Maximum-Bipartite-Matching and Hall's marriage theorem

Special Mathematics Lecture Graph Theory (2024 Spring)

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Introduction

In this report, I aim to understand Matching more deeply through solving some fundamental questions of matching in the book "Graph Theory and Its Applications, Third Edition". Before solving questions, let us recall the Hall's marriage theorem.

Hall's marriage theorem indicates that in a bipartite graph G with bipartition subsets V1 and V2, necessary and sufficient condition for the existence of perfect matching from V1 to V2 is that any subset U of V1 has $|U| \leq |N(U)|$, where $|U|$ and $|N(U)|$ denote the cardinality of these sets. Therefore, if we can find $|U| > |N(U)|$ in one subset U, we can say the bipartite graph has not perfect matching from V1 to V2.

Questions & Solutions

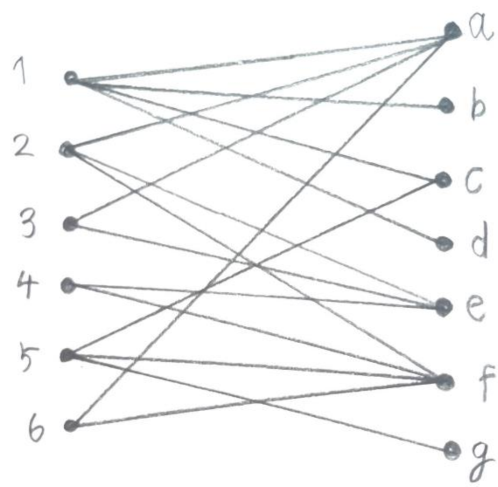
Question 1: The Art History Department would like to offer six courses during the fall semester. There are seven professors in the department, each of whom is willing to teach certain courses, as shown in the table. Is there an assignment of professors to courses so that no professor teaches more than one course? (Graph Theory and Its Applications, Third Edition p.479 10.4.6)

Course	Professor
Greek & Roman	Shargaa, Ward, Johnson, Pate
Renaissance	Maupin, Shargaa, Margeson
Baroque	Maupin, Shargaa
Impressionism	Maupin, Margeson
Early Modern	Vigorito, Johnson, Margeson
Contemporary	Shargaa, Margeson

Solution: First, we treat the set of courses as V1 and the set of professors as V2, and decide symbols and numbers as follows. Then, we wrote a bipartite graph with bipartition subsets V1 and V2.

<V1>
1:Greek & Roman
2:Rrenaissance
3:Baroque
4:Impressionism
5:Early Modern
6:Contemporary

<V2>
a:Shargaa
b:Ward
c:Johnson
d:Pate
e:Maupin
f:Margeson
g:Vigorito



This graph has no perfect matching from V1 to V2. The proof is as follows.

Proof:

We treat the subset of V1 as subset U and consider about subset U= {2,3,4,6}

N(2)= {a,e,f}
N(3)= {a,e}
N(4)= {e,f}
N(6)= {a,f}

Hence, N(U)= {a,e,f}
|N(U)|=3

Since in U= {2,3,4,6} , |U|=4 and |N(U)|=3 , in other words, |U|>|N(U)|, this graph has no perfect matching. □

Therefore, there is no assignment of professors to courses which each professors teach one course.

Question 2: Determine whether Hall's Condition for the existence of a transversal is met by the given family of subsets ({1, 2, 5}, {1, 5}, {1, 2}, {2, 5}). If the condition is met, then find a transversal; otherwise, show how the condition is violated. (Graph Theory and Its Applications, Third Edition p.480 10.4.8)

Solution: We added symbols to each subset as follows.

a= {1,2,5}
b= {1,5}
c= {1,2}
d= {2,5}

and we treat subset U= {a,b,c,d}
Since N(U)= {1,2,5} , |U|=4 and |N(U)|=3, in other words, |U|>|N(U)|.
Therefore, as the reason for Hall's marriage theorem, this family of subset violate Hall's condition. □

References

[1] Serge Richard. Special Mathematics Lecture: Graph Theory, 2024.
[2] Jonathan L. Gross, Jay Yellen, Mark Anderson," Graph Theory and Its Applications" third edition CRC press, 2019.