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Mod-two APS index and domain-wall fermion

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Abstract
We show a mathematical relation between the mod-two Atiyah–Patodi–Singer (APS) index of a massless Dirac operator and massive domain-wall fermion determinant. The domain-wall fermion is given on a closed manifold, which is extended from the original manifold with boundary, where we instead give a fermion mass term changing its sign at the location of the original boundary. This new setup does not need the APS boundary condition, which is non-local. A mathematical proof of equivalence between the two different formulations is given by two different evaluations of the same index of a Dirac operator on a higher-dimensional manifold. The domain-wall fermion allows us to separate the edge and bulk mode contributions in a natural but not in a gauge invariant way, which offers a straightforward description of the global anomaly inflow.

Keywords Index theorem · Domain-wall fermion · Anomaly inflow

Mathematics Subject Classification 81T13

1 Introduction

Anomaly [1, 2] has played an important role in studying the low-energy dynamics of gauge theories, since it is always caused by (nearly) massless fields that describes the infra-red physics. As the anomaly is related to topology and thus invariant under the renormalization group flow, we can obtain non-trivial consequences which cannot be analyzed by perturbation. For example, 't Hooft [3] showed that the breaking pattern of the chiral symmetry in QCD is quite limited. Also, from anomaly in electro-weak

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interaction of quarks, we can determine the coefficient of the Wess–Zumino–Witten term [4, 5] in pion effective Lagrangian, which agrees well with experiments.

If a theory has anomaly in its gauge invariance, the theory is considered to be inconsistent, and cannot describe physics. However, the inconsistency may be cured by extending the theory to higher dimensions. For example, the anomalous four-dimensional chiral fermion can be embedded to five-dimensional vector-like gauge theory. In such cases, the anomaly is identified as the gauge current absorbed into the extra dimensions [6, 7]. This is called the anomaly inflow [8], which is recently widely studied not only in particle physics [9–15] but also in condensed matter physics [16–24].

Let us call the massless fermion on the original (even-dimensional) manifold the edge mode, and that living in the extra dimension the bulk mode, which is massive or gapped. The anomaly inflow caused by the edge mode is canceled by the bulk mode. This phenomenon matches well with the so-called bulk-edge correspondence [25, 26] of topological insulators. When the bulk fermion has anomaly on the boundary, the edge mode having the same anomaly with opposite sign must appear. This realization of the bulk-edge correspondence is valid for interacting fermions.

In [8], the notion of the global anomaly is extended using the anomaly inflow. The traditional argument on the global anomaly [27] is given by one-parameter family of gauge fields which connects two gauge equivalent configurations in d dimensions. One can treat this one-parameter as an extra dimension and when the extended $d+1$-dimensional theory has a non-trivial topology, the phase of the chiral fermion determinant cannot be uniquely determined. From the anomaly inflow point of view, this standard set-up is limited is limited spacetimes called mapping tori. In [8, 28, 29], it was claimed that the global anomaly should be extended to the case of any $d+1$-dimensional manifold. If the phase of the fermion determinant depends on the structure of the bulk manifold, we should regard the theory anomalous in that the phase cannot be uniquely determined with d-dimensional information alone.

More concretely, the anomaly inflow is generalized by the η invariant of a Dirac operator on the $d+1$-dimensional manifold [30, 31], where a non-trivial boundary condition known as the APS boundary condition [32–34] is imposed (see [35] for a physical description). However, the appearance of the APS boundary condition is somewhat puzzling in physics as it is non-locally imposed, and therefore, it is unlikely to be realized in any physical fermion systems. Moreover, the APS boundary condition allows no edge-localized mode to exist, which makes it difficult to separate the η-invariant into bulk and edge contributions.

Yonekura and Witten [36] explained that there is no need for imposing unphysical boundary condition on the physical fermion system. Instead, the APS boundary condition can be introduced as an intermediate state when we rotate the normal direction to the surface to the "time" direction. Then, the unphysical properties cancel out between bra and ket states and the physically reasonable local boundary condition is imposed without any difficulty. But it was not fully answered why the massless Dirac operator with unphysical properties must be introduced in the massive fermion system. Since the spatial boundary is lost in the rotation, it is still difficult to understand the role of the edge or bulk modes separately in the total η invariant.

We have been investigating more physicist-friendly alternative understanding of the anomaly inflow without introducing the unphysical boundary conditions at all. For pseudo-real fermions, the η invariant is reduced to an integer called the APS index [32–34]. In Refs. [37–40], we succeeded in reformulating this APS index using the domain-wall fermions [7, 42, 43]. We add an “outside” to the original boundary and consider a closed manifold in which the two domains are separated by a wall. Interestingly, only the half region is shared by the manifold on which the original APS index is formulated but the same index is obtained. Although the location of the domain-wall coincides with the boundary for the original APS, the boundary conditions imposed on fermions are totally different.

For the mod-two version of the APS index, the issue is more difficult, because the index cannot be expressed by any integral of local curvature functions, and no natural way is known to separate the edge and bulk contributions. As already mentioned, the APS boundary condition allows no edge-localized mode to exist.

In this work, we extend our formulation in [37–39] to the mod-two type index which describes the sign of the real fermion determinant. We will show below a mathematical relation between the domain-wall fermion determinant defined on a closed manifold to the APS index of the massless Dirac operator given on the half of the manifold with boundary, whose location coincides with the domain-wall. In contrast to our previous work limited to even dimensional bulk, this work can be applied to any dimensions.

The rest of the paper is organized as follows. In Sect. 2, we review the global anomaly originally found in [27], as well as recent development leading to the mod-two APS index, which, however, requires an unphysical boundary condition. In Sect. 3, we summarize our previous work where we achieved an alternative expression of the standard APS index using domain-wall fermions without introducing any non-local conditions. Then, we mathematically prove that the mod-two APS index can also be expressed by the domain-wall fermion Dirac operator in Sect. 4 and describe how the bulk-edge correspondence of the anomaly is embedded in the index in Sect. 5. In Sect. 6, we give a summary and discuss possible applications to higher-order topological insulators and lattice gauge theory.

2 Review of global anomaly

In this section, we review the global anomaly, where the mod-two index theorem plays a key role. Starting from the Witten’s $SU(2)$ anomaly [27], we also discuss a modern view of the anomaly as the current inflow to the higher-dimensional bulk. In this point of view, the anomaly can be identified as the η invariant of the massless Dirac operator on a manifold with boundary, as it was shown in [30, 31] that the η invariant satisfies properties required to describe the topological field theory on the manifold, which appears as an effective action of the bulk fermions.

The mod-two APS index naturally appears as a special case of the η invariant. However, it requires a non-local boundary condition on the fermion fields, which cannot be directly applied to the physical fermion systems.

2.1 Global anomaly and mod-two index

In [27], the first example of global gauge anomaly was shown, in which the sign of a Weyl fermion path-integral in the fundamental representation of the $SU(2)$ gauge group cannot be determined in a gauge invariant way. The same discussion applies to general Weyl or Majorana fermions whose Dirac operator is real and anti-symmetric.

Let us consider a real Dirac operator D_X on a manifold X and assume that it has no zero eigenvalue. The complex conjugate[1] of D_X is given as $D_X^* = CD_XC^{-1}$ with a unitary symmetric operator C [2]. Every nonzero eigenvalue of it makes a $\pm$ pair since for $D_X\phi = i\lambda\phi$, we have $D_XC^{-1}\phi^* = -i\lambda C^{-1}\phi^*$ (where λ is real). The Weyl or Majorana fermion Lagrangian is expressed as

$$\mathcal{L} = \frac{1}{2}\psi^T CD_X\psi, \tag{1}$$

with a Grassmannian variable ψ. One can choose a basis so that $C = 1$ and D_X is real anti-symmetric operator. In this basis, the path-integral is the Pfaffian of D_X, or $\mathrm{Pf}\, D_X$, which ends up with a product of half of eigenvalues taking one from all eigenvalue pairs. Since $\det D_X = (\mathrm{Pf}\, D_X)^2$ is real and positive, $\mathrm{Pf}\, D_X$ is real. This means that there is no perturbative gauge anomaly, always appearing as a variation in the complex phase. The sign of the $\mathrm{Pf}\, D_X$ is, thus, the only possible source of the anomaly, which is essentially non-perturbative.

Let us consider two gauge equivalent configurations A and A^g smoothly connected by a one-parameter family, say, parameterized by s: $A^s = (1-s)A + sA^g$. Here, the configuration A^g is obtained from A by an $SU(2)$ gauge transformation g. Since A and A^g are gauge equivalent, exactly the same spectrum of the Dirac eigenvalues is shared. However, some pairs of eigenvalues may be exchanged crossing zero somewhere in $0 < s < 1$, which is called the spectral flow. As $\mathrm{Pf}\, D_X$ is determined by only half of the eigenvalue pairs, if this spectral flow is odd, $\mathrm{Pf}\, D_X$ changes its sign.

Identifying the infinity in $\mathbb{R}^4$ as one point, or compactifying the spacetime to S^4, the gauge transformations are classified by the homotopy group $\pi_4(SU(2)) = \mathbb{Z}_2$. In [27], it was shown that when the gauge transformation is in the non-trivial class of $\pi_4(SU(2))$, the eigenvalues must change the sign by odd times and the spectral flow is odd. Therefore, the sign of $\mathrm{Pf}\, D_X$ is not determined in a gauge invariant way.

The proof was given using the mod-two Atiyah–Singer (AS) index. The one-parameter family s given above can be treated as the fifth dimension, to which the gauge connection A^s is naturally introduced. As the two points $s = 0$ and $s = 1$ are gauge equivalent, the extended spacetime we consider is equivalent to $S^4 \times S^1$, which is called a mapping torus. On this mapping torus, the Dirac operator D is still real, and the number of zero modes mod 2 is known as the mod-two AS index. It was

[1] In this work, we denote the complex conjugate by the superscript $*$ and the Hermitian conjugate by $\dagger$.

[2] C may contain a non-trivial operator on the gauge fields. For example, in the four-dimensional $SU(2)$ gauge theory with fermions in the fundamental representation, the gamma matrices act as pseudo-real operators: $\gamma_\mu^* = E\gamma_\mu E^{-1}$ with an anti-symmetric operator $E = \gamma_1\gamma_3$ (in the chiral representation). So do the gauge fields: we have $A^* = \varepsilon A \varepsilon^{-1}$ with $\varepsilon = \sigma_2$ (the second Pauli matrix). Then $C = E \otimes \varepsilon$, which is a unitary symmetric operator.

proved that the mod-two AS index always agrees with the spectral flow of original four-dimensional D_X as follows. Let us introduce another one-parameter family t, which connects A^0 at $t = -\infty$ and A^1 at $t = \infty$, where the t dependence is mild. The zero modes of D satisfies

$$\partial_t \Psi(t, x) = -\gamma^t D_X(t)\Psi(t, x), \tag{2}$$

where γ^t is the gamma matrix in the t-direction. In this adiabatic situation, the solution is approximated by $\Psi(t, x) = \phi(t)\psi_t(x)$, where $\psi_t(x)$ satisfies the four-dimensional Dirac equation $\gamma^t D_X(t)\psi_t(x) = \lambda(t)\psi_t(x)$, with the eigenvalue $i\lambda(t)$ of $D_X(t)$ at the time slice t. The solution $\phi(t)$ is formally given by

$$\phi(t) = \phi(0) \exp\left[-\int_0^t dt'\lambda(t')\right], \tag{3}$$

but this is normalizable only when $\lambda(t) > 0$ for $t \to \infty$ and $\lambda(t) < 0$ for $t \to -\infty$. Therefore, the number of zero modes of iD always agrees with the spectral flow of D_X. It was also shown that the index is always odd for A^t when the gauge transformation g is in the non-trivial class of $\pi_4(SU(2))$.

In [27], however, the direct equivalence between the mod-two index and the element of $\pi_4(SU(2))$ is not shown explicitly. Let us address this issue in a simpler case with S^5. In K-theory, a $USp(2n)$ bundle can be viewed as a quaternionic vector bundle, which is stably classified by KSp^0. In particular, an $SU(2)(\simeq USp(2))$ bundle over S^5 determines a class in $KSp^0(S^5) \simeq \pi_5(BUSp(\infty)) \simeq \pi_4(USp(\infty)) \simeq \pi_4(SU(2))$. The mod-two AS index corresponds to an isomorphic map from $KSp^0(S^5) \simeq KSp^{-5}(\text{point}) \simeq \mathbb{Z}_2$ to $KO^{-1}(\text{point}) \simeq \mathbb{Z}_2$.

The standard argument of global anomaly is similar to the anomaly inflow of the perturbative anomalies in that the extra dimension and associated Dirac operator are introduced. However, as the extra direction introduced is limited to S^1, it is difficult to treat the original Weyl fermion as the edge localized mode of the total system. The physical role of the bulk massive fermion is not obvious, either. In fact, in the next subsection, the notion of global anomaly is extended to incorporate general bulk manifold attached to the original spacetime. In the mathematical language, the extension is from the mod-two AS index to the mod-two APS index[3].

2.2 Global anomaly inflow (from mod-two AS to mod-two APS)

To understand the anomaly inflow, it is instructive to go back to the perturbative anomaly. It is well known that a single Weyl fermion in a complex representation of $SU(N)$ $(N > 2)$ gauge interactions suffers from anomaly and the theory is inconsistent. However, the anomaly is exactly the same as the surface term of the variation of the Chern–Simons (CS) action and, therefore, the gauge invariance can be recovered by adding a five-dimensional bulk fermion whose effective action contains the CS action to cancel the anomaly of the Weyl fermion. In this extension, known as

[3] In [41], the mod-two indices on non-compact manifolds $\mathbb{R}^d$ (for $d = 1, 2, 3, 4$) were considered.

the Callan–Harvey mechanism [7] the original anomaly can be regarded as a current escaping into the extra dimension, which is, in total, conserved in the five-dimensional system.

The extended theory is still "anomalous" since the theory is no longer defined on the original four-dimensional manifold. The theory is anomaly free when (the total sum of) CS action is zero.

In [8], it was argued that the Callan–Harvey mechanism can be applied to the global anomaly, as well. The anomalous n-dimensional fermion path integral can be cured by extending the theory to $(n+1)$ dimensions where the total phase is given by $\exp(i\pi\eta(iD))$, where $\eta(iD)$ is the η invariant of the Dirac operator iD, on the extended manifold [30, 31]. Here, the η invariant of a Hermitian operator H is given by a regularized sum of the sign of all eigenvalues λ,

$$\eta(H) = \sum_{\lambda} \mathrm{sgn}\lambda + h, \tag{4}$$

where h is the number of zero modes (namely, we count the zero modes as positive eigenvalues.). As the CS action is a perturbative part of the η invariant, the perturbative anomaly is properly included in this anomaly inflow argument.

The η invariant is gauge invariant, and the total theory is, thus, gauge invariant. The theory is still "anomalous," since the theory is no more defined on the original n-dimensional manifold X, but depends on the extended $(n+1)$-dimensional bulk. The theory is anomaly-free or consistent as an n-dimensional theory, only when the (total) η invariant is independent of the bulk. Using the gluing property of the η invariant, this anomaly-free condition is simply given by $\eta = 0$ (mod 2) on any closed manifold which is constructed by gluing two $(n+1)$-dimensional manifolds sharing X, the same n-dimensional boundary.

When D is real, the η invariant is reduced to the number of zero modes, h (Remember that all nonzero modes have $\pm$ pairs.). Namely, this index is the mod-two APS index on a $(n+1)$-dimensional manifold with the n-dimensional boundary. The notion of anomaly is extended in that we can put any $(n+1)$-dimensional bulk, in contrast to the traditional global anomaly limited to the mapping tori[4].

As is the case with perturbative anomaly, if we can relate $\exp(i\pi\eta(iD))$ to the path-integral of the massive fermion, we may be able to unite the notion of anomaly as the symmetry breaking of the n-dimensional massless edge modes, which is canceled by the bulk massive fermions. However, this is not straightforward since the definition of $\eta(iD)$ requires a special type of boundary condition, known as the APS boundary condition, to guarantee the Hermiticity of iD.

2.3 Non-local boundary condition

In the previous subsection, we have introduced $\eta(iD)$, which describes the phase of the fermion path-integral in $(n+1)$ dimension in a gauge invariant way. When iD acts

[4] Even in the framework of the mapping tori, a new-type of anomaly in the four-dimensional $SU(2)$ gauge theory was found [29].

on a field in a real representation, the mod-two APS index appeared as a special case of the η invariant whose nonzero eigenvalues cancel out. But we have not discussed in detail what kind of boundary conditions should be imposed on the $(n+1)$-dimensional fermions.

Before going into details, let us discuss yet another fermion species, or those in a pseudo-real representation under $Spin(n)$ and other symmetry group transformations. This pseudo-real fermion is special in that it allows the mass term. Therefore, any kind of gauge anomaly can be essentially removed by the Pauli–Villars regularization, for example. However, if we “require” the time-reversal (T) symmetry, which is known to be incompatible with gauge symmetry in odd dimensions and for odd number of Dirac fermions, the situation is exactly the same as the previous complex and real fermion examples. The gauge invariance needs bulk fermions.

In the pseudo-real fermion case, it is more natural to consider the anomaly inflow as the one for T symmetry, rather than gauge anomaly. Let us consider a three-dimensional manifold X and massless Dirac fermion with the $SU(N)(N > 2)$ gauge interaction on it, as an example,

$$\lim_{\Lambda\to\infty} \det \frac{D_X}{D_X+\Lambda} = \lim_{\Lambda\to\infty} \prod_{\lambda} \frac{i\lambda}{i\lambda+\Lambda} \propto \exp\left[-i\frac{\pi}{2}\eta(iD_X)\right], \tag{5}$$

where we have employed the Pauli–Villars regularization and $i\lambda$ denotes the eigenvalue of D_X. The η invariant appears since the phase of the determinant is essentially given by how many times i and $-i$ are multiplied, which correspond to the number of positive λ and negative λ, respectively. The T symmetry is broken as the T transformation flips the sign of the mass Λ, and thus the sign of $\eta(iD_X)$.

It is known that the smooth part of $\eta(iD_X)$ is the Chern–Simons action, half of which coincides the surface term of the instanton number density integrated over a four-dimensional manifold Y, whose boundary is the original three-dimensional manifold X. Thus we can add the bulk fermion so that its effective action becomes this instanton number density. The total phase

$$\exp\left[i\pi\left\{P-\frac{1}{2}\eta(iD_X)\right\}\right] = \exp(i\pi\mathcal{I}), \tag{6}$$

where P an integral of local function of curvature[5] over Y, is now guaranteed to be T invariant, as $\mathcal{I}$ is an integer known as the APS index. The APS index $\mathcal{I} = n_+ - n_-$ is defined by the number of zero modes $n_\pm$ with positive/negative chirality, respectively, of the Dirac operator iD on Y. This index is again a special case of $\eta(iD) = h = n_+ + n_- = \mathcal{I} + 2n_-$, where $2n_-$ is irrelevant to the fermion determinant phase. Note that the nonzero modes of iD make $\pm$ pairs by the chirality operator and do not contribute. This APS index beautifully explains the bulk-edge correspondence of the topological insulator where the T symmetry is protected by cancellation of the T anomaly.

[5] In four-dimensional flat space, it is well known that $P = \frac{1}{32\pi^2}\int_Y d^4x\epsilon^{\mu\nu\sigma\rho}\mathrm{Tr}F_{\mu\nu}F_{\nu\rho}$.

Now let us go back to the boundary condition of D. For a complete set $\{\phi_i\}$ of the operand of D, a natural choice would be $\gamma_\tau \phi_i|_X := n^\mu \gamma_\mu \psi_i|_X = \pm\phi_i|_X$, where n^μ is a normal vector to the surface X. This condition is local and respects rotational symmetry of X when it exists. However, this condition spoils the anti-Hermiticity of D by the surface contribution as

$$\int_Y d^{n+1}x\varphi_2^\dagger(x)D\varphi_1(x)+\int_Y d^{n+1}x(D\varphi_2(x))^\dagger\varphi_1(x)=\int_X d^n x\varphi_2^\dagger(x)\gamma_\tau\varphi_1(x), \quad (7)$$

for general $\varphi_1(x)$, $\varphi_2(x)$ satisfying the same boundary condition.

Instead, the original work by APS [32–34] chooses a different boundary condition, known as the APS boundary condition. Assuming a product structure in the metric near the boundary, and denoting the Dirac operator as $D = \gamma_\tau(\partial^\tau + A)$, they require the boundary modes to satisfy[6]

$$\frac{A + |A|}{|A|}\varphi_i(x)|_X = 0. \quad (8)$$

As A anticommutes with γ_τ, the surface contribution in Eq. (7) disappears to keep the anti-Hermiticity of D. Moreover, A commutes with the chirality operator and the index of D in terms of the chiral zero modes is well-defined. In [30, 31] a modified version was used, but the essential properties of APS are inherited.

However, as discussed in detail in [37, 38], the APS boundary condition is unnatural and unlikely to be realized in the materials. In particular, the boundary condition has little relation to the physics of topological insulators. Let us examine if the edge localized solution $\exp(-\lambda\tau)$ can exist near the boundary. If λ is an eigenvalue of A, the Dirac equation holds. But the solution is normalizable only when λ is positive, which is not allowed by the APS boundary condition. Namely, the APS condition prohibits the edge-localized modes to exist. This makes it difficult to understand the bulk-edge correspondence or anomaly inflow, in particular, in the mod-two APS index, as it has no intuitive separation of the bulk and edge contributions, in contrast to the T anomaly inflow in the standard APS index, or the perturbative gauge anomaly inflow of complex fermions. It is also unnatural to lose the rotational symmetry at the surface due to the gauge field dependence of A. Above all, the operator $|A|$ is non-local, which makes the causal structure of the system doubtful.

In [36], Witten and Yonekura explained that we do not need to impose any unphysical boundary condition in the fermion system but introduce the APS boundary condition just as an intermediate "state," rotating the normal direction to the "time." If the gap of the system is big enough, the overlap between the physical boundary state and the APS state is controlled by the ground state of the system and the unphysical features of the APS cancel out between the bra and ket states. Their argument justifies the use of the massless Dirac operator with the APS boundary condition even in the physical system. However, the fundamental question why the index or η invariant with such an unphysical property appears in the materials is not clear.

[6] A more mathematical definition will be given in Sect. 4

A natural question is whether we can reformulate the APS index, or η invariant without relying on the non-local boundary condition. To this question, a positive answer was partly given in our previous works. The key idea is to consider the so-called domain-wall fermion, as discussed in the next section.

3 Domain-wall fermion and standard APS index

In the original work by Callan and Harvey [7], where the anomaly inflow was first discussed, they considered a spacetime Y without any boundary, rather than a manifold with boundary. Instead, they introduced a space-dependent fermion mass (as a vacuum expectation value of scalar field) whose sign flips at some co-dimension one manifold X, which divides Y into $Y_+ \cup Y_-$. Here $Y_\pm$ denotes the region with positive/negative fermion mass. This is the so-called domain-wall fermion. As we will see below, the domain-wall fermion is a good model to describe the physics of topological insulators. The region Y_- corresponds to inside of topological insulator, and Y_+ is normal insulator. This setup is more realistic than a manifold with boundary, since any boundary in our world has "outside" of it.

Let us assume that Y is an odd-dimensional manifold, and X is located at $\tau = 0$ with a simple product structure of the metric of Y near X. Then, the Dirac equation becomes

$$0 = (D + m\kappa)\psi = (\gamma_\tau \partial_\tau + D_X + m\kappa)\, \psi = 0, \tag{9}$$

where $\kappa = \mathrm{sgn}(\tau)$ is a sign function such that $\mathrm{sgn}(\pm t) = \pm 1$ for $t > 0$, D_X is the Dirac operator on X, and $m > 0$ is a real constant. At the leading order of adiabatic approximation assuming slow τ dependence of the gauge field, the above equation has an edge-localized solution [42]:

$$\psi(x, \tau) = \phi(x) \exp(-m|\tau|), \quad \gamma_\tau \phi(x) = \phi(x), \quad D_X \phi(x) = 0, \tag{10}$$

where x is a local coordinate of X. The last two conditions show that the edge mode has positive chirality, and satisfies the massless Dirac equation on the domain-wall X.

In [7], it was shown that the edge-localized modes suffer from gauge anomaly, but it is precisely canceled by the surface term of the Chern–Simons action appearing as an effective action of the massive bulk modes in the region Y_-. As the total massive Dirac fermion determinant in Y can be regularized in a gauge invariant way, with Pauli–Villars fields, for instance, this anomaly cancellation is guaranteed at all order of perturbation. See [44] for a recent recomputation of this anomaly cancellation in a more microscopic and subtle treatment of edge and bulk modes near the domain-wall.

In our recent work [37–39], we have successfully described the anomaly inflow using the domain-wall fermion when Y is an even-dimensional manifold. Let us consider a determinant of the domain-wall fermion with Pauli–Villars regulator

$$\frac{\det(D + \kappa m)}{\det(D + m)} = \frac{\det i\gamma(D + \kappa m)}{\det i\gamma(D + m)}, \tag{11}$$

where we have taken the physical mass and the Pauli–Villars mass the same value for simplicity. The sign function κ again takes ± 1 on $Y_{\pm}$. Thanks to the existence of the chirality operator γ, the determinant is always real since $\det(D + \kappa m) = \det \gamma(D + \kappa m)\gamma = \det(D^{\dagger} + \kappa m)$.

From the right-hand side of Eq. (11), one can see that the sign of the determinant is controlled by the η invariant of the Hermitian operators $\gamma(D + \kappa m)$ and $\gamma(D + m)$. And it coincides with the APS index $\mathrm{Ind}_{\mathrm{APS}} D$, on the half of the manifold Y_{-} with the APS boundary condition is imposed on X. Namely, we have

$$\mathrm{Ind}_{\mathrm{APS}} D|_{Y_{-}} = -\frac{1}{2}\eta(\gamma(D + \kappa m)) + \frac{1}{2}\eta(\gamma(D + m)). \tag{12}$$

This non-trivial equivalence was perturbatively shown by three of the present authors [37, 38]. Then, the other three of the present authors who are mathematicians joined the collaboration and gave a mathematical proof [39] that the agreement is not a coincidence but generally true on any even-dimensional curved manifold when m is large enough.

In our reformulation of the APS index, we put the Dirac operator on a closed even-dimensional manifold Y, which ensures the anti-Hermiticity of D, and no specific boundary condition is needed. Instead, the local and rotational symmetric boundary condition is automatically given on the domain-wall. We have shown that the boundary η invariant entirely comes from the edge-modes localized on the wall, and the curvature integral part in the index is from the bulk modes. Thus, the bulk-edge correspondence is manifest in our reformulation. The non-local feature of the boundary η invariant is also naturally explained by the masslessness of the edge modes. This formulation is so physicist-friendly that even the application to the lattice gauge theory is achieved [40].

In this work, we pursue the mod-two version of APS index, which applies to the real fermions in odd dimensions. The most general case with complex fermions is still an open question, although we expect a similar relation between the domain-wall fermion and $\eta(D)$ with the APS boundary condition.

4 Main theorem

Here, we describe our main theorem using a mathematically precise language. The physics consequence is discussed in the next section.

4.1 Mod-two APS indices

In this subsection, for completeness, we will define the mod-two APS index for real skew-adjoint operators on manifolds with boundaries, which is a slight modification of the original APS index [32–34] for self-adjoint operators on manifolds with boundaries[7]. $\mathcal{H}_{\mathbb{R}}$ denotes a separable real Hilbert space, and $\mathcal{H}_{\mathbb{C}} := \mathcal{H}_{\mathbb{R}} \otimes \mathbb{C}$ its

[7] The definition of the mod-two APS index is easy, but it is important that a mod-two version of the APS index *theorem* does not exist.

complexification. A $\mathbb{C}$-linear operator D on $\mathcal{H}_{\mathbb{C}}$ is called real if it coincides with its complex conjugate, and skew-adjoint if $D^\dagger = -D$. The complexification of an $\mathbb{R}$-linear operator D on $\mathcal{H}_{\mathbb{R}}$ is also denoted by D, which is a real operator. The spectrum of an $\mathbb{R}$-linear operator on $\mathcal{H}_{\mathbb{R}}$ is always understood to be the spectrum of its complexification.

Recall that, for a real skew-adjoint Fredholm operator D on a separable real Hilbert space, the dimension mod 2 of its kernel is a deformation invariant [45]. So we define its *mod-two index* by

$$\mathrm{Ind}(D) := \dim\ker D \pmod 2.$$

For a closed manifold equipped with a real vector bundle, the mod-two index of a skew-adjoint elliptic operator is defined in the above way and studied by the mod-two index theorem of Atiyah and Singer [46]. Here we would like to formulate the mod-two APS index for the case with boundaries.

Let Y_- be a compact Riemannian manifold with boundary $X = \partial Y_-$, and S be a real Euclidean vector bundle over Y_-. We assume the collar structure $(-\epsilon, 0]_\tau \times X$ near the boundary of Y_-, and there exists a real Euclidean vector bundle S_X over X with the isomorphism $S_X \simeq S$ over the collar. We assume that S_X is equipped with a self-adjoint endomorphism $\gamma_X \in \mathrm{End}(S_X)$ with $\gamma_X^2 = \mathrm{id}_{S_X}$. Let D be a $\mathbb{R}$-linear formally skew-adjoint elliptic operator on $C^\infty(Y_-; S)$. Assume that, on the collar, D is of the form

$$D = \gamma_X \partial_\tau + D_X,$$

for some $\mathbb{R}$-linear skew-adjoint elliptic operator D_X on $C^\infty(X; S_X)$ which anti-commutes with γ_X, i.e., $\gamma_X D_X + D_X \gamma_X = 0$. In order to define the mod-two APS index, we assume that D_X *is invertible*[8].

In this setting, the APS boundary condition defined in [32–34] is the following. Note that $\gamma_X D_X$ is self-adjoint on $L^2(X; S_X)$. Let $P := \chi_{[0,\infty)}(\gamma_X D_X)$ denote the spectral projection onto the nonnegative eigenspaces of $\gamma_X D_X$.

Definition 1 (the APS boundary condition and mod-two APS indices) In the above settings, a smooth section $f \in C^\infty(Y_-; S)$ satisfies the APS boundary condition [32–34] if it satisfies

$$Pf|_X = 0.$$

The closure of this operator on $L^2(Y_-; S)$ with the above boundary condition, still denoted by D, is Fredholm. Moreover, if D_X is invertible, D is skew-adjoint. We define the mod-two APS index $\mathrm{Ind}_{\mathrm{APS}}(D) \in \mathbb{Z}_2$ of D by its mod-two index.

The mod-two APS indices have another formulation as follows. We consider $Y_{\mathrm{cyl}} := Y_- \cup [0, +\infty) \times X$ with the standard cylindrical-end metric. The bundle S and the operator D naturally extend to Y_{cyl}, which is denoted by S_{cyl} and D_{cyl}.

[8] The APS boundary condition is defined also in the case where D_X has a non-trivial kernel, but the resulting operator is not skew-adjoint.

Proposition 1 ([32,Proposition 3.11]) *If D_X is invertible, D_{cyl} is a skew-adjoint Fredholm operator on $L^2(Y_{\text{cyl}}; S_{\text{cyl}})$. Let us denote by $\text{Ind}(D_{\text{cyl}})$ its mod-two index. We have*

$$\text{Ind}_{\text{APS}}(D) = \text{Ind}(D_{\text{cyl}}).$$

4.2 The statement of the main theorem

Let Y be a closed Riemannian manifold whose dimension can be odd or even. Let S be a real Euclidean vector bundle on Y. Let $D : C^\infty(Y; S) \to C^\infty(Y; S)$ be a first-order, formally skew-adjoint, elliptic partial differential operator. Let $X \subset Y$ be a separating submanifold that decomposes Y into two compact manifolds Y_+ and Y_- with common boundary X. Let $\kappa : Y \to [-1, 1]$ be the L^∞–function such that $\kappa \equiv \pm 1$ on $Y_\pm \backslash X$. We define $D_{\text{DW}} = D + \kappa m \text{id}_S$ with a real positive number m as a domain-wall Dirac operator, where id_S is an identity matrix on S. We also define $D_{\text{PV}} = D + m\text{id}_S$, whose determinant corresponds to the Pauli–Villars regulator.

We assume that X has a collar neighborhood isometric to the standard product $(-4, 4) \times X$ and satisfying $((-4, 4) \times X) \cap Y_- = (-4, 0] \times X$. We denote the coordinate along $(-4, 4)$ by τ. We assume the collar structure on S and D explained in Sect. 4.1.

In the collar region, D_{DW} can be written as

$$D_{\text{DW}} = \gamma_X(\partial_X + \gamma_X \kappa m \text{id}_S + \gamma_X D_X). \tag{13}$$

For m large enough, D_{DW} is invertible. This can be shown in the same way as [39, Proposition 9], and can be understood as follows. In the large m limit, we have edge-localized modes proportional to $\exp(-m|\tau|)$ in a $\gamma_X = +1$ subspace, on which the domain-wall Dirac operator operates as $D_{\text{DW}} = D_X P_+$, where $P_+ = (1+\gamma_X)/2$[9]. When D_X at $\tau = 0$ has no zero eigenvalue, it is guaranteed that D_{DW} is invertible.

Theorem 1 *If D_X on $C^\infty(X; S_X)$ is invertible, then there exists a constant $m_0 > 0$ that depends only on X, S, and D such that for any $m > m_0$ we have,*

$$\text{Ind}_{\text{APS}}(D|_{Y_-}) = \frac{1 - \text{sgn}\det(D_{\text{DW}} D_{\text{PV}}^{-1})}{2} \pmod 2. \tag{14}$$

where $\text{sgn}\det(D_{\text{DW}} D_{\text{PV}}^{-1})$ *will be defined in Definition 2 below.*

Here, "sgn det" in the right hand side of (14) needs an explanation, because the operator $D_{\text{DW}} D_{\text{PV}}^{-1}$ is defined on infinite-dimensional Hilbert space. Note that the real invertible operator $D_{\text{DW}} D_{\text{PV}}^{-1}$ differs from the identity operator by a compact operator. For such operators, we define "sgn det" which generalizes the usual signature of the determinants of invertible real operators on finite-dimensional Hilbert spaces

[9] Note that D_X is self-adjoint but $D_X P_+$, which operates on the edge Weyl fermions, is not.

as follows. For a real Hilbert space $\mathcal{H}_{\mathbb{R}}$, let

$$\mathcal{C}(\mathcal{H}_{\mathbb{R}}) := \{A \in \mathrm{Id}_{\mathcal{H}_{\mathbb{R}}} + \mathcal{K}(\mathcal{H}_{\mathbb{R}}) \mid A \text{ is invertible }\}, \tag{15}$$

where $\mathcal{K}(\mathcal{H}_{\mathbb{R}})$ denotes the space of compact operators on $\mathcal{H}_{\mathbb{R}}$. The space $\mathcal{C}(\mathcal{H}_{\mathbb{R}})$, equipped with the norm topology, has two connected components [45, Proposition 3.3].

Definition 2 (sgn det) We define a map

$$\mathrm{sgn}\det : \mathcal{C}(\mathcal{H}_{\mathbb{R}}) \to \{1, -1\}$$

by letting $\mathrm{sgn}\det(A) := 1$ if A belongs to the same connected component of $\mathcal{C}(\mathcal{H}_{\mathbb{R}})$ with the identity, and $\mathrm{sgn}\det(A) := -1$ otherwise.

This map is a generalization of the "sgn det" for finite-dimensional case. Indeed, if $A \in \mathcal{C}(\mathcal{H}_{\mathbb{R}})$ is of the form $A = A_V \oplus \mathrm{id}_{V^\perp}$ for some finite-dimensional subspace $V \subset \mathcal{H}_{\mathbb{R}}$, then the value $\mathrm{sgn}\det(A)$ defined in Definition 2 coincides with the signature of the determinant of A_V.

4.3 Example on a closed manifold

Before giving a general proof, let us consider a special case with $Y_- = Y$ or $\kappa = -1$ on whole Y and there is no domain-wall. In this case, we obtain the mod-two AS index on whole Y.

Corollary 1 *For any $m > 0$, we have*

$$\mathrm{Ind}_{\mathrm{AS}}(D) = \frac{1 - \mathrm{sgn}\det\left[(D - m\mathrm{id}_S)(D + m\mathrm{id}_S)^{-1}\right]}{2} \pmod 2, \tag{16}$$

where $\mathrm{Ind}_{\mathrm{AS}}(D) = \dim\ker(D) \pmod 2$.

This corollary can be easily checked by a direct evaluation of the massive fermion determinant. Remembering that every nonzero eigenvalue $i\lambda$ of D makes a pair with another eigenvalue $-i\lambda$ (where λ is real) the ratio of the determinant is expressed as

$$\begin{aligned}\det\left[(D - m\mathrm{id}_S)(D + m\mathrm{id}_S)^{-1}\right] &= \frac{(-m)^{N_0}\prod_{\lambda>0}(\lambda^2 + m^2)}{m^{N_0}\prod_{\lambda>0}(\lambda^2 + m^2)} \\ &= (-1)^{N_0},\end{aligned}$$

where N_0 is the number of zero modes or $N_0 = Ind_{\mathrm{AS}}(D)$ mod 2.

4.4 Mathematical preparations: mod-two spectral flows and indices on cylinders

In this subsection, we give mathematical preparations necessary for the proof of the main theorem. In [47], Carey, Phillips and Schulz-Baldes introduced mod-two spectral flow for paths of real skew-adjoint Fredholm operators. After recalling it and its necessary properties, we relate it with "sgn det" in Definition 2. We also explain its relation with mod-two indices of operators on cylinders[10].

We have to deal with unbounded operators on Hilbert spaces. Unbounded operators appearing below are always assumed to be closed and densely defined. We topologize the set of unbounded closed densely defined Fredholm operators by the Riesz topology (see for example [48] for topologies on the space of unbounded Fredholm operators). In this topology, a family $\{D_x\}_{x\in X}$ of Fredholm operators is continuous if and only if the families $\{D_x(1+D_x^\dagger D_x)^{-1/2}\}_{x\in X}$ and $\{D_x^\dagger(1+D_x D_x^\dagger)^{-1/2}\}$ are both continuous with respect to the norm topology. Restricted to the subspace of bounded Fredholm operators, it coincides with the norm topology.

Now, we recall the definition of mod-two spectral flows for continuous paths of real skew-adjoint operators following [47]. The spectrum of a real skew-adjoint operator D lies in $\sqrt{-1}\mathbb{R}$ and satisfies $\mathrm{Spec}(D) = -\mathrm{Spec}(D)$. In generic cases, the mod-two spectral flow counts the parity of the number of crossings of eigenvalue pairs at 0.

First assume that $\mathcal{H}_\mathbb{R}$ is finite-dimensional. Given two invertible real skew-adjoint operators D_{-1} and D_1, the mod-two spectral flow between them is defined as follows. Choose an operator A on $\mathcal{H}_\mathbb{R}$ such that

$$D_1 = A^\dagger D_{-1} A.$$

Then, the mod-two spectral flow in the finite-dimensional case is simply

$$\mathrm{Sf}(D_{-1}, D_1) := \frac{1-\mathrm{sgn}\det(A)}{2} \in \mathbb{Z}_2. \tag{17}$$

Next we pass to the infinite-dimensional case. The definition in [47] is given for bounded families, but it is straightforward to extend it to the unbounded case[11]. The precise definition of mod-two spectral flow consists of subdividing a path into pieces small enough, and applying the definition for finite-dimensional paths for each pieces. Assume we are given a continuous family $\{D_t\}_{t\in[a,b]}$ of real skew-adjoint Fredholm operators on $\mathcal{H}_\mathbb{R}$, parameterized by a finite interval $[a,b] \subset \mathbb{R}$. We assume that D_a and D_b are invertible. For $\lambda > 0$ and $t \in [a,b]$, we define the corresponding spectral projection by

$$Q_\lambda(t) := \chi_{(-\lambda,\lambda)}(\sqrt{-1}D_t),$$

where $\chi_{(-\lambda,\lambda)}$ is the characteristic function of $(-\lambda,\lambda)$. $Q_\lambda(t)$ is a real projection. By Fredholmness of D_t, for λ small enough $Q_\lambda(t)\mathcal{H}_\mathbb{R}$ is a finite-dimensional subspace of $\mathcal{H}_\mathbb{R}$. For each t, take an arbitrary skew-adjoint operator R_t on the kernel of

10 The equivalence between the APS index and that on a non-compact manifold with cylindrical ends is proved in [32–34]. For the latter setup, non-local boundary condition is not needed but we still consider it unphysical, since having exact copies of gauge fields are not allowed by causality in physics.

11 In [49], the authors extend the definition of spectral flows to paths of operators with general Clifford symmetries. There, they also define spectral flows for paths of unbounded Fredholm operators.

$Q_\lambda(t)D_tQ_\lambda(t)$[12]. Let us denote

$$D_t^{(\lambda)} := Q_\lambda(t)D_tQ_\lambda(t) + R_t.$$

This is a real skew-adjoint invertible operator on $Q_\lambda(t)\mathcal{H}_{\mathbb{R}}$.

We choose a subdivision of the interval as $a = t_0 < t_1 < \cdots < t_N = b$, and a sequence of positive numbers $\{\lambda_n\}_{n=1}^N$ such that $Q_{\lambda_n}(t)$ is of constant finite rank on the interval $[t_{n-1}, t_n]$ for all n, and the orthogonal projection

$$V_n\colon Q_{\lambda_n}(t_{n-1})\mathcal{H}_{\mathbb{R}} \to Q_{\lambda_n}(t_n)\mathcal{H}_{\mathbb{R}}$$

is a bijection for all n. Using these data, the mod-two spectral flow of the path $\{D_t\}_t$ is defined as follows.

Definition 3 (Mod-two spectral flows)][[47, Definition 4.1]] Let $\{D_t\}_{t\in[a,b]}$ be a continuous path of real skew-adjoint possibly unbounded Fredholm operators on $\mathcal{H}_{\mathbb{R}}$. We assume that D_a and D_b are both invertible. Choosing additional data as above, we define the spectral flow of the path $\{D_t\}_{t\in[a,b]}$ by

$$\mathrm{Sf}(\{D_t\}_t) := \sum_{n=1}^N \mathrm{Sf}(D_{t_{n-1}}^{(\lambda_n)}, V_n^\dagger D_{t_n}^{(\lambda_n)} V_n).$$

Remark 1 The mod-two spectral flow in Definition 3 gives the family index of the path of real skew-adjoint operators, which is in $KO^{-1}([a,b],\{a,b\}) \simeq KO^{-2}(\mathrm{pt}) = \mathbb{Z}_2$. However, if we apply the same definition for a path of *odd* real skew-adjoint operators as we will do below, we actually get its family index which is an element of $KO^0([a,b],\{a,b\}) \simeq KO^{-1}(\mathrm{pt}) = \mathbb{Z}_2$. This is because, using the Atiyah–Bott–Shapiro description of $KO^{p-q}(\mathrm{pt})$ via Clifford modules $M_{p,q-1}/M_{p,q}$, there is a forgetful map $M_{1,1}/M_{1,2} \to M_{0,1}/M_{0,2}$ that sends generators to generators [49, Theorem 2.6]

For an unbounded path $\{D_t\}_{t\in[a,b]}$, we can also take the bounded transform $\{D_t(1+D_t^\dagger D_t)^{-1/2}\}_{t\in[a,b]}$ to get a bounded path and consider its mod-two spectral flow. We easily see that

$$\mathrm{Sf}(\{D_t\}_t) = \mathrm{Sf}(\{D_t(1+D_t^\dagger D_t)^{-1/2}\}_t). \tag{18}$$

4.4.1 The case of paths consisting of bounded operators

In this subsubsection, we deal with paths consisting of bounded operators. We relate "sgn det" in Definition 2 with a certain type of mod-two spectral flows.

In general, spectral flows are not determined by the operators at the endpoints, but depend on the choice of the paths. However, continuous deformations of the paths do

[12] The family $\{R_t\}_t$ is necessary in order to define mod-two spectral flows in the case where the original family has kernels on an interval of positive length.

not change the spectral flows, as long as they fix the endpoints [47, Theorem 4.3]. This implies the following.

Lemma 1 *Given two bounded paths* $\{D_t\}_{t\in[a,b]}$ *and* $\{D'_t\}_{t\in[a,b]}$ *satisfying the conditions in Definition 3, assume* $D_a = D'_a$, $D_b = D'_b$, *and that* $D_t - D'_t$ *is a compact operator for all* $t \in [a, b]$. *Then, we have*

$$\mathrm{Sf}(\{D_t\}_t) = \mathrm{Sf}(\{D'_t\}_t).$$

Proof Since the Fredholmness is preserved by adding compact operators, we get a continuous deformation between two paths $\{D_t\}_t$ and $\{D'_t\}_t$ by the linear homotopy.□

Thus, if we are given two invertible real skew-adjoint Fredholm operators D_{-1} and D_1 which differ by a compact operator, we get a distinguished value of spectral flows between them; namely those of paths consisting of compact perturbations between them.

Definition 4 Let D_{-1} and D_1 be two invertible real skew-adjoint bounded Fredholm operators on $\mathcal{H}_{\mathbb{R}}$. Assume that $(D_1 - D_{-1})$ is a compact operator. Take any path $\{D_t\}_{t\in[-1,1]}$ of real skew-adjoint Fredholm operators connecting D_{-1} and D_1, such that $(D_t - D_{-1})$ is a compact operator for all $t \in [-1, 1]$. Then we define

$$\mathrm{Sf}_{\mathrm{cpt}}(D_{-1}, D_1) := \mathrm{Sf}(\{D_t\}_{t\in[-1,1]}).$$

This does not depend on the choice of the path by Lemma 1.

For $\mathrm{Sf}_{\mathrm{cpt}}$, we have a similar formula as (17), which expresses the spectral flow by "sgn det" of operators defined in Definition 2.

Proposition 2 *Let* D_{-1} *and* D_1 *be two invertible real skew-adjoint bounded operators on* $\mathcal{H}_{\mathbb{R}}$. *Assume that there exists an element* $A \in \mathcal{C}(\mathcal{H}_{\mathbb{R}})$ *such that*

$$D_1 = A^\dagger D_{-1} A.$$

In particular, this means that $D_1 - D_{-1}$ *is compact. Then, we have*

$$\mathrm{Sf}_{\mathrm{cpt}}(D_{-1}, D_1) := \frac{1 - \mathrm{sgn}\det(A)}{2},$$

where $\mathrm{sgn}\det(A)$ *is defined in Definition 2.*

Proof Choose $\lambda > 0$ so that the spectrum of $\sqrt{-1}D_{-1}$ is discrete on the interval $[-\lambda, \lambda]$. The Hilbert space $\mathcal{H}_{\mathbb{R}}$ is decomposed as

$$\mathcal{H}_{\mathbb{R}} = Q_\lambda(-1)\mathcal{H}_{\mathbb{R}} \oplus (1 - Q_\lambda(-1))\mathcal{H}_{\mathbb{R}}$$

with the first component finite-dimensional. Choose a continuous path $\{A_t\}_{t\in[1,2]}$ in $\mathcal{C}(\mathcal{H}_{\mathbb{R}})$ such that $A = A_1$ and A_2 is of the form

$$A_2 = A_2|_{Q_\lambda(-1)\mathcal{H}_{\mathbb{R}}} \oplus \mathrm{id}_{(1-Q_\lambda(-1))\mathcal{H}_{\mathbb{R}}}.$$

This implies

$$\operatorname{sgn}\det(A) = \operatorname{sgn}\det(A_2|_{Q_\lambda(-1)\mathcal{H}_\mathbb{R}}). \tag{19}$$

Consider a path $\{D_t\}_{t\in[-1,2]}$ defined as

$$D_t := \begin{cases} \frac{1-t}{2}D_{-1} + \frac{t+1}{2}D_1 & \text{if } t \in [-1,1], \\ A_t^\dagger D_{-1} A_t & \text{if } t \in [1,2]. \end{cases}$$

Then, we have

$$\mathrm{Sf}_{\mathrm{cpt}}(D_{-1}, D_2) = \mathrm{Sf}(\{D_t\}_{t\in[-1,1]}) + \mathrm{Sf}(\{D_t\}_{t\in[1,2]}) = \mathrm{Sf}_{\mathrm{cpt}}(D_{-1}, D_1),$$

where the second equality follows from the invertibility of the family $\{D_t\}_{t\in[1,2]}$. Note that $\mathrm{Sf}_{\mathrm{cpt}}(D_{-1}, D_2)$ is equal to the spectral flow of the linear path between D_{-1} and D_2. Applying Definition 3 to this linear path, we see that

$$\mathrm{Sf}_{\mathrm{cpt}}(D_{-1}, D_2) = \frac{1 - \operatorname{sgn}\det(A_2|_{Q_\lambda(-1)\mathcal{H}_\mathbb{R}})}{2}.$$

Combining these with (19), we get the result. □

4.4.2 The case of paths consisting of elliptic pseudodifferential operators

In this subsubsection, we deal with the paths $\{D_t\}_t$ consisting of first order elliptic pseudodifferential operators on closed manifolds. Let us assume that $\mathcal{H}_\mathbb{R} = L^2(Y; S)$, where Y is a closed manifold and S is an $\mathbb{R}$-vector bundle with inner product over Y. Using the relation (18), we have the corresponding notion of $\mathrm{Sf}_{\mathrm{cpt}}$ in this setting.

Definition 5 Let Y and S as above. Let D_{-1} and D_1 be two invertible real skew-adjoint first order elliptic pseudodifferential operators on $L^2(Y; S)$. Assume that $D_1 - D_{-1}$ is of zeroth order. Take any path $\{D_t\}_{t\in[-1,1]}$ of real skew-adjoint elliptic operators connecting D_{-1} and D_1, such that $D_t - D_{-1}$ is of zeroth order for all $t \in [-1, 1]$. Then, we define

$$\mathrm{Sf}_{\mathrm{cpt}}(D_{-1}, D_1) := \mathrm{Sf}(\{D_t\}_{t\in[-1,1]}).$$

This does not depend on the choice of the path by Lemma 1 and (18).

We see that $\mathrm{Sf}_{\mathrm{cpt}}$ is also compatible with the bounded transform,

$$\mathrm{Sf}_{\mathrm{cpt}}(D_{-1}, D_1) = \mathrm{Sf}_{\mathrm{cpt}}(D_{-1}(1 + D_{-1}^\dagger D_{-1})^{-1/2}, D_1(1 + D_1^\dagger D_1)^{-1/2}), \tag{20}$$

where the left hand side is defined in Definition 5 and the right hand side is defined in Definition 4.

4.4.3 A relation between mod-two APS indices on cylinders and mod-two spectral flows

In this subsection, we assume that $\mathcal{H}_\mathbb{R}$ is $\mathbb{Z}_2$-graded. Let $\gamma \in O(\mathcal{H}_\mathbb{R})$ denote the $\mathbb{Z}_2$-grading operator. We deal with both cases where a family $\{D_t\}_t$ is bounded and unbounded. We explain a relation between mod two spectral flows of *odd* real skew-adjoint Fredholm operators and mod-two indices of certain operators on $\mathbb{R}$.

Proposition 3 *Let $\mathcal{H}_\mathbb{R}$ be $\mathbb{Z}_2$-graded with the grading operator γ. Let $\{D_t\}_{t\in[a,b]}$ be a continuous path of odd (i.e., $\gamma D_t + D_t\gamma = 0$ for all t) real skew-adjoint possibly unbounded Fredholm operators on $\mathcal{H}_\mathbb{R}$. We assume that D_a and D_b are both invertible.*

We construct a real skew-adjoint operator $\hat{D}$ on $L^2(\mathbb{R}_t) \otimes \mathcal{H}_\mathbb{R}$ as follows. By a continuous homotopy which fixes the endpoints, we perturb the path $\{D_t\}_{t\in[a,b]}$ into a smooth path $\{D_t^{\mathrm{sm}}\}_{t\in[a,b]}$ which is constant near the endpoints. We extend the path to $\{D_t^{\mathrm{sm}}\}_{t\in\mathbb{R}}$ by letting $D_t^{\mathrm{sm}} = D_a$ for $t < a$ and $D_t^{\mathrm{sm}} = D_b$ for $t > b$. We define $\hat{D}$ as

$$\hat{D} := \gamma\partial_t + D_t^{\mathrm{sm}}.$$

Then, $\hat{D}$ is Fredholm and its mod-two index does not depend on the choice of the smoothing $\{D_t^{\mathrm{sm}}\}_{t\in[a,b]}$. We have,

$$\mathrm{Ind}(\hat{D}) = \mathrm{Sf}(\{D_t\}_{t\in[a,b]}) \in \mathbb{Z}_2. \tag{21}$$

Proof The independence of $\mathrm{Ind}(\hat{D})$ on the choice of smoothings follows from the deformation invariance of indices.

We reduce the proof of (21) to finite-dimensional cases. In order to do this, we need the following easy properties of the indices of operators on cylinders.

(a) Given a path $\{D_t\}_{t\in[a,b]}$ as above, if D_t is invertible for all $t \in [a, b]$, we have $\mathrm{Ind}(\hat{D}) = 0$.
(b) Given a path $\{D_t\}_{t\in[a,b]}$ as above, assume that the path is divided into two paths as $\{D_t\}_{t\in[a,b]} = \{D'_t\}_{t\in[a,c]} \cup \{D''_t\}_{t\in[c,a]}$ with D_c invertible. We construct the operators $\hat{D}'$ and $\hat{D}''$ on $L^2(\mathbb{R}) \otimes \mathcal{H}_\mathbb{R}$ using $\{D'_t\}_t$ and $\{D''_t\}_t$, respectively, in the same way. Then, we have

$$\mathrm{Ind}(\hat{D}) = \mathrm{Ind}(\hat{D}') + \mathrm{Ind}(\hat{D}'').$$

(c) Given two paths $\{D_t\}_{t\in[a,b]}$ and $\{D'_t\}_{t\in[a,b]}$ as above, assume that $D_a = D'_a$ and $D_b = D'_b$, and that the two paths are connected by a continuous homotopy leaving the endpoints fixed. Then we have

$$\mathrm{Ind}(\hat{D}) = \mathrm{Ind}(\hat{D}').$$

Indeed, (a) is because $\hat{D}$ is invertible in such cases, (b) follows from the gluing property of the indices, and (c) follows from the deformation invariance of the indices.

Using the definition of mod-two spectral flows and the above properties of indices of operators on cylinders, as well as the corresponding properties of mod-two spectral flows [47, Theorem 4.2, 4.3], we can easily reduce to the case where $\mathcal{H}_{\mathbb{R}}$ is finite-dimensional. Moreover, using the above properties again, we are reduced to the case where $\mathcal{H}_{\mathbb{R}} = \mathbb{R}^2$,

$$\gamma = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad D_t = t\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \quad t \in [-1, 1].$$

In this case, we have $\mathrm{Sf}(\{D_t\}_{t\in[-1,1]}) = 1$. On the other hand, the L^2-kernel of $\hat{D}$ is one-dimensional, spanned by an element which is asymptotically $e^{-t}(1, -1)$ on $t \gg 1$ and $e^t(1, -1)$ on $t \ll -1$, so we have $\mathrm{Ind}(\hat{D}) = 1$. Thus, we get (21) and the result follows. □

Now assume that we are given two invertible odd real skew-adjoint operators D_{-1} and D_1 which differ by compact (resp. zero-th order) in the bounded case (resp. first-order elliptic case). In this case, we get a canonical choice of operator A satisfying $D_1 = A^\dagger D_{-1} A$. Namely, with respect to the $\mathbb{Z}_2$-grading, we decompose D_t, $t = \pm 1$, as

$$D_t = \begin{pmatrix} 0 & D_{+,t} \\ -(D_{+,t})^\dagger & 0 \end{pmatrix}. \tag{22}$$

Then, we can choose A to be,

$$A := \begin{pmatrix} (D^\dagger_{+,-1})^{-1} D^\dagger_{+,1} & 0 \\ 0 & \mathrm{id} \end{pmatrix}.$$

By the assumption on the difference between D_1 and D_{-1}, we see that $A \in \mathcal{C}(\mathcal{H}_{\mathbb{R}})$. In the bounded case, by Proposition 2 and the obvious identity $\mathrm{sgn}\det(A) = \mathrm{sgn}\det(A^\dagger)$, we get the following.

Proposition 4 *Let $\mathcal{H}_{\mathbb{R}}$ be $\mathbb{Z}_2$-graded with the grading operator γ. Assume we are given two invertible odd real skew-adjoint bounded operators D_{-1} and D_1 with $D_1 - D_{-1}$ compact. Then we have*

$$\mathrm{Sf}_{\mathrm{cpt}}(D_{-1}, D_1) = \frac{1 - \mathrm{sgn}\det(D_{+,1}(D_{+,-1})^{-1})}{2}.$$

Here $D_{+,t}$ is defined in (22).

In the first-order elliptic case, we have the corresponding result.

Proposition 5 *Let Y be a closed manifold and S be a $\mathbb{Z}_2$-graded real Euclidean vector bundle over Y. Assume we are given two invertible odd real skew-adjoint first-order elliptic operators D_{-1} and D_1 on $L^2(Y; S)$. Suppose that $D_1 - D_{-1}$ is of zeroth order. Then, we have*

$$\mathrm{Sf}_{\mathrm{cpt}}(D_{-1}, D_1) = \frac{1 - \mathrm{sgn}\det(D_{+,1}(D_{+,-1})^{-1})}{2}.$$

Here $D_{+,t}$ is defined in (22).

Proof For an unbounded Fredholm operator, let us denote by $\chi(D) := D(1 + D^\dagger D)^{-1/2}$ its bounded transform. Note that if D is odd and skew-adjoint, $\chi(D)$ is also odd and skew-adjoint. By (20) and Proposition 4, we get

$$\mathrm{Sf}_{\mathrm{cpt}}(D_{-1}, D_1) = \frac{1 - \operatorname{sgn}\det(\chi(D_{+,1})(\chi(D_{+,-1}))^{-1})}{2}.$$

Since the operator $(1 + D_{+,1}^\dagger D_{+,1})^{1/2}(1 + D_{+,-1}^\dagger D_{+,-1})^{-1/2}$ lies in the same connected component of $\mathcal{C}(\mathcal{H}_\mathbb{R})$ as the identity, we see that

$$\operatorname{sgn}\det(D_{+,1}(D_{+,-1})^{-1}) = \operatorname{sgn}\det(\chi(D_{+,1})(\chi(D_{+,-1}))^{-1}).$$

Thus, we get the result. □

4.5 Proof of main theorem

In this subsection, we prove Theorem 1. The proof given here relies on the techniques developed in our previous work [39]. We will see that, by modifying the proof in that paper appropriately, essentially the same proof works in the mod-two case. We give an alternative simpler and self-contained proof in Appendix.

First, in order to deal with smooth operators in the proof, we perturb the L^∞-function $\kappa\colon Y \to [-1, 1]$ to a smooth function $\kappa^{\mathrm{sm}}\colon Y \to [-1, 1]$ so that $\kappa^{\mathrm{sm}} \equiv \pm 1$ on $Y_\pm \setminus (-4, 4) \times X$ (recall the collar parameter introduced before the statement of Theorem 1). Consider the corresponding smoothed domain-wall Dirac operator,

$$D_{\mathrm{DW}}^{\mathrm{sm}} = D + \kappa^{\mathrm{sm}} m \mathrm{id}_S.$$

For m large enough, we have

$$\operatorname{sgn}\det(D_{\mathrm{DW}} D_{\mathrm{PV}}^{-1}) = \operatorname{sgn}\det(D_{\mathrm{DW}}^{\mathrm{sm}} D_{\mathrm{PV}}^{-1}).$$

This is because, for m large enough, the linear path connecting D_{DW} and $D_{\mathrm{DW}}^{\mathrm{sm}}$ consists of invertible operators.

Let us consider a $\mathbb{Z}_2$-graded vector bundle $S \oplus S$ over Y with the natural real structure, with the $\mathbb{Z}_2$-grading given by $\gamma = \mathrm{diag}(\mathrm{id}_S, -\mathrm{id}_S)$. Choose any smooth function $\hat{\kappa}^{\mathrm{sm}}\colon \mathbb{R} \times Y \to [-1, 1]$ such that $\hat{\kappa}_t^{\mathrm{sm}} := \hat{\kappa}^{\mathrm{sm}}(t, \cdot) = +1$ for $t < -0.5$ and $\hat{\kappa}_t^{\mathrm{sm}} = \kappa^{\mathrm{sm}}$ for $t > 0.5$. Let $D_t\colon L^2(Y; S \oplus S) \to L^2(Y; S \oplus S)$ be a one-parameter family of odd real skew-adjoint elliptic operators defined by

$$D_t := \begin{pmatrix} 0 & D + m\hat{\kappa}_t^{\mathrm{sm}} \mathrm{id}_S \\ D - m\hat{\kappa}_t^{\mathrm{sm}} \mathrm{id}_S & 0 \end{pmatrix}. \tag{23}$$

Note that

$$D_{-1} = \begin{pmatrix} 0 & D_{\mathrm{PV}} \\ -(D_{\mathrm{PV}})^{\dagger} & 0 \end{pmatrix}, \; D_1 = \begin{pmatrix} 0 & D_{\mathrm{DW}}^{\mathrm{sm}} \\ -(D_{\mathrm{DW}}^{\mathrm{sm}})^{\dagger} & 0 \end{pmatrix}, \tag{24}$$

and these two operators are both invertible. The path $\{D_t\}_{t\in[-1,1]}$ satisfies the condition in Definition 5; namely, $D_t - D_{-1}$ is of zeroth-order for all $t \in [-1, 1]$. By Proposition 5, we get

$$\mathrm{Sf}(\{D_t\}_{t\in[-1,1]}) = \mathrm{Sf}_{\mathrm{cpt}}(D_{-1}, D_1) = \frac{1 - \mathrm{sgn}\det(D_{\mathrm{DW}}^{\mathrm{sm}} D_{\mathrm{PV}}^{-1})}{2}. \tag{25}$$

Applying Proposition 3, we get the following. We introduce a real skew-adjoint operator $\hat{D}_m$ on $C^{\infty}(\mathbb{R} \times Y; S \oplus S)$ defined by

$$\hat{D}_m := \gamma\partial_t + D_t = \begin{pmatrix} \partial_t & D + m\hat{\kappa}_t^{\mathrm{sm}}\mathrm{id}_S \\ D - m\hat{\kappa}_t^{\mathrm{sm}}\mathrm{id}_S & -\partial_t \end{pmatrix}. \tag{26}$$

Then, we have

$$\mathrm{Ind}(\hat{D}_m) = \mathrm{Sf}(\{D_t\}_{t\in[-1,1]}). \tag{27}$$

By (25) and (27), we are left to prove the following.

$$\mathrm{Ind}_{\mathrm{APS}}(D|_{Y_-}) = \mathrm{Ind}(\hat{D}_m). \tag{28}$$

Now the proof is just a small modification of that of the main theorem of our previous work [39], so we summarize the main points here and refer the details to it. The strategy is to embed $Y_{\mathrm{cyl}} := Y_- \cup [0, +\infty) \times X$ into $\mathbb{R} \times Y$ in a certain way, and use localization argument to prove (28).

First, since D_X is assumed to be invertible, we can apply Proposition 1 and get

$$\mathrm{Ind}_{\mathrm{APS}}(D|_{Y_-}) = \mathrm{Ind}(D_{\mathrm{cyl}}). \tag{29}$$

Here S and D are extended to the cylinder to S_{cyl} and D_{cyl}, respectively, in a canonical way.

Moreover, let us consider a bundle $S_{\mathrm{cyl}} \oplus S_{\mathrm{cyl}}$ on $\mathbb{R}_s \times Y_{\mathrm{cyl}}$ and introduce a higher-dimensional Dirac operator

$$\hat{D}_{\mathrm{cyl}} = \begin{pmatrix} \partial_s & D_{\mathrm{cyl}} + m \;\mathrm{sgn}\; \mathrm{id}_S, \\ D_{\mathrm{cyl}} + m \;\mathrm{sgn}\; \mathrm{id}_S & -\partial_s \end{pmatrix}, \tag{30}$$

where $\mathrm{sgn} : \mathbb{R} \times Y_{\mathrm{cyl}} \to [-1, 1]$ is the L^{∞}-function[13] such that $\mathrm{sgn} = -1$ on $(-\infty, 0) \times Y_{\mathrm{cyl}}$ and $\mathrm{sgn} = 1$ on $(0, \infty) \times Y_{\mathrm{cyl}}$ (see Fig. 1). In the same way as [39, Section 3.3] we have $\dim \mathrm{Ker}\,\hat{D}_{\mathrm{cyl}} = \dim \mathrm{Ker}\, D_{\mathrm{cyl}}$.

[13] Here, the operator is not smooth, but it causes no problem. We may also use a smoothing of the function sgn if we like.

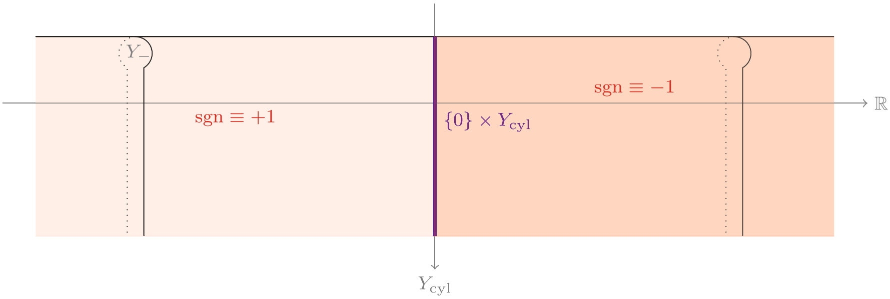

Fig. 1 $\mathbb{R}_s \times Y_{\text{cyl}}$, where the domain-wall is put at $s = 0$

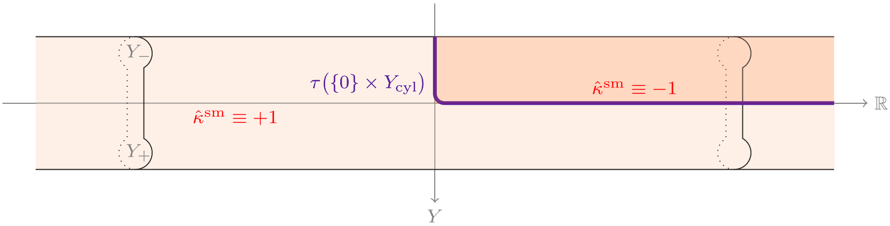

Fig. 2 $\mathbb{R} \times Y$, where $\{0\} \times Y_{\text{cyl}}$ is smoothly embedded

Now we recall the construction of a smooth embedding in [39, Section 3.4]. We define an embedding

$$\bar{\tau} : (-2, 2) \times Y_{\text{cyl}} \to \mathbb{R} \times Y.$$

Roughly speaking, the cylinder $\{0\} \times Y_{\text{cyl}} \subset (-2, 2) \times Y_{\text{cyl}}$ goes to a smoothing of the subset $\{0\} \times Y_- \cup [0, \infty) \times X \subset \mathbb{R} \times Y$.

Let $R_1 := (-2, 2) \times (-4, \infty)$ and $R_2 = \mathbb{R} \times (-4, 4)$. We denote the coordinate of R_1 by $(-\tau, t)$, and that of R_2 by (s, u). Fix an embedding $\tau_{\mathbb{R}^2} : R_1 \to R_2$ such that $\tau_{\mathbb{R}^2} \equiv \text{id}$ for $t < -2$ and

$$(-\tau, t) \mapsto (t, \tau)$$

for $t > 100$. Since X has a collar neighborhood isometric to $(-4, 4) \times X$, we can regard $R_1 \times X$ and $R_2 \times Y$ as open subsets of $(-2, 2) \times Y_{\text{cyl}}$ and $\mathbb{R} \times Y$, respectively. Using this, we can define an embedding $\bar{\tau}$ so that $\bar{\tau} \equiv \text{id}_{\mathbb{R}} \times \text{id}_Y$ on $(-2, 2) \times Y_-$ and $\bar{\tau} \equiv \tau_{\mathbb{R}^2} \times \text{id}_X$ on $R_1 \times X$. Note that $\bar{\tau}$ is an isometry outside a compact set $((-2, 2) \times (-2, 100) \times X)$ (Fig. 2).

Here, the important point for the localization argument is the following. We view $\bar{\tau}(\{0\} \times Y_{\text{cyl}})$ as a domain-wall in $\mathbb{R} \times Y$, which separates $\mathbb{R} \times Y$ into two connected components. Then, the smooth function $\hat{\kappa}^{\text{sm}} : \mathbb{R} \times Y \to [-1, 1]$ is a smoothing of the domain-wall function which takes value ± 1 on the two connected components, respectively.

In our previous work, we have shown that there is a one-parameter family of Riemannian metric connecting the induced metric by $\bar{\tau}$ and the original one on $\mathbb{R} \times Y$ so that the low lying spectrum is unchanged for $m > m_0$ with some real positive number m_0. Thus, for such m we have

$$\dim \operatorname{Ker} \hat{D}_{\text{cyl}} = \dim \operatorname{Ker} \hat{D}_m. \tag{31}$$

By (29) and (31), we get (28) and the result follows.

5 Anomaly inflow and bulk-edge correspondence in the mod-two APS index

In the previous section, we have proved that for any mod-two APS index of a Dirac operator on a manifold Y_- with boundary X, there exists a domain-wall Dirac fermion determinant

$$\det(D_{\text{DW}} D_{\text{PV}}^{-1}) = \det\left(\frac{D + \kappa m \mathrm{id}_S}{D + m \mathrm{id}_S}\right), \tag{32}$$

and the quantity $(1 - \operatorname{sgn} \det(D_{\text{DW}} D_{\text{PV}}^{-1}))/2$ coincides with the original index (mod 2). In the latter setup, instead of the boundary X, we put the "outside" Y_+ to form a closed manifold Y, and the mass term is introduced in such a way that the sign flips at the original location of X.

Contrary to the original APS's massless Dirac operator, which requires a non-local and unphysical boundary condition, the operator D in the domain-wall fermion determinant is kept anti-Hermitian (skew-adjoint) without any difficulty. The local and rotational symmetric boundary condition, which is commonly expected in the fermion system of topological insulators, is automatically satisfied on the domain-wall.

In this section, we discuss another physicist-friendly aspect of the domain-wall fermion formulation: it allows a "natural decomposition" of the index into bulk and edge contributions. To this end, we introduce another domain-wall fermion in the trivial representation with the opposite sign of the mass. Its free edge fermion has the opposite chirality to our target fermion. Then, we can add a set of Pauli–Villars fields where the target domain-wall field and the free field are coupled with another "mass" μ. This procedure corresponds to the traditional Pauli–Villars regularization of the Weyl fermion determinant. For complex fermions, this regularization is known to produce the consistent anomaly, satisfying the Wess–Zumino condition, which can be canceled precisely by the bulk Chern–Simons action. Below we show that this traditional treatment works even in the case of global anomaly.

Let us introduce a free fermion field, or a trivial bundle S_0 on Y, where we assume by an appropriate regularization, that $(S_0)_y$ the fiber at $y \in Y$, is isomorphic[14] to $(S)_y$ (but we do not assume a smooth isomorphism on whole Y). Then, we define the domain-wall Dirac operator with the opposite sign of the mass to the original fermion:

[14] For example, $(S_0)_y$ is isomorphic to $(S)_y$ at each site y in the lattice regularization.

$\partial_{\rm DW} = \partial - \kappa m \mathrm{id}_{S_0} : C^\infty(Y; S_0) \to C^\infty(Y; S_0)$ with a free Dirac operator ∂. As in the case with $D_{\rm DW}$, this new operator $\partial_{\rm DW}$ also has edge-localized eigenstates but with opposite chirality $\gamma_\tau = -1$. Here, we assume that $\partial_{\rm DW}$ is invertible, which is achieved by, for instance, choosing a spin structure such that the fermion obeys the anti-periodic boundary condition around a non-trivial cycle on X. We further assume that $\mathrm{sgn}\det \partial_{\rm DW}\partial_{\rm PV}^{-1} = +1$ with the free Pauli–Villars operator $\partial_{\rm PV} = \partial + m\mathrm{id}_{S_0}$. Namely, the corresponding mod-two index is always trivial.

Now we can decompose the $\mathrm{sgn}\det(D_{\rm DW}D_{\rm PV}^{-1})$ in Eq. (14) as follows.

$$\begin{aligned}\mathrm{sgn}\det(D_{\rm DW}D_{\rm PV}^{-1}) &= \mathrm{sgn}\left[\det(D_{\rm DW}D_{\rm PV}^{-1})\det(\partial_{\rm DW}\partial_{\rm PV}^{-1})\right]\\ &= \mathrm{sgn}\left[\det\begin{pmatrix} D_{\rm DW} & 0\\ 0 & \partial_{\rm DW}\end{pmatrix}\det\begin{pmatrix} D_{\rm PV} & 0\\ 0 & \partial_{\rm PV}\end{pmatrix}^{-1}\right]\\ &= \mathrm{sgn}\left[\det D_{\rm edge}\right]\mathrm{sgn}\left[\det D_{\rm bulk}\right], \end{aligned} \tag{33}$$

where $D_{\rm edge/bulk} : C^\infty(Y; S\oplus S_0) \to C^\infty(Y; S\oplus S_0)$ are defined as

$$D_{\rm edge} := \begin{pmatrix} D_{\rm DW} & 0\\ 0 & \partial_{\rm DW}\end{pmatrix}\begin{pmatrix} D_{\rm DW} & \mu I\\ \mu I^{-1} & \partial_{\rm DW}\end{pmatrix}^{-1}, \tag{34}$$

$$D_{\rm bulk} := \begin{pmatrix} D_{\rm DW} & \mu I\\ \mu I^{-1} & \partial_{\rm DW}\end{pmatrix}\begin{pmatrix} D_{\rm PV} & 0\\ 0 & \partial_{\rm PV}\end{pmatrix}^{-1}, \tag{35}$$

with a positive constant μ and a trivial isomorphism $I = \mathrm{diag}(1, 1...) : (S_0)_y \to (S)_y$ at each $y \in Y$. Note that both of $D_{\rm edge}$ and $D_{\rm bulk}$ are real operators, and therefore, $\mathrm{sgn}\left[\det D_{\rm edge}\right]$ and $\mathrm{sgn}\left[\det D_{\rm bulk}\right]$ are both well-defined in the same sense as that for the original operator $D_{\rm DW}D_{\rm PV}^{-1}$.

Now let us take a hierarchical limit $\lambda_{\rm edge} \ll \mu \ll m$, where $\lambda_{\rm edge}$ denotes a typical energy scale of the edge localized modes. In this limit, $\det D_{\rm edge}$ is dominated by contribution from the edge modes, since $D_{\rm edge}$ operates as $\mathrm{id}_{S\oplus S_0}$ up to μ/m corrections on the bulk modes. Similarly, $\det D_{\rm bulk}$ is essentially described by the bulk modes.

It is important to remark here that $D_{\rm edge/bulk}$ and their signs are not gauge invariant, due to the new mass term μI and its inverse. Therefore, $\mathrm{sgn}\left[\det D_{\rm edge}\right]$ depends on the choice of the gauge, and its gauge transformation can change its sign. This is exactly what we expect for the global anomaly. In their product $\mathrm{sgn}\left[\det D_{\rm edge}\right]\mathrm{sgn}\left[\det D_{\rm bulk}\right]$, however, the μ dependence precisely cancels out and the total index is gauge invariant. Now, we have manifestly achieved the global anomaly inflow, decomposing the mod-two APS index:

$$\begin{aligned}\mathrm{Ind}_{\rm APS}(D|_{Y_-}) &= I_{\rm edge} + I_{\rm bulk} \pmod 2,\\ I_{\rm edge} &= \frac{1-\mathrm{sgn}\left[\det D_{\rm edge}\right]}{2},\\ I_{\rm bulk} &= \frac{1-\mathrm{sgn}\left[\det D_{\rm bulk}\right]}{2}, \end{aligned} \tag{36}$$

where the gauge dependence of I_{edge} is canceled by that of I_{bulk}. Or equivalently, we can say that the gauge invariance of the APS index guarantees the bulk-edge correspondence of the global anomalies.

6 Summary and discussion

In this work, we gave a mathematical proof that for any APS index $\text{Ind}_{\text{APS}}(D)$ of a massless Dirac operator D on a manifold Y_- with boundary X, there exists a domain-wall Dirac fermion determinant, whose sign coincides with $(-1)^{\text{Ind}_{\text{APS}}(D)}$.

Our domain-wall fermion Dirac operator is formulated on a closed manifold extended from Y_-. Instead, the mass term flips its sign at the original location of X. Unlike the original APS setup, where an unphysical boundary condition is needed to keep the chiral symmetry and edge localized modes are not allowed to exist, the domain-wall fermion keeps many essential features to understand the physics of topological matters. No specific boundary condition is imposed *a priori*, but a local and physically sensible one having rotational symmetry is automatically imposed on the domain-wall. The distinction of the massless edge-localized modes and the bulk massive modes is manifest. Moreover, we find a natural decomposition of the mod-two APS index into edge and bulk contributions. Each of them is given by a non-gauge invariant integer, and therefore, contains a global anomaly. The gauge invariance of the mod-two APS index guarantees its cancellation or the bulk-edge correspondence of the global anomalies. Thus, our theorem indicates that the domain-wall fermion determinant (with Pauli–Villars regularization) can be used as a physicist-friendly "reformulation" of the mod-two APS index.

The mathematical proof was given introducing a higher $(d+2)$-dimensional Dirac operator $\hat{D}_m$[15], whose mod-two index is equal to the original $\text{Ind}_{\text{APS}}(D)$ and also equal to the spectral flow of a skew-adjoint operator D_t, which coincides with $(1 - \operatorname{sgn}\det(D_{\text{DW}}D_{\text{PV}}^{-1}))/2$. What is the physical meaning of $\hat{D}_m$? An interesting observation is that $\text{Ind}(\hat{D}_m)$ is equal to $\text{Ind}_{\text{APS}}(\hat{D}_m|_{Z_-})$, where $Z_- = Y \times [-1,1]$. Then, denoting $Z = Y \times \mathbb{R}$ and $Z_+ = Z \backslash Z_-$, and introducing $\rho : Z \to [-1,1]$ by $\rho \equiv \pm 1$ on $Z_\pm \backslash Y$, we can recursively use our main theorem to obtain

$$\text{Ind}(\hat{D}_m) = \frac{1 - \operatorname{sgn}\left[\det(\hat{D}_{\text{DW}}\hat{D}_{\text{PV}}^{-1})\right]}{2}, \tag{37}$$

where $\hat{D}_{\text{DW/PV}} : C^\infty(Z; S \oplus S) \to C^\infty(Z; S \oplus S)$ are defined by $\hat{D}_{\text{DW}} = \hat{D}_m + M\rho\text{id}_{S\oplus S}$ and $\hat{D}_{\text{PV}} = \hat{D}_m + M\text{id}_{S\oplus S}$, respectively, with a positive constant M, which is sufficiently larger than m. This new domain-wall fermion Dirac operator

$$\hat{D}_{\text{DW}} := \begin{pmatrix} \partial_t + M\rho\text{id}_S & D + m\hat{\kappa}\text{id}_S \\ D - m\hat{\kappa}\text{id}_S & -\partial_t + M\rho\text{id}_S \end{pmatrix} \tag{38}$$

[15] The physical role of $(d+2)$-dimensional Dirac operator was also discussed in our previous work [39].

in the large m and M limits, has an "edge-of-edge" solution, whose asymptotic behavior near $(\tau, t) = (0, 1)$ is given by

$$\begin{aligned} &\Psi(x, \tau, t) = \Phi(x) \exp(-m|\tau|) \exp(-M|t-1|), \\ &\mathrm{id} \otimes \gamma_\tau \Phi(x) = \Phi(x), \quad \gamma \otimes \mathrm{id}_S \Phi(x) = -\Phi(x), \quad D\Phi(x) = 0. \end{aligned} \tag{39}$$

Thus, our domain-wall fermion formulation naturally contains a mathematical structure that gapless states appear at a boundary of the system of codimension larger than one [50–52], which may be useful to understand the physics of higher-order topological insulators [53, 54].

Another interesting application is the formulation in lattice gauge theory[16]. On a flat Euclidean lattice with periodic boundary conditions, whose continuum limit corresponds to T^{d+1}, we can construct a lattice Dirac operator having the same properties as D_{DW} above. For example, in the $SU(2)$ gauge theory on a hyper-cubic five-dimensional lattice $Y^{\mathrm{lat}} = L^5$, the domain-wall Dirac operator $D^{\mathrm{lat}}_{\mathrm{DW}} : Y^{\mathrm{lat}} \otimes S^{\mathrm{lat}} \to Y^{\mathrm{lat}} \otimes S^{\mathrm{lat}}$ on a fermion field in the fundamental representation denoted by S^{lat} can be defined as

$$D^{\mathrm{lat}}_{\mathrm{DW}}(x, y) = D_W(x, y) + \kappa m \mathrm{id}_{S^{\mathrm{lat}}} \delta_{x,y}, \tag{40}$$

where $x = (x_1, x_2, x_3, x_4, x_5)$ and $y = (y_1, y_2, y_3, y_4, y_5)$ represent discrete lattice points on Y^{lat}, $\kappa = \mathrm{sgn}(x_5 + 1/2)\mathrm{sgn}(L/2 - x_5 - 1/2)$, the mass is in a range $0 < m < 2$ (to avoid contribution from doublers), and $D_W(x, y)$ is the Wilson Dirac operator

$$\begin{aligned} D_W &= \sum_{\mu=1}^{5} \gamma_\mu \frac{\nabla^f_\mu + \nabla^b_\mu}{2} - \sum_{\mu=1}^{5} \frac{\nabla^f_\mu \nabla^b_\mu}{2}, \\ \nabla^f_\mu(x, y) &= U_\mu(x)\delta_{x+1,y} - \delta_{x,y}, \\ \nabla^b_\mu(x, y) &= \delta_{x,y} - U^\dagger_\mu(y)\delta_{x-1,y}. \end{aligned} \tag{41}$$

Here, we take the lattice spacing unity. Note that the link variables $U_\mu(x)$ in the fundamental representation of $SU(2)$ is pseudo-real: $U_\mu(x)^* = \mathcal{E} U_\mu(x) \mathcal{E}$ with the second Pauli matrix $\mathcal{E} = i\tau_2$, which is anti-symmetric. This is also the case for $\gamma^*_\mu = C\gamma_\mu C$ with $C = \gamma_2\gamma_4\gamma_5$ (for the chiral representation), which is also anti-symmetric. Therefore, $D^{\mathrm{lat}}_{\mathrm{DW}}$ is real: $(D^{\mathrm{lat}}_{\mathrm{DW}})^* = C\mathcal{E} D^{\mathrm{lat}}_{\mathrm{DW}} C\mathcal{E}$. Then, we can "define" the mod-two APS index on the lattice by

$$\frac{1 - \mathrm{sgn}\left[\det(D^{\mathrm{lat}}_{\mathrm{DW}})\right]}{2} \mod 2, \tag{42}$$

and it is natural to conjecture that this lattice index for sufficiently large L and smooth link variables coincides with the continuum one on $T^4 \times [-L/2, 0]$. Note that the

[16] For the standard APS index, index a lattice formulation was proposed in [40] using the Wilson Dirac operator. Here we consider the mod-two version.

application to the mod-two AS index, which was already mathematically defined in [55], is straightforward, setting $\kappa = -1$ to define the mod-two AS index[17] on T^5 by

$$\frac{1 - \mathrm{sgn}\left[\det(D_W - m\mathrm{id}_{S^{\mathrm{lat}}})\right]}{2} \mod 2. \tag{43}$$

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A An alternative proof of the main theorem

In this appendix, we sketch an alternative simpler proof for the main theorem (Theorem 1). The proof given here does not rely on the techniques developed in our previous work [39], and is self-contained. In particular, we do not use the embedding of cylinder $Y_{\mathrm{cyl}} := Y_- \cup [0, +\infty) \times X$ into $\mathbb{R} \times Y$ or the localization argument.

We use the same notations as in Sect. 4.5. We proceed in the same way to get (25), and using Proposition 1, we are left to prove the equality

$$\mathrm{Sf}(\{D_t\}_{t\in[-1,1]}) = \mathrm{Ind}(D_{\mathrm{cyl}}). \tag{44}$$

We proceed in a different way from here.

First consider the following three operators acting on $L^2(Y_{\mathrm{cyl}}; S_{\mathrm{cyl}} \oplus S_{\mathrm{cyl}})$,

$$D_{\mathrm{cyl},-1} := \begin{pmatrix} 0 & D_{\mathrm{cyl}} + m\mathrm{id}_S \\ D_{\mathrm{cyl}} - m\mathrm{id}_S & 0 \end{pmatrix}, \tag{45}$$

$$D_{\mathrm{cyl},0} := \begin{pmatrix} 0 & D_{\mathrm{cyl}} - m\mathrm{id}_S \\ D_{\mathrm{cyl}} + m\mathrm{id}_S & 0 \end{pmatrix}, \tag{46}$$

$$D_{\mathrm{cyl},1} := \begin{pmatrix} 0 & D_{\mathrm{cyl}} + m\kappa^{\mathrm{sm}}_{\mathrm{cyl}}\mathrm{id}_S \\ D_{\mathrm{cyl}} - m\kappa^{\mathrm{sm}}_{\mathrm{cyl}}\mathrm{id}_S & 0 \end{pmatrix}. \tag{47}$$

Here $\kappa^{\mathrm{sm}}_{\mathrm{cyl}} \colon Y_{\mathrm{cyl}} \to [-1, 1]$ is a smooth function with $\kappa^{\mathrm{sm}}_{\mathrm{cyl}} \equiv -1$ on Y_- and $\kappa^{\mathrm{sm}}_{\mathrm{cyl}} \equiv 1$ on $X \times (4, +\infty)$. Let $\{D^s_{\mathrm{cyl},t}\}_{(s,t)\in[0,1]\times[-1,1]}$ denote the two-parameter family of operators defined as

$$D^s_{\mathrm{cyl},t} := \frac{1-t}{2} D_{\mathrm{cyl},-1} + \frac{(1-s)(1+t)}{2} D_{\mathrm{cyl},0} + \frac{s(1+t)}{2} D_{\mathrm{cyl},1}.$$

This family consists of real and formally skew-adjoint operators. Moreover, by the invertibility of D_X each operator is Fredholm, and the family is continuous, by the same argument as that in [57, Section 2]. We regard this as a path, parameterized by $s \in$

17 The mod-two AS index on a mapping torus was introduced to explain global anomaly by Witten [27]. Its lattice version was considered in Ref [56], where the Weyl fermion determinant is regularized by the overlap fermion on a lattice with a definite chirality projection, and its one parameter family connecting two gauge-equivalent configurations was discussed.

[0, 1], of paths $\{D^s_{\mathrm{cyl},t}\}_{t\in[-1,1]}$ of real skew-adjoint Fredholm operators. Obviously, $D^s_{\mathrm{cyl},-1} = D_{\mathrm{cyl},-1}$ is invertible. Using the invertibility of D_X, for m large enough, we can also see that $D^s_{\mathrm{cyl},1}$ are all invertible for all $s \in [0,1]$: this can be shown in the same way as [39, Proposition 9]. Thus, by the deformation invariance of spectral flows, we get

$$\mathrm{Sf}(\{D^0_{\mathrm{cyl},t}\}_{t\in[-1,1]}) = \mathrm{Sf}(\{D^1_{\mathrm{cyl},t}\}_{t\in[-1,1]}). \tag{48}$$

Moreover, at $s = 0$, we see directly from the definition of spectral flow that

$$\mathrm{Sf}(\{D^0_{\mathrm{cyl},t}\}_{t\in[-1,1]}) = \mathrm{Ind}(D_{\mathrm{cyl}}). \tag{49}$$

So we get

$$\mathrm{Sf}(\{D^1_{\mathrm{cyl},t}\}_{t\in[-1,1]}) = \mathrm{Ind}(D_{\mathrm{cyl}}). \tag{50}$$

Note that, restricted on the cylindrical end $X \times (4, \infty)$, the family $\{D^1_{\mathrm{cyl},t}\}_t$ does not depend on t.

In order to pass to the closed manifold Y, we consider the manifold $Y_{+,\mathrm{cyl}} := (-\infty, 0)\times X \cup Y_+$ with the corresponding bundle $S_{+,\mathrm{cyl}}$ and $D_{+,\mathrm{cyl}}$. Let $\{D_{+,\mathrm{cyl},t}\}_{t\in[-1,1]}$ the constant family of operators on $L^2(Y_{+,\mathrm{cyl}}; S_{+,\mathrm{cyl}} \oplus S_{+,\mathrm{cyl}})$ defined by

$$D_{+,\mathrm{cyl},t} := \begin{pmatrix} 0 & D_{+,\mathrm{cyl}} + m\mathrm{id}_S \\ D_{+,\mathrm{cyl}} - m\mathrm{id}_S & 0 \end{pmatrix}. \tag{51}$$

Of course, we have

$$\mathrm{Sf}(\{D_{+,\mathrm{cyl},t}\}_{t\in[-1,1]}) = 0. \tag{52}$$

By the gluing property of Fredholm index, we can show the corresponding gluing formula for mod-two spectral flows[18]. If we glue the family $\{D^1_{\mathrm{cyl},t}\}_{t\in[-1,1]}$ and $\{D_{+,\mathrm{cyl},t}\}_{t\in[-1,1]}$ along X, we get the family $\{D_t\}_{t\in[-1,1]}$ on $L^2(Y; S \oplus S)$ defined in (23), and get

$$\mathrm{Sf}(\{D_t\}_{t\in[-1,1]}) = \mathrm{Sf}(\{D^1_{\mathrm{cyl},t}\}_{t\in[-1,1]}) + \mathrm{Sf}(\{D_{+,\mathrm{cyl},t}\}_{t\in[-1,1]}) = \mathrm{Ind}(D_{\mathrm{cyl}}). \tag{53}$$

So the proof is complete.

Remark 2 The authors came up with this simpler proof while writing this paper. We can also prove the main theorem of our previous work [39] in a similar way.

Actually, the two proofs are essentially the same. The relation between them can be understood by comparing Figs. 3, 4 and 5. Figure 3 corresponds to the proof in Appendix, and Fig. 5 corresponds to that in Sect. 4.5. On the pink regions, we have

[18] One simplest way to show the gluing of spectral flows here is to use Proposition 3. Using it, we can reduce the problem to the gluing property of indices of operators on $Y_- \times \mathbb{R}$ and $Y_+ \times \mathbb{R}$, which is standard.

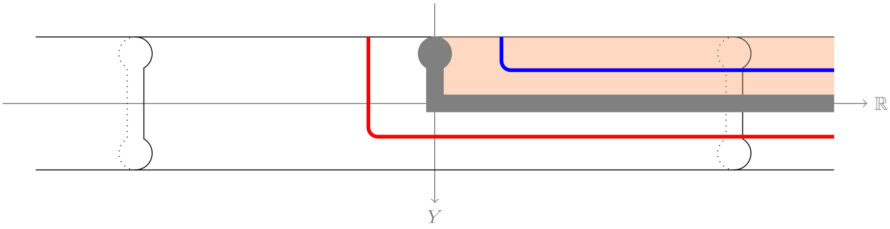

Fig. 3 The proof in Appendix

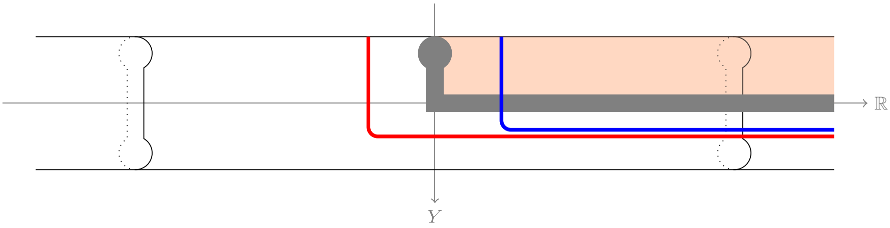

Fig. 4 Deformation of the two reference manifolds

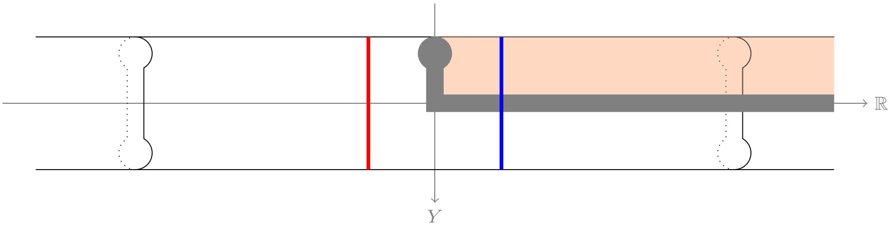

Fig. 5 The proof in Sect. 4.5

$\kappa = 1$, and on the white regions, we have $\kappa = -1$. In the proofs, we identified the mod-two APS index with the mod-two spectral flows between operators defined on red and blue parts. The fact that spectral flows in Figs. 3 and 5 coincide can be understood by moving the red and blue manifolds in the way shown in Fig. 4.

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