On Cycles of a Bipartite Graph

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This report aims to understand one of the properties, the cycle length, of a Bipartite graph. Before delving into the crux of the report, let us recall the definition of a Bipartite graph.

Definition 1. (Bipartite graph) ([2], pg. 4). An undirected graph $G = (V, E)$ is *bipartite* if the set of its vertices can be divided into two subsets V_1 and V_2 such that any $e \in E$ has one endpoint in V_1 and the other endpoint in V_2. The sets V_1 and V_2 are called the *bipartition subsets*.

Theorem 1 ([1], pg. 42)**.** *A graph G is bipartite if and only if it has no cycles of odd length.*

Proof. Necessity ($\Rightarrow$): Assume G is a bipartite graph. Considering that any walk within this graph alternates between the two partitions, an even number of steps is required to circle back to the original partition. Consequently, any cycle within G must comprise an even number of edges.

Sufficiency ($\Leftarrow$): Consider a graph G with at least two vertices, devoid of odd-length cycles. Assuming G is a connected graph, select a vertex u from G. We can then categorize the vertex set V into two groups:

$$X = \{x \mid d(u, x) \text{ is even}\}$$

$$Y = \{y \mid d(u, y) \text{ is odd}\}$$

Should (X, Y) fail to be a valid bipartition due to an edge connecting two vertices within the same set, say v and w, consider the shortest paths from u to v (P_1) and from u to w (P_2). According to the partitioning categorization that we have defined, both paths would be of the same parity in length (both even or both odd). Let's identify the furthest vertex shared by both paths, labeled z. Note that z can coincide with u. For the sketch of the proof refer to **Figure 1**.

Given that P_1 and P_2 are the shortest paths, the segments from u to z on both paths must be equivalent in length, which means that the remaining segments from z to v and z to w also share the same parity. Linking these segments with the edge e would construct a cycle whose length is odd, which is a contradiction to our initial assumption. Therefore, the partition (X, Y) must indeed separate G into two distinct parts, establishing it as a bipartite graph. □

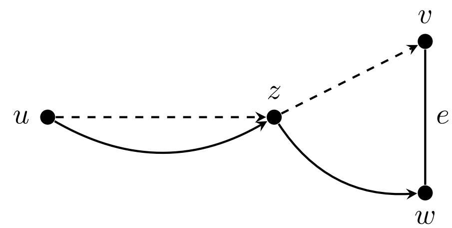

Figure 1: A bipartite graph with vertices u, v, w, and z, and an edge e between v and w.

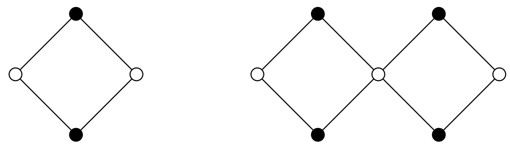

Figure 2: Two examples of bipartite graphs with even cycle lengths, showcasing partitions with different vertex colors.

References

[1] Anderson Gross, Yellen. *Graph Theory and Its Applications*. 2019.

[2] Serge Richard. *Special Mathematics Lecture: Graph Theory*. 2024.