Proof of Brooks' Theorem

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1 Brooks' Theorem

Brooks' Theorem states that for any connected graph G, if G is neither a complete graph nor an odd cycle, then the chromatic number $\chi(G)$ is at most the maximum degree $\Delta(G)$.

2 Proof

We will consider two cases in the proof of Brooks' Theorem: one for general connected graphs and another for k-regular graphs.[1]

2.1 Case 1: General Connected Graphs

If G is a connected graph other than a complete graph or an odd cycle, then $\chi(G) \leq \Delta(G)$.

Proof. Let G be a connected graph and let $k = \Delta(G)$. We may assume $k \geq 3$, since G is a complete graph when $k \leq 1$, and G is bipartite when $k = 2$, in which case the bound holds.

Our aim is to order the vertices so that each has at most $k-1$ lower-indexed neighbors; greedy coloring for such an ordering yields the bound.

When G is not k-regular, we can choose a vertex of degree less than k as v_n. Since G is connected, we can grow a spanning tree of G from v_n, assigning indices in decreasing order as we reach vertices. Each vertex other than v_n in the resulting ordering $v_1, \ldots, v_n$ has a higher-indexed neighbor along the path to v_n in the tree. Hence each vertex has at most $k-1$ lower-indexed neighbors, and the greedy coloring uses at most k colors. □

Figure 1: case 1

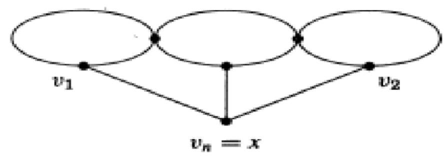

Figure 2: case 2

2.2 Case 2: k-Regular Graphs

If G is a connected k-regular graph with $k \geq 3$, then $\chi(G) \leq k$.

Proof. In the remaining case, G is k-regular. Suppose first that G has a cut-vertex x, and let G' be a subgraph consisting of a component of $G - x$ together with its edges to x. The degree of x in G' is less than k, so the method above provides a proper k-coloring of G'. By permuting the names of colors in the subgraphs resulting in this way from components of $G - x$, we can make the colorings agree on x to complete a proper k-coloring of G.

We may thus assume that G is 2-connected. In every vertex ordering, the last vertex has k earlier neighbors. The greedy coloring idea may still work if we arrange that two neighbors of v_n get the same color.

In particular, suppose that some vertex v_n has neighbors v_1, v_2 such that $v_1 \neq v_2$ and $G - \{v_1, v_2\}$ is connected. In this case, we index the vertices of a spanning tree of $G - \{v_1, v_2\}$ using $3, \ldots, n$ such that labels increase along paths to the root v_n. As before, each vertex before v_n has at most $k-1$ lower-indexed neighbors. The greedy coloring also uses at most $k-1$ colors on neighbors of v_n, since v_1 and v_2 receive the same color.

Hence it suffices to show that every 2-connected k-regular graph with $k \geq 3$ has such a triple v_1, v_2, v_n. Let's denote the vertex connectivity of a graph G by $\kappa(G)$. Choose a vertex x. If $\kappa(G - x) \geq 2$, let v_1 be x and let v_2 be a vertex with distance 2 from x. Such a vertex v_2 exists because G is regular and is not a complete graph; let v_n be a common neighbor of v_1 and v_2.

If $\kappa(G - x) = 1$, let $v_n = x$. Since G has no cut-vertex, x has a neighbor in every leaf block of $G - x$. Neighbors v_1, v_2 of x in two such blocks are nonadjacent. Also, $G - \{x, v_1, v_2\}$ is connected, since blocks have no cut-vertices. Since $k \geq 3$, vertex x has another neighbor, and $G - \{v_1, v_2\}$ is connected. □

References

[1] Cheng, J. (2024). *Graph Theory & Algorithms.* Tsinghua University Press.