with respect to the second variable ϕ at the origin $(0,0)$ is given by

$$\partial_2 \Phi_{(0,0)} = d_A^+ d_A^* = \frac{1}{2}\big(\nabla_A^*\nabla_A + S/3\big) : \mathcal{C}^{2,\alpha}\big(\Lambda^+(\operatorname{ad} E)\big) \to \mathcal{C}^{0,\alpha}\big(\Lambda^+(\operatorname{ad} E)\big). \tag{47}$$

Here we have used the Weitzenböck formula (see (11)).

Proposition 10.1. *The map* $(\nabla_A^*\nabla_A + S/3) : \mathcal{C}^{2,\alpha}(\Lambda^+(\operatorname{ad} E)) \to \mathcal{C}^{0,\alpha}(\Lambda^+(\operatorname{ad} E))$ *is isomorphic.*

Proof. The injectivity follows from the L^∞-estimate of Proposition 9.7. So the problem is the surjectivity. First we prove the following lemma.

Lemma 10.2. *Suppose that* $\eta \in \mathcal{C}^{0,\alpha}(\Lambda^+(\operatorname{ad} E))$ *is compactly supported. Then there exists* $\phi \in \mathcal{C}^{2,\alpha}(\Lambda^+(\operatorname{ad} E))$ *satisfying* $(\nabla_A^*\nabla_A + S/3)\phi = \eta$ *and* $\|\phi\|_{\mathcal{C}^{2,\alpha}} \leqslant \mathrm{const}\cdot \|\eta\|_{\mathcal{C}^{0,\alpha}}$.

Proof. Set $L^2_1 := \{\xi \in L^2(\Lambda^+(\operatorname{ad} E)) \mid \nabla_A \xi \in L^2\}$. For $\xi_1, \xi_2 \in L^2_1$, set $(\xi_1,\xi_2)_{S/3} := (S/3)(\xi_1,\xi_2)_{L^2} + (\nabla_A\xi_1, \nabla_A\xi_2)_{L^2}$. Since S is a positive constant, this inner product defines a norm equivalent to the standard L^2_1-norm. η defines a bounded linear functional $(\cdot,\eta)_{L^2} : L^2_1 \to \mathbb{R}$, $\xi \mapsto (\xi,\eta)_{L^2}$. From the Riesz representation theorem, there uniquely exists $\phi \in L^2_1$ satisfying $(\xi,\phi)_{S/3} = (\xi,\eta)_{L^2}$ for any $\xi \in L^2_1$. This implies that $(\nabla_A^*\nabla_A + S/3)\phi = \eta$ in the sense of distributions. Moreover we have $\|\phi\|_{L^2_1} \leqslant \mathrm{const}\|\eta\|_{L^2}$. From the elliptic regularity (see Gilbarg and Trudinger [9, Chapter 9]) and the Sobolev embedding $L^2_1 \hookrightarrow L^4$,

$$\begin{aligned}
\|\phi\|_{L^4_2(U_{n\lambda})} &\leqslant \mathrm{const}_\lambda\big(\|\phi\|_{L^4(U'_{n\lambda})} + \|\eta\|_{L^4(U'_{n\lambda})}\big) \\
&\leqslant \mathrm{const}_\lambda\big(\|\phi\|_{L^2_1(U'_{n\lambda})} + \|\eta\|_{L^4(U'_{n\lambda})}\big) \\
&\leqslant \mathrm{const}_\lambda\big(\|\eta\|_{L^2} + \|\eta\|_{L^4}\big).
\end{aligned}$$

Here const_λ are constants depending on $\lambda = 1,2,\ldots,\Lambda$. The important point is that they are independent of $n \in \mathbb{Z}$. This is because we have the $T\mathbb{Z}$-symmetry of the equation. From the Sobolev embedding $L^4_2 \hookrightarrow L^\infty$, we have

$$\|\phi\|_{L^\infty} \leqslant \mathrm{const}\cdot \sup_{n,\lambda} \|\phi\|_{L^4_2(U_{n\lambda})} \leqslant \mathrm{const}\big(\|\eta\|_{L^2} + \|\eta\|_{L^4}\big) < \infty.$$

Using the Schauder interior estimate (see Gilbarg and Trudinger [9, Chapter 6]), we get

$$\|\phi\|_{\mathcal{C}^{2,\alpha}(\bar{U}_{n\lambda})} \leqslant \mathrm{const}_\lambda\big(\|\phi\|_{L^\infty} + \|\eta\|_{\mathcal{C}^{0,\alpha}(\bar{U}'_{n\lambda})}\big).$$

From Proposition 9.7, we get $\|\phi\|_{L^\infty} \leqslant (24/S)\|\eta\|_{L^\infty}$. It is easy to see that

$$\sup_{n,\lambda} \|\eta\|_{\mathcal{C}^{0,\alpha}(\bar{U}'_{n\lambda})} \leqslant \mathrm{const}\|\eta\|_{\mathcal{C}^{0,\alpha}}. \tag{48}$$

(Recall $\|\eta\|_{\mathcal{C}^{0,\alpha}} = \sup_{n,\lambda} \|\eta\|_{\mathcal{C}^{0,\alpha}(\bar{U}_{n\lambda})}$.) Hence $\|\phi\|_{\mathcal{C}^{2,\alpha}} \leqslant \mathrm{const}(\|\eta\|_{L^\infty} + \|\eta\|_{\mathcal{C}^{0,\alpha}}) \leqslant \mathrm{const}\|\eta\|_{\mathcal{C}^{0,\alpha}}$. □

Let $\eta \in \mathcal{C}^{0,\alpha}(\Lambda^+(\mathrm{ad}\, E))$ (not necessarily compactly supported). Let φ_k $(k = 1, 2, \ldots)$ be cut-off functions such that $0 \leqslant \varphi_k \leqslant 1$, $\varphi_k = 1$ over $|t| \leqslant k$ and $\varphi_k = 0$ over $|t| \geqslant k + 1$. Set $\eta_k := \varphi_k \eta$. From the above Lemma 10.2, there exists $\phi_k \in \mathcal{C}^{2,\alpha}(\Lambda^+(\mathrm{ad}\, E))$ satisfying $(\nabla_A^* \nabla_A + S/3)\phi_k = \eta_k$. From the L^∞-estimate (Proposition 9.7), we get

$$\|\phi_k\|_{L^\infty} \leqslant (24/S)\|\eta_k\|_{L^\infty} \leqslant (24/S)\|\eta\|_{L^\infty}.$$

From the Schauder interior estimate, we get

$$\|\phi_k\|_{\mathcal{C}^{2,\alpha}(\bar{U}_{n\lambda})} \leqslant \mathrm{const}_\lambda \cdot \left(\|\phi_k\|_{L^\infty(U'_{n\lambda})} + \|\eta_k\|_{\mathcal{C}^{0,\alpha}(\bar{U}'_{n\lambda})}\right) \leqslant \mathrm{const}\left(\|\eta\|_{L^\infty} + \|\eta_k\|_{\mathcal{C}^{0,\alpha}(\bar{U}'_{n\lambda})}\right).$$

We have $\eta_k = \eta$ over each $U'_{n\lambda}$ for $k \gg 1$. Hence $\|\phi_k\|_{\mathcal{C}^{2,\alpha}(\bar{U}_{n\lambda})}$ $(k \geqslant 1)$ is bounded for each (n, λ). Therefore, if we take a subsequence, ϕ_k converges to a $\mathcal{C}^2$-section ϕ of $\Lambda^+(\mathrm{ad}\, E)$ in the $\mathcal{C}^2$-topology over every compact subset. ϕ satisfies $(\nabla_A^* \nabla_A + S/3)\phi = \eta$ and $\|\phi\|_{L^\infty} \leqslant (24/S)\|\eta\|_{L^\infty}$. The Schauder interior estimate gives

$$\|\phi\|_{\mathcal{C}^{2,\alpha}(\bar{U}_{n\lambda})} \leqslant \mathrm{const}_\lambda\left(\|\phi\|_{L^\infty} + \|\eta\|_{\mathcal{C}^{0,\alpha}(\bar{U}'_{n\lambda})}\right).$$

By (48), we get $\|\phi\|_{\mathcal{C}^{2,\alpha}} \leqslant \mathrm{const}\|\eta\|_{\mathcal{C}^{0,\alpha}} < \infty$. □

Since the map (47) is isomorphic, the implicit function theorem implies that there exist $\delta_2 > 0$ and $\delta_3 > 0$ such that for any $a \in H_A^1$ with $\|a\|_{L^\infty} \leqslant \delta_2$ there uniquely exists $\phi_a \in \mathcal{C}^{2,\alpha}(\Lambda^+(\mathrm{ad}\, E))$ with $\|\phi_a\|_{\mathcal{C}^{2,\alpha}} \leqslant \delta_3$ satisfying $F^+(A + a + d_A^*\phi_a) = 0$, i.e.,

$$d_A^+ d_A^* \phi_a + \left[a \wedge d_A^* \phi_a\right]^+ + \left(d_A^* \phi_a \wedge d_A^* \phi_a\right)^+ = -(a \wedge a)^+. \tag{49}$$

Here the "uniqueness" means that if $\phi \in \mathcal{C}^{2,\alpha}(\Lambda^+(\mathrm{ad}\, E))$ with $\|\phi\|_{\mathcal{C}^{2,\alpha}} \leqslant \delta_3$ satisfies $F^+(A + a + d_A^*\phi) = 0$ then $\phi = \phi_a$. From the elliptic regularity, ϕ_a is smooth. We have $\phi_0 = 0$ and

$$\|\phi_a\|_{\mathcal{C}^{2,\alpha}} \leqslant \mathrm{const}\|a\|_{L^\infty}, \qquad \|\phi_a - \phi_b\|_{\mathcal{C}^{2,\alpha}} \leqslant \mathrm{const}\|a - b\|_{L^\infty}, \tag{50}$$

for any $a, b \in H_A^1$ with $\|a\|_{L^\infty}, \|b\|_{L^\infty} \leqslant \delta_2$. The map $a \mapsto \phi_a$ is T-equivariant, i.e., $\phi_{T^*a} = T^*\phi_a$ where $T : S^3 \times \mathbb{R} \to S^3 \times \mathbb{R}$, $(\theta, t) \mapsto (\theta, t + T)$.

We have $F(A + a + d_A^*\phi_a) = F(A + a) + d_A d_A^* \phi_a + [a \wedge d_A^*\phi_a] + d_A^*\phi_a \wedge d_A^*\phi_a$. From (45), if we choose $\delta_2 > 0$ sufficiently small,

$$\left\|F\left(A + a + d_A^*\phi_a\right)\right\|_{L^\infty} \leqslant \left\|F(A)\right\|_{L^\infty} + \mathrm{const} \cdot \delta_2 \leqslant d. \tag{51}$$

Moreover we can choose $\delta_2 > 0$ so that, for any $a \in H^1_A$ with $\|a\|_{L^\infty} \leqslant \delta_2$,

$$\left\|a + d_A^* \phi_a\right\|_{L^\infty} \leqslant \mathrm{const} \cdot \delta_2 \leqslant \varepsilon_2, \tag{52}$$

where ε_2 is the positive constant introduced in Proposition 9.6.

Lemma 10.3. *We can take the above constant $\delta_2 > 0$ sufficiently small so that, if $a, b \in H^1_A$ with $\|a\|_{L^\infty}, \|b\|_{L^\infty} \leqslant \delta_2$ satisfy $a + d_A^*\phi_a = b + d_A^*\phi_b$, then $a = b$.*

Proof. By (49),

$$\begin{aligned}
&\frac{1}{2}\big(\nabla_A^*\nabla_A + S/3\big)(\phi_a - \phi_b) \\
&\quad = d_A^+ d_A^*(\phi_a - \phi_b) \\
&\quad = \big(b \wedge (b-a)\big)^+ + \big((b-a)\wedge a\big)^+ + \big[b \wedge \big(d_A^*\phi_b - d_A^*\phi_a\big)\big]^+ + \big[(b-a)\wedge d_A^*\phi_a\big]^+ \\
&\qquad + \big(d_A^*\phi_b \wedge \big(d_A^*\phi_b - d_A^*\phi_a\big)\big)^+ + \big(\big(d_A^*\phi_b - d_A^*\phi_a\big)\wedge d_A^*\phi_a\big)^+.
\end{aligned} \tag{53}$$

Its $\mathcal{C}^{0,\alpha}$-norm is bounded by

$$\begin{aligned}
&\mathrm{const}\big(\|a\|_{\mathcal{C}^{0,\alpha}} + \|b\|_{\mathcal{C}^{0,\alpha}} + \left\|d_A^*\phi_a\right\|_{\mathcal{C}^{0,\alpha}}\big)\|a-b\|_{\mathcal{C}^{0,\alpha}} \\
&\quad + \mathrm{const}\big(\|b\|_{\mathcal{C}^{0,\alpha}} + \left\|d_A^*\phi_a\right\|_{\mathcal{C}^{0,\alpha}} + \left\|d_A^*\phi_b\right\|_{\mathcal{C}^{0,\alpha}}\big)\left\|d_A^*\phi_a - d_A^*\phi_b\right\|_{\mathcal{C}^{0,\alpha}}.
\end{aligned}$$

From (50), this is bounded by $\mathrm{const} \cdot \delta_2\|a-b\|_{L^\infty}$. Then Proposition 10.1 implies

$$\|\phi_a - \phi_b\|_{\mathcal{C}^{2,\alpha}} \leqslant \mathrm{const} \cdot \delta_2 \|a-b\|_{L^\infty}.$$

Hence, if $a + d_A^*\phi_a = b + d_A^*\phi_b$ then

$$\|a-b\|_{L^\infty} = \left\|d_A^*\phi_a - d_A^*\phi_b\right\|_{L^\infty} \leqslant \mathrm{const} \cdot \delta_2\|a-b\|_{L^\infty}.$$

If δ_2 is sufficiently small, then this implies $a = b$. □

For $r > 0$, we set $B_r(H^1_A) := \{a \in H^1_A \mid \|a\|_{L^\infty} \leqslant r\}$.

Lemma 10.4. *Let $\{a_n\}_{n\geqslant 1} \subset B_{\delta_2}(H^1_A)$ and suppose that this sequence converges to $a \in B_{\delta_2}(H^1_A)$ in the topology of uniform convergence over compact subsets, i.e., for any compact set $K \subset S^3 \times \mathbb{R}$, $\|a_n - a\|_{L^\infty(K)} \to 0$ as $n \to \infty$. Then $d_A^*\phi_{a_n}$ converges to $d_A^*\phi_a$ in the $\mathcal{C}^\infty$-topology over every compact subset in $S^3 \times \mathbb{R}$.*

Proof. It is enough to prove that there exists a subsequence (also denoted by $\{a_n\}$) such that $d_A^*\phi_{a_n}$ converges to $d_A^*\phi_a$ in the topology of $\mathcal{C}^\infty$-convergence over compact subsets in $S^3 \times \mathbb{R}$. From the elliptic regularity, a_n converges to a in the $\mathcal{C}^\infty$-topology over every compact subset. Hence, for each $k \geqslant 0$ and each compact subset K in X, the $\mathcal{C}^k$-norms of ϕ_{a_n} over K ($n \geqslant 1$) are bounded by Eq. (49) and $\|\phi_{a_n}\|_{\mathcal{C}^{2,\alpha}} \leqslant \delta_3$. Then a subsequence of ϕ_{a_n} converges to some ϕ in the $\mathcal{C}^\infty$-topology over every compact subset. We have $\|\phi\|_{\mathcal{C}^{2,\alpha}} \leqslant \delta_3$ and $F^+(A+a+d_A^*\phi)=0$. Then the uniqueness of ϕ_a implies $\phi=\phi_a$. □

Consider the following map (cf. the description of $\mathcal{M}_d$ in Remark 1.3):

$$B_{\delta_2}\big(H_A^1\big) \to \mathcal{M}_d, \quad a \mapsto \big[E, A+a+d_A^*\phi_a\big]. \tag{54}$$

Note that we have $|F(A+a+d_A^*\phi_a)| \leqslant d$ (see (51)), and hence this map is well defined. $B_{\delta_2}(H_A^1)$ is equipped with the topology of uniform convergence over compact subsets. ($B_{\delta_2}(H_A^1)$ becomes compact and metrizable.) The map (54) is continuous by Lemma 10.4. $T\mathbb{Z}$ naturally acts on $B_{\delta_2}(H_A^1)$, and the map (54) is $T\mathbb{Z}$-equivariant. ($\mathcal{M}_d$ is equipped with the action of $T\mathbb{Z}$ induced by the action of $\mathbb{R}$.)

Lemma 10.5. *The map* (54) *is injective for sufficiently small* $\delta_2 > 0$.

Proof. Let $a, b \in B_{\delta_2}(H_A^1)$, and suppose that there exists a gauge transformation $g: E \to E$ satisfying $g(A+a+d_A^*\phi_a) = A+b+d_A^*\phi_b$. We have $d_A^*(a+d_A^*\phi_a) = d_A^*(b+d_A^*\phi_b) = 0$ and $\|a+d_A^*\phi_a\|_{L^\infty}, \|b+d_A^*\phi_b\|_{L^\infty} \leqslant \varepsilon_2$ (see (52)). Then Proposition 9.6 implies $a+d_A^*\phi_a = b+d_A^*\phi_b$. Then we have $a=b$ by Lemma 10.3. □

Therefore the map (54) becomes a $T\mathbb{Z}$-equivariant topological embedding. Hence

$$\dim_{[E,A]}(\mathcal{M}_d : T\mathbb{Z}) \geqslant \dim_0\big(B_{\delta_2}\big(H_A^1\big) : T\mathbb{Z}\big). \tag{55}$$

The right-hand side is the local mean dimension of $(B_{\delta_2}(H_A^1), T\mathbb{Z})$ at the origin. We define a distance on $B_{\delta_2}(H_A^1)$ by

$$\operatorname{dist}(a,b) := \sum_{n\geqslant 0} 2^{-n}\|a-b\|_{L^\infty(|t|\leqslant (n+1)T)} \quad \big(a, b \in B_{\delta_2}\big(H_A^1\big)\big).$$

Set $\Omega_n := \{0, T, 2T, \dots, (n-1)T\} \subset T\mathbb{Z}$ ($n \geqslant 1$). $\{\Omega_n\}_{n\geqslant 1}$ is an amenable sequence in $T\mathbb{Z}$. For $a, b \in B_{\delta_2}(H_A^1)$,

$$\operatorname{dist}_{\Omega_n}(a,b) \geqslant \|a-b\|_{L^\infty(0\leqslant t\leqslant nT)}. \tag{56}$$

For each $n \geqslant 1$, let $\pi_n : S^3 \times (\mathbb{R}/nT\mathbb{Z}) \to S^3 \times (\mathbb{R}/T\mathbb{Z})$ be the natural n-fold covering, and set $E_n := \pi_n^*(\underline{E})$ and $A_n := \pi_n^*(\underline{A})$. We define $H_{A_n}^1$ as the space of $a \in \Omega^1(\operatorname{ad} E_n)$ over $S^3 \times (\mathbb{R}/nT\mathbb{Z})$ satisfying $(d_{A_n}^+ + d_{A_n}^*)a = 0$. We can identify $H_{A_n}^1$ with the subspace of H_A^1 consisting of nT-invariant elements. The index formula gives $\dim H_{A_n}^1 = 8nc_2(\underline{E})$. (We have $H_{A_n}^0 = H_{A_n}^2 = 0$.) From (56), for $a, b \in B_{\delta_2}(H_{A_n}^1) := \{u \in H_{A_n}^1 \mid \|u\|_{L^\infty(X)} \leqslant \delta_2\}$

$$\operatorname{dist}_{\Omega_n}(a,b) \geqslant \|a-b\|_{L^\infty(X)}. \tag{57}$$

Let $0 < r < \delta_2$. Since $\operatorname{dist}_{T\mathbb{Z}}(a,b) \leqslant 2\|a-b\|_{L^\infty}$, we have $B_{r/2}(H_A^1) \subset B_r(0; B_{\delta_2}(H_A^1))_{T\mathbb{Z}}$. Here $B_r(0; B_{\delta_2}(H_A^1))_{T\mathbb{Z}}$ is the closed r-ball centered at 0 in $B_{\delta_2}(H_A^1)$ with respect to the distance $\operatorname{dist}_{T\mathbb{Z}}(\cdot,\cdot)$. From (57) and Lemma 2.1, for $\varepsilon < r/2$

$$\begin{aligned}\mathrm{Widim}_\varepsilon\big(B_r\big(0;\, B_{\delta_2}\big(H_A^1\big)\big)_{T\mathbb{Z}}, \mathrm{dist}_{\Omega_n}\big) &\geqslant \mathrm{Widim}_\varepsilon\big(B_{r/2}\big(H_A^1\big), \mathrm{dist}_{\Omega_n}\big)\\ &\geqslant \mathrm{Widim}_\varepsilon\big(B_{r/2}\big(H_{A_n}^1\big), \|\cdot\|_{L^\infty}\big) = \dim H_{A_n}^1 = 8nc_2(\underline{\mathrm{E}}).\end{aligned}$$

Hence, for $\varepsilon < r/2$,

$$\begin{aligned}&\mathrm{Widim}_\varepsilon\big(B_r\big(0;\, B_{\delta_2}\big(H_A^1\big)\big)_{T\mathbb{Z}} \subset B_{\delta_2}\big(H_A^1\big) : T\mathbb{Z}\big)\\ &\quad\geqslant \limsup_{n\to\infty}\left(\frac{1}{n}\mathrm{Widim}_\varepsilon\big(B_r\big(0;\, B_{\delta_2}\big(H_A^1\big)\big)_{T\mathbb{Z}}, \mathrm{dist}_{\Omega_n}\big)\right) \geqslant 8c_2(\underline{\mathrm{E}}).\end{aligned}$$

Let $\varepsilon \to 0$. Then

$$\dim\big(B_r\big(0;\, B_{\delta_2}\big(H_A^1\big)\big)_{T\mathbb{Z}} \subset B_{\delta_2}\big(H_A^1\big) : T\mathbb{Z}\big) \geqslant 8c_2(\underline{\mathrm{E}}).$$

Let $r \to 0$. We get $\dim_0(B_{\delta_2}(H_A^1) : T\mathbb{Z}) \geqslant 8c_2(\underline{\mathrm{E}})$. From (55) and Proposition 2.11,

$$\dim_{[E,A]}(\mathcal{M}_d : \mathbb{R}) = \dim_{[E,A]}(\mathcal{M}_d : T\mathbb{Z})/T \geqslant 8c_2(\underline{\mathrm{E}})/T = 8\rho(A).$$

Therefore we get the conclusion:

Theorem 10.6. *If $\boldsymbol{A}$ is a periodic ASD connection on $\boldsymbol{E}$ satisfying $\|F(\boldsymbol{A})\|_{L^\infty(X)} < d$, then*

$$\dim_{[\boldsymbol{A}]}(\mathcal{M}_d : \mathbb{R}) = 8\rho(\boldsymbol{A}).$$

Proof. The upper bound $\dim_{[\boldsymbol{A}]}(\mathcal{M}_d : \mathbb{R}) \leqslant 8\rho(\boldsymbol{A})$ was already proved in Section 8.2. If $\boldsymbol{A}$ is not flat, then the above argument shows that we also have the lower bound $\dim_{[\boldsymbol{A}]}(\mathcal{M}_d : \mathbb{R}) \geqslant 8\rho(\boldsymbol{A})$. If $\boldsymbol{A}$ is flat, then $\dim_{[\boldsymbol{A}]}(\mathcal{M}_d : \mathbb{R}) \geqslant 0 = 8\rho(\boldsymbol{A})$. Hence $\dim_{[\boldsymbol{A}]}(\mathcal{M}_d : \mathbb{R}) = 8\rho(\boldsymbol{A})$. □

We have completed all the proofs of Theorems 1.1 and 1.2.

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Appendix A. Green kernel

In this appendix, we prepare some basic facts on a Green kernel over $S^3 \times \mathbb{R}$. Let $a > 0$ be a positive constant. Some constants introduced in this appendix depend on a, but we don't explicitly write their dependence on a for simplicity of the explanation. In the main body of the paper we have $a = S/3$ (S is the scalar curvature of $S^3 \times \mathbb{R}$), and its value is fixed throughout the argument. Hence we don't need to care about the dependence on $a = S/3$.

A.1. $(\Delta + a)$ *on functions*

Let $\Delta := \nabla^*\nabla$ be the Laplacian on functions over $S^3 \times \mathbb{R}$. (Notice that the sign convention of our Laplacian $\Delta = \nabla^*\nabla$ is "geometric". For example, we have $\Delta = -\sum_{i=1}^4 \partial^2/\partial x_i^2$ on the Euclidean space $\mathbb{R}^4$.) Let $g(x, y)$ be the Green kernel of $\Delta + a$;

$$(\Delta_y + a)g(x, y) = \delta_x(y).$$

This equation means that

$$\phi(x) = \int_{S^3\times\mathbb{R}} g(x, y)(\Delta_y + a)\phi(y)\, d\mathrm{vol}(y),$$

for compactly supported smooth functions ϕ. The existence of $g(x, y)$ is essentially standard [2, Chapter 4]. We briefly explain how to construct it. We fix $x \in S^3 \times \mathbb{R}$ and construct a function $g_x(y)$ satisfying $(\Delta + a)g_x = \delta_x$. As in [2, Chapter 4, Section 2], by using a local coordinate around x, we can construct (by hand) a compactly supported function $g_{0,x}(y)$ satisfying

$$(\Delta + a)g_{0,x} = \delta_x - g_{1,x},$$

where $g_{1,x}$ is a compactly supported continuous function. Moreover $g_{0,x}$ is smooth outside $\{x\}$ and it satisfies

$$\mathrm{const}_1/d(x, y)^2 \leqslant g_{0,x}(y) \leqslant \mathrm{const}_2/d(x, y)^2,$$

for some positive constants const_1 and const_2 in some small neighborhood of x. Here $d(x, y)$ is the distance between x and y. Since $(\Delta + a) : L^2_2 \to L^2$ is isomorphic, there exists $g_{2,x} \in L^2_2$ satisfying $(\Delta + a)g_{2,x} = g_{1,x}$. ($g_{2,x}$ is of class $\mathcal{C}^1$.) Then $g_x := g_{0,x} + g_{2,x}$ satisfies $(\Delta + a)g_x = \delta_x$, and $g(x, y) := g_x(y)$ becomes the Green kernel. $g(x, y)$ is smooth outside the diagonal. Since $S^3 \times \mathbb{R} = SU(2) \times \mathbb{R}$ is a Lie group and its Riemannian metric is two-sided invariant, we have $g(x, y) = g(zx, zy) = g(xz, yz)$ for $x, y, z \in S^3 \times \mathbb{R}$. $g(x, y)$ satisfies

$$c_1/d(x, y)^2 \leqslant g(x, y) \leqslant c_2/d(x, y)^2 \quad \bigl(d(x, y) \leqslant \delta\bigr), \tag{58}$$

for some positive constants c_1, c_2, δ.

Lemma A.1. $g(x, y) > 0$ *for* $x \neq y$.

Proof. Fix $x = (\theta_0, t_0) \in S^3 \times \mathbb{R}$. We have $(\Delta + a)g_x = 0$ outside $\{x\}$, and hence (by elliptic regularity)

$$\bigl|g_x(\theta, t)\bigr| \leqslant \mathrm{const}\|g_x\|_{L^2(S^3\times[t-1,t+1])} \quad \bigl(|t - t_0| > 1\bigr).$$

Since the right-hand side goes to zero as $|t| \to \infty$, g_x vanishes at infinity. Let $R > 0$ be a large positive number and set $\Omega := S^3 \times [-R, R] \setminus B_\delta(x)$. ($\delta$ is a positive constant in (58).) Since $g_x(y) \geqslant c_1/d(x, y)^2 > 0$ on $\partial B_\delta(x)$, we have $g_x \geqslant -\sup_{t=\pm R} |g_x(\theta, t)|$ on $\partial\Omega$. Since $(\Delta +$

$a)g_x = 0$ on Ω, we can apply the weak maximum (minimum) principle to g_x (Gilbarg and Trudinger [9, Chapter 3, Section 1]) and get

$$g_x(y) \geqslant -\sup_{t=\pm R} \left|g_x(\theta, t)\right| \quad (y \in \Omega).$$

The right-hand side goes to zero as $R \to \infty$. Hence we have $g_x(y) \geqslant 0$ for $y \neq x$. Since g_x is not constant, the strong maximum principle [9, Chapter 3, Section 2] implies that g_x cannot achieve zero. Therefore $g_x(y) > 0$ for $y \neq x$. □

Lemma A.2. *There exists $c_3 > 0$ such that*

$$0 < g(x, y) \leqslant c_3 e^{-\sqrt{a} d(x,y)} \quad \big(d(x, y) \geqslant 1\big).$$

In particular,

$$\int_{S^3\times\mathbb{R}} g(x, y)\, d\mathrm{vol}(y) < \infty.$$

The value of this integral is independent of $x \in S^3 \times \mathbb{R}$ because of the symmetry of $g(x, y)$.

Proof. We fix $x_0 = (\theta_0, 0) \in S^3 \times \mathbb{R}$. Since $S^3 \times \mathbb{R}$ is homogeneous, it is enough to show that $g_{x_0}(y) = g(x_0, y)$ satisfies

$$g_{x_0}(y) \leqslant \mathrm{const} \cdot e^{-\sqrt{a}|t|} \quad \big(y = (\theta, t) \in S^3 \times \mathbb{R} \text{ and } |t| \geqslant 1\big).$$

Let $C := \sup_{|t|=1} g_{x_0}(\theta, t) > 0$, and set $u := Ce^{\sqrt{a}(1-|t|)} - g_{x_0}(y)$ $(|t| \geqslant 1)$. We have $u \geqslant 0$ at $t = \pm 1$ and $(\Delta + a)u = 0$ $(|t| \geqslant 1)$. u goes to zero at infinity. (See the proof of Lemma A.1.) Hence we can apply the weak minimum principle (see the proof of Lemma A.1) to u and get $u \geqslant 0$ for $|t| \geqslant 1$. Thus $g_{x_0}(y) \leqslant Ce^{\sqrt{a}(1-|t|)}$ $(|t| \geqslant 1)$. □

The following technical lemma will be used in the next subsection.

Lemma A.3. *Let f be a smooth function over $S^3 \times \mathbb{R}$. Suppose that there exist non-negative functions $f_1, f_2 \in L^2$, $f_3 \in L^1$ and $f_4, f_5, f_6 \in L^\infty$ such that $|f| \leqslant f_1 + f_4$, $|\nabla f| \leqslant f_2 + f_5$ and $|\Delta f + af| \leqslant f_3 + f_6$. Then we have*

$$f(x) = \int_{S^3\times\mathbb{R}} g(x, y)(\Delta_y + a) f(y)\, d\mathrm{vol}(y).$$

Proof. We fix $x \in S^3 \times \mathbb{R}$. Let ρ_n $(n \geqslant 1)$ be cut-off functions satisfying $0 \leqslant \rho_n \leqslant 1$, $\rho_n = 1$ over $|t| \leqslant n$ and $\rho_n = 0$ over $|t| \geqslant n + 1$. Moreover $|\nabla \rho_n|, |\Delta \rho_n| \leqslant \mathrm{const}$ (independent of $n \geqslant 1$). Set $f_n := \rho_n f$. We have

$$f_n(x) = \int g(x, y)(\Delta_y + a) f_n(y)\, d\mathrm{vol}(y),$$
$$(\Delta + a) f_n = \Delta\rho_n \cdot f - 2\langle \nabla\rho_n, \nabla f\rangle + \rho_n(\Delta + a) f.$$

Note that $g_x(y) = g(x, y)$ is smooth outside $\{x\}$ and exponentially decreases as y goes to infinity. Hence for $n \gg 1$,

$$\int g_x |\Delta\rho_n \cdot f|\, d\mathrm{vol} \leqslant C\sqrt{\int_{\mathrm{supp}(d\rho_n)} f_1^2\, d\mathrm{vol}} + C\int_{\mathrm{supp}(d\rho_n)} g_x f_4\, d\mathrm{vol}(y).$$

Since $\mathrm{supp}(d\rho_n) \subset \{t \in [-n-1, -n] \cup [n, n+1]\}$ and $f_1 \in L^2$ and $f_4 \in L^\infty$, the right-hand side goes to zero as $n \to \infty$. In the same way, we get

$$\int g_x \bigl|\langle \nabla\rho_n, \nabla f\rangle\bigr|\, d\mathrm{vol} \to 0 \quad (n \to \infty).$$

We have $g_x|\rho_n(\Delta + a) f| \leqslant g_x|\Delta f + af|$, and

$$\int g_x(y)|\Delta f + af|\, d\mathrm{vol} \leqslant \int_{d(x,y)\leqslant 1} g_x(y)|\Delta f + af|\, d\mathrm{vol} + \Bigl(\sup_{d(x,y)>1} g_x(y)\Bigr)\int_{d(x,y)>1} f_3\, d\mathrm{vol}$$
$$+ \int_{d(x,y)>1} g_x f_6\, d\mathrm{vol} < \infty.$$

Hence Lebesgue's theorem implies

$$\lim_{n\to\infty}\int g_x\rho_n(\Delta + a) f\, d\mathrm{vol} = \int g_x(\Delta + a) f\, d\mathrm{vol}.$$

Therefore we get

$$f(x) = \int g_x(\Delta + a) f\, d\mathrm{vol}. \qquad \square$$

A.2. $(\nabla^*\nabla + a)$ *on sections*

Let E be a real vector bundle over $S^3 \times \mathbb{R}$ with a fiberwise metric and a connection ∇ compatible with the metric.

Lemma A.4. *Let ϕ be a smooth section of E such that $\|\phi\|_{L^2}$, $\|\nabla\phi\|_{L^2}$ and $\|\nabla^*\nabla\phi + a\phi\|_{L^\infty}$ are finite. Then ϕ satisfies*

$$\bigl|\phi(x)\bigr| \leqslant \int_{S^3\times\mathbb{R}} g(x, y)\bigl|\nabla^*\nabla\phi(y) + a\phi(y)\bigr|\, d\mathrm{vol}(y).$$

Proof. The following argument is essentially due to Donaldson [5, p. 184]. Let $\underline{\mathbb{R}}$ be the product line bundle over $S^3 \times \mathbb{R}$ with the product metric and the product connection. Set $\phi_n := (\phi, 1/n)$ (a section of $E \oplus \underline{\mathbb{R}}$). Then $|\phi_n| \geqslant 1/n$ and hence $\phi_n \neq 0$ at all points. We want to apply Lemma A.3 to $|\phi_n|$. $|\phi_n| \leqslant |\phi| + 1/n$ where $|\phi| \in L^2$ and $1/n \in L^\infty$. $\nabla\phi_n = (\nabla\phi, 0)$ and $\nabla^*\nabla\phi_n = (\nabla^*\nabla\phi, 0)$. We have the Kato inequality $|\nabla|\phi_n|| \leqslant |\nabla\phi_n|$. Hence $\nabla|\phi_n| \in L^2$. From $\Delta|\phi_n|^2/2 = (\nabla^*\nabla\phi_n, \phi_n) - |\nabla\phi_n|^2$,

$$(\Delta + a)|\phi_n| = \big(\nabla^*\nabla\phi_n + a\phi_n, \phi_n/|\phi_n|\big) - \frac{|\nabla\phi_n|^2 - |\nabla|\phi_n||^2}{|\phi_n|}. \tag{59}$$

Hence (by using $|\phi_n| \geqslant 1/n$ and $|\nabla|\phi_n|| \leqslant |\nabla\phi_n|$)

$$\big|(\Delta + a)|\phi_n|\big| \leqslant \big|\nabla^*\nabla\phi_n + a\phi_n\big| + n|\nabla\phi_n|^2 \leqslant \big|\nabla^*\nabla\phi + a\phi\big| + a/n + n|\nabla\phi|^2.$$

$|\nabla^*\nabla\phi + a\phi| + a/n \in L^\infty$ and $n|\nabla\phi|^2 \in L^1$. Therefore we can apply Lemma A.3 to $|\phi_n|$ and get

$$\big|\phi_n(x)\big| = \int g(x, y)(\Delta_y + a)\big|\phi_n(y)\big|\, d\mathrm{vol}(y).$$

From (59) and the Kato inequality $|\nabla|\phi_n|| \leqslant |\nabla\phi_n|$,

$$(\Delta_y + a)\big|\phi_n(y)\big| \leqslant \big|\nabla^*\nabla\phi_n + a\phi_n\big| \leqslant \big|\nabla^*\nabla\phi + a\phi\big| + a/n.$$

Thus

$$\big|\phi_n(x)\big| \leqslant \int g(x, y)\big|\nabla^*\nabla\phi(y) + a\phi(y)\big|\, d\mathrm{vol}(y) + \frac{a}{n}\int g(x, y)\, d\mathrm{vol}(y).$$

Let $n \to \infty$. Then we get the desired bound. □

Proposition A.5. *Let ϕ be a section of E of class $\mathcal{C}^2$, and suppose that ϕ and $\eta := (\nabla^*\nabla + a)\phi$ are contained in L^∞. Then*

$$\|\phi\|_{L^\infty} \leqslant (8/a)\|\eta\|_{L^\infty}.$$

Proof. There exists a point $(\theta_1, t_1) \in S^3 \times \mathbb{R}$ where $|\phi(\theta_1, t_1)| \geqslant \|\phi\|_{L^\infty}/2$. We have

$$\Delta|\phi|^2 = 2\big(\nabla^*\nabla\phi, \phi\big) - 2|\nabla\phi|^2 = 2(\eta, \phi) - 2a|\phi|^2 - 2|\nabla\phi|^2.$$

Set $M := \|\phi\|_{L^\infty}\|\eta\|_{L^\infty}$. Then

$$(\Delta + 2a)|\phi|^2 \leqslant 2(\eta, \phi) \leqslant 2M.$$

Define a function f on $S^3 \times \mathbb{R}$ by $f(\theta, t) := (2M/a)\cosh\sqrt{a}(t - t_1) = (M/a)(e^{\sqrt{a}(t-t_1)} + e^{\sqrt{a}(-t+t_1)})$. Then $(\Delta + a)f = 0$, and hence $(\Delta + 2a)f = af \geqslant 2M$. Therefore

$$(\Delta + 2a)\big(f - |\phi|^2\big) \geqslant 0.$$

Since $|\phi|$ is bounded and f goes to $+\infty$ at infinity, we have $f - |\phi|^2 > 0$ for $|t| \gg 1$. Then the weak minimum principle [9, Chapter 3, Section 1] implies $f(\theta_1, t_1) - |\phi(\theta_1, t_1)|^2 \geqslant 0$. This means that $\|\phi\|_{L^\infty}^2/4 \leqslant |\phi(\theta_1, t_1)|^2 \leqslant (2M/a) = (2/a)\|\phi\|_{L^\infty}\|\eta\|_{L^\infty}$. Thus $\|\phi\|_{L^\infty} \leqslant (8/a)\|\eta\|_{L^\infty}$. □

Lemma A.6. *Let η be a compactly supported smooth section of E. Then there exists a smooth section ϕ of E satisfying $(\nabla^*\nabla + a)\phi = \eta$ and*

$$\big|\phi(x)\big| \leqslant \int\limits_{S^3\times\mathbb{R}} g(x, y)\big|\eta(y)\big|\, d\mathrm{vol}(y).$$

Proof. Set $L_1^2(E) := \{\xi \in L^2(E) \mid \nabla\xi \in L^2\}$ and $(\xi_1, \xi_2)_a := (\nabla\xi_1, \nabla\xi_2)_{L^2} + a(\xi_1, \xi_2)_{L^2}$ for $\xi_1, \xi_2 \in L_1^2(E)$. (Since $a > 0$, this inner product defines a norm equivalent to the standard L_1^2-norm.) η defines the bounded functional

$$(\cdot, \eta)_{L^2} : L_1^2(E) \to \mathbb{R}, \quad \xi \mapsto (\xi, \eta)_{L^2}.$$

From the Riesz representation theorem, there uniquely exists $\phi \in L_1^2(E)$ satisfying $(\xi, \phi)_a = (\xi, \eta)_{L^2}$ for any $\xi \in L_1^2(E)$. Then we have $(\nabla^*\nabla + a)\phi = \eta$ in the sense of distribution. From the elliptic regularity, ϕ is smooth. ϕ and $\nabla\phi$ are in L^2, and $(\nabla^*\nabla + a)\phi = \eta$ is in L^∞. Hence we can apply Lemma A.4 to ϕ and get

$$\big|\phi(x)\big| \leqslant \int g(x, y)\big|\nabla^*\nabla\phi(y) + a\phi(y)\big|\, d\mathrm{vol}(y) = \int g(x, y)\big|\eta(y)\big|\, d\mathrm{vol}(y). \qquad \square$$

Proposition A.7. *Let η be a smooth section of E satisfying $\|\eta\|_{L^\infty} < \infty$. Then there exists a smooth section ϕ of E satisfying $(\nabla^*\nabla + a)\phi = \eta$ and*

$$\big|\phi(x)\big| \leqslant \int\limits_{S^3\times\mathbb{R}} g(x, y)\big|\eta(y)\big|\, d\mathrm{vol}(y). \tag{60}$$

(Hence ϕ is in L^∞.) In particular, if η vanishes at infinity, then ϕ also vanishes at infinity. Moreover, if a smooth section $\phi' \in L^\infty(E)$ satisfies $(\nabla^*\nabla + a)\phi' = \eta$ (η does not necessarily vanishes at infinity), then $\phi' = \phi$.

Proof. Let ρ_n ($n \geqslant 1$) be the cut-off functions introduced in the proof of Lemma A.3, and set $\eta_n := \rho_n\eta$. From Lemma A.6, there exists a smooth section ϕ_n satisfying $(\nabla^*\nabla + a)\phi_n = \eta_n$ and

$$\big|\phi_n(x)\big| \leqslant \int g(x, y)\big|\eta_n(y)\big|\, d\mathrm{vol}(y) \leqslant \int g(x, y)\big|\eta(y)\big|\, d\mathrm{vol}(y). \tag{61}$$

Hence $\{\phi_n\}_{n\geqslant 1}$ is uniformly bounded. Then by using the Schauder interior estimate [9, Chapter 6], for any compact set $K \subset S^3 \times \mathbb{R}$, the $\mathcal{C}^{2,\alpha}$-norms of ϕ_n over K are bounded $(0 < \alpha < 1)$. Hence there exists a subsequence $\{\phi_{n_k}\}_{k\geqslant 1}$ and a section ϕ of E such that $\phi_{n_k} \to \phi$ in the $\mathcal{C}^2$-topology over every compact subset in $S^3 \times \mathbb{R}$. Then ϕ satisfies $(\nabla^*\nabla + a)\phi = \eta$. ϕ is smooth by the elliptic regularity, and it satisfies (60) from (61).

Suppose η vanishes at infinity. Set $K := \int g(x,y)\,d\mathrm{vol}(y) < \infty$ (independent of x). For any $\varepsilon > 0$, there exists a compact set $\Omega_1 \subset S^3 \times \mathbb{R}$ such that $|\eta| \leqslant \varepsilon/(2K)$ on the complement of Ω_1. There exists a compact set $\Omega_2 \supset \Omega_1$ such that for any $x \notin \Omega_2$

$$\|\eta\|_{L^\infty} \int_{\Omega_1} g(x,y)\,d\mathrm{vol}(y) \leqslant \varepsilon/2.$$

Then from (60), for $x \notin \Omega_2$,

$$\big|\phi(x)\big| \leqslant \int_{\Omega_1} g(x,y)\big|\eta(y)\big|\,d\mathrm{vol}(y) + \int_{\Omega_1^c} g(x,y)\big|\eta(y)\big|\,d\mathrm{vol}(y) \leqslant \varepsilon/2 + \varepsilon/2 = \varepsilon.$$

This shows that ϕ vanishes at infinity.

Suppose that smooth $\phi' \in L^\infty(E)$ satisfies $(\nabla^*\nabla + a)\phi' = \eta$. We have $(\nabla^*\nabla + a)(\phi - \phi') = 0$, and $\phi - \phi'$ is contained in L^∞. Then the L^∞-estimate in Proposition A.5 implies $\phi - \phi' = 0$. □

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