Ramsey Theorem

Course: SML - Graph Theory

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1 Ramsey's theorem

In this report, we consider a different type of edge coloring where 2 adjacent edges are not required to be assigned with 2 different colors. Hence, we consider simply edge labeling where the labels are colors. Also, a **k-edge coloring** of a graph is the assignment of one of k colors $c_1, c_2, \dots, c_k$ to each edge of the graph. Because our notion of edge coloring is more general, we have the following definition which is very essential for our discussion in this report.

Definition 1. Any graph with an edge coloring contains a **monochromatic subgraph** if it contains a subgraph with all edges assigned with the same color. In this case, the subgraph is **monochromatic**.

In the next definition, we note that "a graph G contains a graph G_0" means that the graph G contains a subgraph that is isomorphic to G_0.

Definition 2. Let $G_1, G_2, \dots, G_k$ be k simple graphs with at least one edge for $k \geq 2$. A positive integer $R(G_1, G_2, \dots, G_k)$ is called a **Ramsey number** if it is the smallest number such that for any $N \geq R(G_1, G_2, \dots, G_k)$, every k-edge coloring of K_N must contain G_j which is monochromatic in color c_j, for some j such that $1 \leq j \leq k$. If all of these graphs are complete graphs, i.e. $G_1 = K_{p_1}$, $G_2 = K_{p_2}$, ..., $G_k = K_{p_k}$, then we set $R(G_1, G_2, \dots, G_k) = R(p_1, p_2, \dots, p_k)$. Also, if all p_j for $1 \leq j \leq k$ are the number p, then we set $R(p_1, p_2, \dots, p_k) = R_k(p)$.

At this stage, we do not know whether the Ramsey number $R(G_1, G_2, \dots, G_k)$ exists or not for any graphs $G_1, G_2, \dots, G_k$. First of all, we prove that $R(p, q)$ exists for any $p, q \geq 2$. To do this, we need the following lemma.

Lemma 3. *For any positive integer $n \geq 2$, the Ramsey numbers $R(n, 2)$ and $R(2, n)$ exist and $R(n, 2) = R(2, n) = n$.*

Proof. We observe that K_2 has only one edge which can be colored by c_1 or c_2. For any $N \geq 2$, a complete graph K_N does not contain K_2 which is monochromatic in c_1 only when K_N is monochromatic in c_2. Hence, if $N \geq n$, then K_N contains K_n which is monochromatic in c_2, which implies that $R(2, n)$ exists and $R(2, n) = n$. Similarly, we can show that $R(n, 2)$ exists and $R(n, 2) = n$. □

The following theorem which was proved by Frank Ramsey in 1928 shows the existence of Ramsey numbers $R(p, q)$ for any $p, q \geq 2$.

Theorem 4 (Ramsey theorem)**.** *Let p, q be any positive integers such that $p, q \geq 2$. There exists a smallest positive integer $R(p, q)$ such that for any $N \geq R(p, q)$, every edge coloring of K_N in two colors c_1 and c_2 must contain either K_p which is monochromatic in the color c_1 or K_q which is monochromatic in the color c_2.*

Proof. By Lemma 3, we have $R(2, 2) = 2$ and $R(2, 3) = R(3, 2) = 3$. We prove this theorem by induction on the number $p + q$ and the base step is to prove the theorem for the case $p + q = 5$. Since $p, q \geq 2$, the solutions to $p + q = 5$ are $p = 2, q = 3$ and $p = 3, q = 2$. Hence, the theorem is true for the case $p + q = 5$ by Lemma 3.

Next, assume that the theorem is true for any p, q such that $5 \leq p + q < n$, for some integer n. Let $P, Q \geq 2$ be any positive integers such that $P + Q = n$. Because $(P - 1) + Q = n - 1 < n$ and $P+(Q-1) = n-1 < n$, there exist two Ramsey numbers $R(P-1, Q)$ and $R(P, Q-1)$. Consider any edge labeling of a complete graph K_N in two colors c_1 and c_2, where $N \geq R(P-1, Q)+R(P, Q-1)$. We select any vertex v of this graph. Let the number of c_1-colored and c_2-colored edges incident on v be k and l, respectively. If k and l both satisfy $k \leq R(P-1, Q)-1$ and $l \leq R(P, Q-1)-1$, then

$$\deg(v) = k + l \leq \big(R(P - 1, Q) - 1\big) + \big(R(P, Q - 1) - 1\big).$$

However, we have

$$\deg(v) = N - 1 \geq R(P - 1, Q) + R(P, Q - 1) - 1 > \big(R(P - 1, Q) - 1\big) + \big(R(P, Q - 1) - 1\big),$$

which is a contradiction. Therefore, either $k > R(P-1, Q)-1$ or $l > R(P, Q-1)-1$. Because k and l are integers, so either $k \geq R(P-1, Q)$ or $l \geq R(P, Q-1)$. Hence, the vertex v is either connected to $K_{R(P-1,Q)}$ by $R(P-1, Q)$ edges in the color c_1 or connected to $K_{R(P,Q-1)}$ by $R(P, Q-1)$ edges in the color c_2. In the former case, either $K_{R(P-1,Q)}$ contains K_{P-1} monochromatic in the color c_1, in which case K_{P-1} and the vertex v form K_P monochromatic in the color c_1, or $K_{R(P-1,Q)}$ contains K_Q which is monochromatic in the color c_2. Similarly, in the latter case, either $K_{R(P,Q-1)}$ contains K_{Q-1} monochromatic in the color c_2, in which case K_{Q-1} and the vertex v form K_Q monochromatic in the color c_2, or $K_{R(P,Q-1)}$ contains K_P which is monochromatic in the color c_1. Thus, we have found a finite upper bound for $R(P, Q)$:

$$R(P, Q) \leq R(P - 1, Q) + R(P, Q - 1),$$

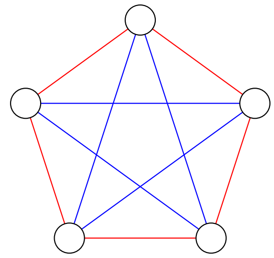
which implies that the Ramsey number $R(P, Q)$ exists. □

Remark. We can interpret this theorem in another way. For any 2-edge coloring of a complete graph K_n, we can remove all edges with the color c_2 to create a new graph G. If $n = |G| \geq R(p, q)$, then we can find either a clique with p vertices or an independent set consisting of q elements (a monochromatic K_q with all edges removed). In this case, we have either $w(G) \geq p$, or $\alpha(G) \geq q$ where $w(G)$ and $\alpha(G)$ are the clique number and the independence number of the graph G.

Example 5. In this example, we will prove that $R_2(3) = 6$, which is a very famous result called the theorem on friends and strangers. From the proof of Theorem 4, we have proven that $R(p, q) \leq R(p - 1, q) + R(p, q - 1)$. With $p = q = 3$, we have

$$R_2(3) = R(3, 3) \leq R(2, 3) + R(3, 2) = 3 + 3 = 6,$$

where we have used Lemma 3. Now, we need to show that $R_2(3) \geq 6$. Consider an edge coloring of K_5 in two colors, red and blue, as follows.



We can see that there is no monochromatic triangle K_3 in this edge coloring of K_5. Hence, we have $R_2(3) > 5$ and therefore, $R_2(3) = 6$.

To explain the origin of the name of the theorem, consider a party with n people, each of whom is represented by a vertex in the complete graph K_n. An edge in this graph is blue if the two people know each other, and red if the two people do not know each other. Then, $R_2(3) = 6$ means that there are always either 3 people knowing each other or 3 people not knowing each other if $n \geq 6$.

Ramsey's theorem initiated an entire combinatorial field called Ramsey theory even though the theorem is not the first result in this theory (we will see a result preceding Ramsey's theorem in the next section). A more general version of Ramsey's theorem is as follows.

Theorem 6. *Let $G_1, G_2, \dots, G_k$ be any k simple graphs with at least one edge for $k \geq 2$. There exists a smallest positive integer $R(G_1, G_2, \dots, G_k)$ such that for any $N \geq R(G_1, G_2, \dots, G_k)$, every k-edge coloring of K_N must contain G_j which is monochromatic in color c_j, for some j such that $1 \leq j \leq k$.*

Proof. Since a complete graph contains every graph with the same number of vertices, we have

$$R(G_1, G_2, \dots, G_k) \leq R(|G_1|, |G_2|, \dots, |G_k|).$$

Therefore, it suffices to prove only the existence of Ramsey numbers $R(p_1, p_2, \dots, p_k)$ for any positive integers $p_1, p_2, \dots, p_k$ and $k \geq 2$. We prove this by induction on k and start from the case $k = 2$, which follows from Theorem 4.

Assume that the theorem is true for any k such that $2 \leq k < n$ for some positive integer n. Then, there exists a Ramsey number $R(p_1, p_2, \dots, p_{n-1})$ for any positive integers $p_1, p_2, \dots, p_{n-1} \geq 2$. Now, choose any positive integer $p_n \geq 2$ and consider any n-edge coloring of the complete graph K_N where

$$N \geq R\left(p_0, p_n\right), \qquad p_0 = R(p_1, p_2, \dots, p_{n-1}).$$

We recolor any edge with any color other than c_n by a new color c_0. By Theorem 4, the graph K_N must contain either K_{p_0} monochromatic in the color c_0 or K_{p_n} monochromatic in the color c_n. In the former case, the original edge coloring of K_{p_0} is a $(n-1)$-edge coloring, so it must contain K_{p_j} which is monochromatic in the color c_j for some $1 \leq j \leq n-1$ by the induction hypothesis. Therefore, we have found a finite upper bound for $R(p_1, p_2, \dots, p_n)$:

$$R(p_1, p_2, \dots, p_n) \leq R\Big(R(p_1, p_2, \dots, p_{n-1}), p_n\Big),$$

which implies that the Ramsey number $R(p_1, p_2, \dots, p_n)$ exists. □

The study of Ramsey numbers in the case where all graphs are complete is called **classical Ramsey theory**. Otherwise, in the case where more general graphs are considered, it is called **generalized Ramsey theory**.

2 An application in number theory

For the next theorem, the set $\{1, 2, \dots, n\}$ is denoted by $[1, n]$. A r-coloring of $[1, n]$ means that every integer in $[1, n]$ is assigned one of the r colors c_k $(1 \leq k \leq r)$.
Theorem 7 (Schur's theorem)**.** *For any $r \geq 1$, there exists a smallest positive integer $s = s(r)$ such that for any r-coloring of $[1, s]$, there is a monochromatic solution in $[1, s]$ to the equation $x + y = z$. The number $s(r)$ is called a **Schur number**.*
Remark. "Monochromatic solution" means that x, y, z are assigned the same color in the r-coloring of $[1, n]$.

Proof. For the case $r = 1$, we have $s(1) \leq 2$ because if $[1, 2] = \{1, 2\}$ is colored by one color, then $(1, 1, 2)$ is a monochromatic solution $(1 + 1 = 2)$. In addition, we cannot find a monochromatic solution from an element of $[1, 1] = \{1\}$, so $s(1) = 2$.

For the case $r \geq 2$, we consider a complete graph K_{n+1} with each vertex assigned uniquely a number in $[1, n + 1]$ for some positive integer n. Randomly assign a color c_k for $k \in [1, r]$ to each number in $[1, n]$. Then, create a r-edge coloring of this graph where an edge connecting vertices i and j is assigned with the color c_k where c_k is the color of $|i - j|$. This edge coloring is well-defined because $1 \leq |i - j| \leq n$ for any i, j such that $1 \leq i, j \leq n + 1$. If $n + 1 \geq R_r(3)$ or equivalently, $n \geq R_r(3) - 1$, then the graph K_{n+1} must contain a monochromatic K_3 by Theorem 6. Let vertices of this monochromatic subgraph be denoted by a, b, c so that $a < b < c$. Then, $x_1 = b - a$, $y_1 = c - b$, $z_1 = c - a$ have the same color and (x_1, y_1, z_1) is a monochromatic solution to $x + y = z$ because

$$x_1 + y_1 = b - a + c - b = c - a = z_1.$$

Therefore, we have found a finite upper bound for $s(r)$:

$$s(r) \leq R_r(3) - 1,$$

which implies that the Schur number $s(r)$ exists. □

Originally, in 1916, Issac Schur did not prove this theorem by using Ramsey's theorem which was proved in 1928. However, this proof of Schur's theorem shows the relation between Ramsey numbers $R_r(3)$ and Schur numbers $s(r)$: for any $r \geq 2$, we have $s(r) \geq R_r(3) - 1$. In addition, Schur used Theorem 7 as a lemma to prove the next theorem which is interesting. Before stating this theorem, we need to introduce a notion from number theory. For any positive integer p, two integers x and y are said to be congruent modulo p if there exists an integer k such that $x - y = kp$. In this case, we write $x \equiv y \pmod{p}$. The theorem is stated as follows.
Theorem 8. *Let $n \geq 1$. There exists a prime q such that for all primes $p \geq q$ the congruence $x^n + y^n \equiv z^n \pmod{p}$ has a solution in the integers with $xyz \not\equiv 0$.*

We only briefly describe the proof of this theorem here because the detailed proof involve group theory and number theory (we do not want to go too far from graph theory). For any prime

$p > s(n)$, define a subset of $[1, p-1]$:

$$S = \left\{ a \in [1, p-1] \,\middle|\, \exists x \in [1, p-1] \text{ with } a \equiv x^n \pmod{p} \right\}.$$

Then, we partition $[1, p-1]$ into $r = r(n, p)$ sets of the form $d_i S$ for $d_i \in [1, p-1]$ and $1 \leq i \leq r$. Also, we can show that $r \leq n$, so we have $p - 1 \geq s(n) \geq s(r)$. Next, we color every element $a \in [1, p-1]$ by color c_i if $a \in d_i S$ for $1 \leq i \leq r$. Hence, by Theorem 7, there exist $a, b, c \in d_i S$ for some $1 \leq i \leq r$ such that $a + b = c$. As result, we have $a \equiv d_i x^n$, $b \equiv d_i y^n$, and $c \equiv d_i z^n \pmod{p}$ for some $x, y, z \in [1, p-1]$, and we also have

$$d_i x^n + d_i y^n \equiv d_i z^n \pmod{p} \quad \Leftrightarrow \quad x^n + y^n \equiv z^n \pmod{p}.$$

We know from Fermat's last theorem that the equation $x^n + y^n = z^n$ has no integer solution for $n \geq 3$. This result shows that we cannot prove Fermat's last theorem by congruence.

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