Proof of Berge's Theorem

Yinan Rao

July 2024

1 Introduction to Matchings

In graph theory, a **matching** M in a graph $G = (V, E)$ is a set of edges without common vertices. That is, no two edges in M share a vertex. A vertex is **matched** (or **saturated**) if it is incident to an edge in the matching. A **maximum matching** is a matching that contains the largest possible number of edges. A matching is **perfect** if every vertex of the graph is matched.

An **augmenting path** relative to a matching M is a path that starts and ends at free (unsaturated) vertices and whose edges alternate between those not in M and those in M. Berge's theorem provides a characterization of maximum matchings using augmenting paths.

2 Berge's Theorem

[Berge, 1957] A matching M in G is a maximum matching if and only if G contains no M-augmenting path.

3 Proof

Proof. Let M be a matching in G, and suppose that G contains an M-augmenting path $v_0 v_1 \dots v_{2m+1}$. Define $M' \subseteq E$ by

$$M' = (M \setminus \{v_1 v_2, v_3 v_4, \dots, v_{2m-1} v_{2m}\}) \cup \{v_0 v_1, v_2 v_3, \dots, v_{2m} v_{2m+1}\}$$

Then M' is a matching in G, and $|M'| = |M| + 1$. Thus M is not a maximum matching.

Conversely, suppose that M is not a maximum matching, and let M' be a maximum matching in G. Then

$$|M'| > |M| \tag{1}$$

Set $H = G[M \Delta M']$, where $M \Delta M'$ denotes the symmetric difference of M and M'. Each vertex of H has degree either one or two in H, since it can be incident with at most one edge of M and one edge of M'. Thus each component

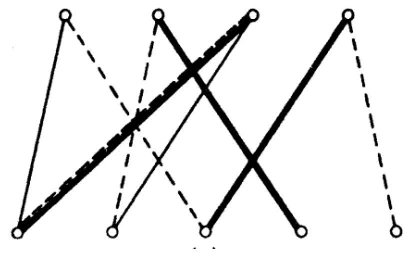

Figure 1: G, M heavy and M' broken

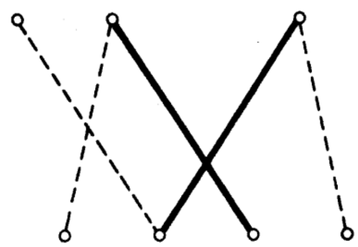

Figure 2: H

of H is either an even cycle with edges alternately in M and M', or else a path with edges alternately in M and M'. By (1), H contains more edges of M' than of M, and therefore some path component P of H must start and end with edges of M'. The origin and terminus of P, being M'-saturated in H, are M-unsaturated in G. Thus P is an M-augmenting path in G. $\square$