Eulerian graph and Optimal postman tour

Special Mathematics Lecture Graph Theory (2024 Spring)

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Introduction

In this report, I aim to learn Eulerian graph more deeply by solving some simple Postal delivery problems about Eulerian graph. Beforehand, let us recall the concepts of Eulerian graph and Postal delivery problems.

Eulerian graph

When we start at one vertex and come back to the vertex passing all edges of a graph each only one time, we call the graph an Eulerian graph or an Eulerian tour. These graphs have no vertices with odd degrees. On the other hand, when although we cannot return to a starting vertex, we can pass all edges of a graph each only one time, we call the graph a semi-Eulerian graph or an Eulerian trail. These graphs have two vertices with odd degrees which are the starting and targeting points.

Postal delivery problems

This is a famous question in Graph theory which consists in finding the most efficient closed walk to deliver luggage to all objective points. Concretely, in connected and finite graphs, we look for a closed walk which is the smallest total edge-weight passing each edge of the graph at least once.

Questions & Solutions

First, I recommend solutions of Postal delivery problems in Eulerian graph.

In Eulerian graph, a closed walk drawn with a single stroke of the brush from starting vertex to the same vertex is the answer to this question. The way finding the closed walk is as follows.

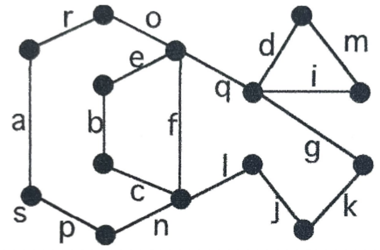
Step 1: Decide the starting vertex and find closed walk starting from the vertex.
Step. 2: Find a closed walk starting a vertex which is an element of the closed walk of Step. 1 and return to the vertex.
Step 3: Until all elements of a graph is included in a closed walk, continue the same process.
Step 4: Finally, we connect the closed walk. (When we arrive at the starting point of a closed walk, follow the closed walk and return to the previous closed walk.)

In semi-Eulerian graph, we need return to targeting vertex in one stoke from starting vertex.

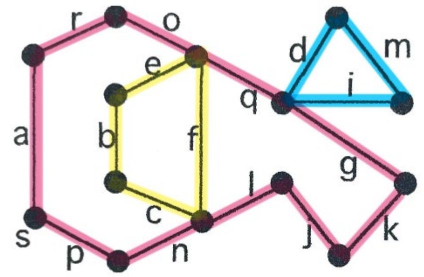
Therefore, in addition to above 4 Steps, we need to do next Step 5.

Step 5: Find the smallest total edge-weight walk from targeting vertex to starting vertex with Dijkstra method and pass the walk.

Question 1: Construct an Eulerian tour of the given graph. Begin the construction at vertex s. (Graph Theory and Its Applications, Third Edition p.263 6.1.9)



Solution: This graph has no vertices with odd degrees, so it is a Eulerian graph. Therefore, we can solve it by following Step 1 to Step 4.



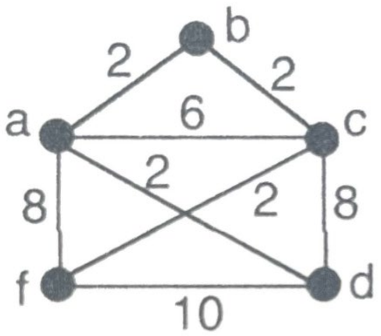
Step 1: (s)→a→r→o→q→g→k→j→l→n→p→(s)

Step 2: f→c→b→e

Step 3: i→m→d

Step 4: (s)→a→r→o→f→c→b→e→q→i→m→d→g→k→j→l→n→p→(s)

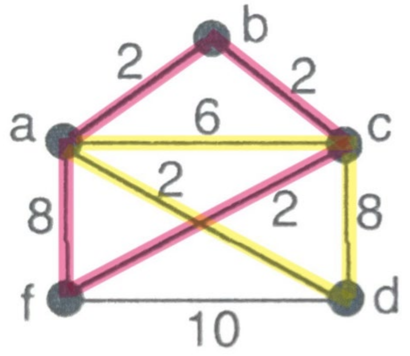
Question 2: Find a minimum-weight postman tour for the given weighted graph. (Graph Theory and Its Applications, Third Edition p.277 6.2.13)



Solution: This graph has two vertices (f, d) with odd degrees, so it is a semi-Eulerian graph. I solved this question as follows. First, I treat f as starting vertex and d as targeting vertex and look for a closed walk drawn with a single stroke of the brush from starting vertex f to the targeting vertex d. The closed walk is f-a-c-d-a-b-c-f-d. Next, I look for a smallest weighted path from d to f. It is d-a-b-c-f.

Therefore, the answer is as follows.

f→a→c→d→a→b→c→f→d→a→b→c→f



References

[1] Serge Richard. Special Mathematics Lecture: Graph Theory, 2024.

[2] Jonathan L. Gross, Jay Yellen, Mark Anderson," Graph Theory and Its Applications" third edition CRC press, 2019.