Proof of the Five Color Theorem

Yinan Rao

July 2024

1 Five Color Theorem

Theorem . Every planar graph can be 5-colored.

2 Proof

Proof: We will do this by induction on the number of vertices.

2.1 Base Case

The simplest connected planar graph consists of a single vertex. Pick a color for that vertex. We are done.

2.2 Induction Step

Assume $k \geq 1$, and assume that every planar graph with k or fewer vertices can be 5-colored. Now consider a planar graph with $k+1$ vertices. From above, we know that the graph has a vertex of degree 5 or fewer[1]. Remove that vertex (and all edges connected to it). By the induction hypothesis, we can 5-color the remaining graph. Put the vertex (and edges) back in. We have a graph with every vertex colored (without conflicts) except for the one.

If the vertex has degree less than 5, or if it has degree 5 and only 4 or fewer colors are used for vertices connected to it, we can pick an available color for it, and we are done (numbers represent colors).

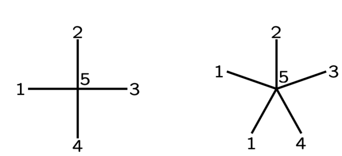

Figure 1: Vertex degree less than 5 or has degree 5 with 4 or fewer colors used

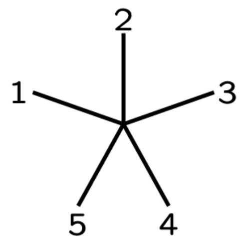

Figure 2: All 5 colors are connected to one vertex

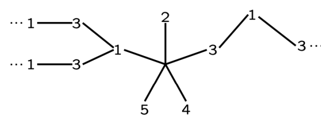

Figure 3: Subgraph with vertices colored 1 and 3

If the vertex has degree 5, and all 5 colors are connected to it, we have a little more work to do. In this case, using numbers 1 through 5 to represent colors, we label the vertices adjacent to the "special" (degree 5) vertex 1 through 5 (in order).

Now make a subgraph out of all the vertices colored 1 or 3 which are connected to the 1 and 3 colored vertices adjacent to the "special" vertex.

If the adjacent vertex colored 1 and the adjacent vertex colored 3 are not connected by a path in this subgraph, simply exchange the colors 1 and 3 throughout the subgraph connected to the vertex colored 1. This will leave color 1 available to color the "special" vertex, and we are done.

On the other hand, if the vertices colored 1 and 3 are connected via a path in the subgraph, we do the same "subgraph" process with vertices colored 2 and 4 adjacent to the "special" vertex. Note that this will be a disconnected pair of subgraphs, separated by a path connecting the vertices colored 1 and 3. Now we can exchange the colors 2 and 4 in the subgraph connected to the adjacent

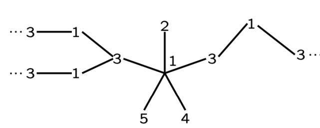

Figure 4: After 1-3 color exchanged

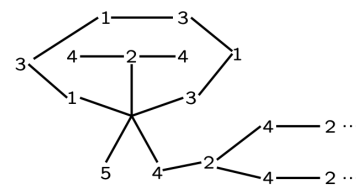

Figure 5: Subgraph with vertices colored 1 and 3 connected by a path

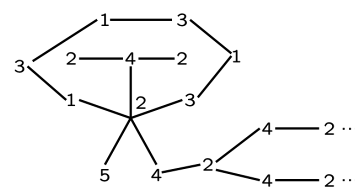

Figure 6: After 2-4 color exchanged

vertex labeled 2. This will leave color 2 for the "special" vertex.

3 References

References

[1] Cheng, J. (2024). *Graph Theory & Algorithms*. Tsinghua University Press.