Proving Catalan Recurrence Theorem

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This report aims to derive a formula for the number of different binary trees on n vertices which is prominently known as the Catalan Numbers. As is frequently the case in enumerative combinatorics, the derivation begins by establishing a recursive formula. We refer to ([1], pg. 163) for the following definitions and explanations behind the Catalan Numbers.

Catalan Numbers and Binary Trees

Let C_n can be denoted as the number of distinct binary trees constructed with n vertices. To simplify, initialize $C_0 = 1$. As the simplest tree, a single vertex without children represents $C_1 = 1$. For $n \geq 1$, any binary tree T consisting of n vertices can be split into a left subtree having j vertices and a right subtree comprising $n - 1 - j$ vertices, where j ranges from 0 to $n - 1$. The count of such trees, C_n, is then determined by the sum of products of the number of trees in the left and right subtrees across all possible splits, given by

$$C_n = C_0 C_{n-1} + C_1 C_{n-2} + \cdots + C_{n-1} C_0$$

This formula, known as the *Catalan recursion*, computes the n-th Catalan number, which finds utility in numerous applications beyond merely counting binary trees.

Example 1. *Applying the Catalan recursion to the cases $n = 2$ and $n = 3$ yields $C_2 = 2$ and $C_3 = 5$. The five different binary trees on 3 vertices are illustrated in Figure 1.*

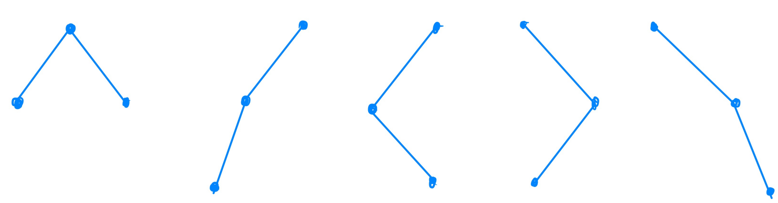

Figure 1: The five different binary trees on 3 vertices.

With the aid of generating functions, it's possible to derive the closed formula of C_n.

Theorem 1. *The number C_n of different binary trees on n vertices is given by*

$$C_n = \frac{1}{n+1}\binom{2n}{n}$$

Proof. First of all, let us consider the following claim

Claim 1. *Define $g(x) = \sum_{k=0}^{\infty} C_k x^k$, the generating function for C_n, representing the number of distinct binary trees with n vertices. Using the Catalan recurrence relation, one has the following equation*

$$1 + xg(x)^2 = g(x).$$

Proof. Let us express $x(g(x))^2$

$$\begin{aligned} x(g(x))^2 &= x\left(\sum_{k=0}^{\infty} C_k x^k\right)\left(\sum_{l=0}^{\infty} C_l x^l\right) \\ &= x\sum_{k=0}^{\infty}\sum_{l=0}^{k} C_l C_{k-l} x^{l+k-l} \quad \text{(Using the Cauchy Product between two infinite sums)} \end{aligned}$$

Observe that the Catalan recursion tells us that

$$C_{k+1} = C_0C_k + C_1C_{k-1} + \cdots + C_kC_0 = \sum_{l=0}^{k} C_l C_{k-l}$$

Hence, one has

$$\begin{aligned} x(g(x))^2 &= \sum_{k=0}^{\infty} C_{k+1} x^{k+1} \\ &= \sum_{k=0}^{\infty} C_k x^k - C_0 \end{aligned}$$

This gives,

$$1 + x(g(x))^2 = g(x) \quad (\because C_0 := 1) \tag{1}$$

□

We have defined a generating function $g(x)$ as a power series. However, one might ask what is the radius of convergence of the power series $g(x)$. Consider the following claim regarding the radius convergence of the generating function $g(x)$.

Claim 2. *The generating function $g(x)$ for C_n that has been defined previously has a radius of convergence of one-quarter ($r = 1/4$).*

Proof. Let us set $g(x) = \sum_{k=0}^{\infty} C_k x^k := \sum_{k=0}^{\infty} u_k$. Suppose that Theorem 1 is true, then one would have the following

$$\begin{aligned}\left|\frac{u_{k+1}}{u_k}\right| &= \left|\frac{C_{k+1}}{C_k}x\right| \\ &= \left|\frac{k+1}{k+2} \cdot \frac{(2k+2)!}{(k+1)!(k+1)!} \cdot \frac{k!k!}{(2k)!}x\right| \\ &= \left|\frac{(2k+1)(2k+2)}{(k+1)(k+2)}x\right|.\end{aligned}$$

Then, if one takes the limit of $k \to \infty$, one would have the following

$$\begin{aligned}\lim_{k\to\infty}\left|\frac{u_{k+1}}{u_k}\right| &= \lim_{k\to\infty}\left|\frac{(2k+1)(2k+2)}{(k+1)(k+2)}x\right| \\ &= |4x|.\end{aligned}$$

Now, if one takes $|x| < 1/4$ we would have a convergent power series. Hence, the radius of convergence of the generating function $g(x)$ is equal to one-quarter ($r = 1/4$). □

Utilizing the quadratic formula to solve the equation (1), we find

$$g(x) = \frac{1 - \sqrt{1-4x}}{2x}.$$

Observe that the validity of the solution above is given by $|x| < 1/4$ which matches with our result in **Claim 2**. According to the Generalized Binomial Theorem, for any real number r, $(1+z)^r = \sum_{k=0}^{\infty} \binom{r}{k} z^k$, where $\binom{r}{k}$ is determined by

$$\binom{r}{k} = \begin{cases} 1, & \text{if } k = 0; \\ \frac{r(r-1)(r-2)\cdots(r-k+1)}{k!}, & \text{if } k \geq 1. \end{cases}$$

Applying this theorem to the expression $(1-4x)^{1/2}$ gives

$$\sqrt{1-4x} = \sum_{k=0}^{\infty} \binom{1/2}{k} (-4x)^k$$

Hence, one has

$$g(x) = \frac{1}{2x}\left(1 - \sum_{k=0}^{\infty} \binom{1/2}{k} (-4x)^k\right).$$

Then, one can modify the index from k into $n+1$ such that one has

$$\begin{aligned} g(x) &= \frac{1}{2x}\left(1 - \sum_{n=-1}^{\infty}\binom{1/2}{n+1}(-4x)^{n+1}\right) \\ &= \frac{1}{2x}\left(1 - \binom{1/2}{0}\right) + \frac{1}{2x}\sum_{n=0}^{\infty}\binom{1/2}{n+1}(-1)^n 2^{2n+2} x^{n+1} \\ &= \sum_{n=0}^{\infty}\binom{1/2}{n+1}(-1)^n 2^{2n+1} x^n. \end{aligned}$$

According to the definition of our generating function $g(x)$, let us deduce that

$$C_n = \binom{1/2}{n+1}(-1)^n 2^{2n+1}.$$

From the definition of $\binom{r}{k}$ that we have defined previously,

$$\begin{aligned} \binom{1/2}{n+1} &= \frac{(1/2)(-1/2)(-3/2)\dots(1/2-n)}{(n+1)!} \\ &= \frac{(-1)^n}{2^{n+1}(n+1)} \cdot \frac{(1)(3)(5)\cdots(2n-1)}{(1)(2)(3)\cdots(n)} \\ &= \frac{(-1)^n}{2^{n+1}(n+1)} \cdot \frac{(1)(2)(3)(4)\cdots(2n-1)(2n)}{(2)(4)(6)\cdots(2n)(1)(2)(3)\cdots(n)} \\ &= \frac{(-1)^n}{2^{n+1}(n+1)} \cdot \frac{(1)(2)(3)(4)\cdots(2n-1)(2n)}{2^n[(1)(2)(3)\cdots(n)](1)(2)(3)\cdots(n)} \\ &= \frac{(-1)^n}{2^{n+1}(n+1)} \cdot \frac{2n!}{2^n n!\cdot n!} \\ &= \frac{(-1)^n}{2^{2n+1}(n+1)}\binom{2n}{n} \end{aligned}$$

Hence, C_n can be expressed as follows

$$C_n = \frac{(-1)^n}{2^{2n+1}(n+1)}\binom{2n}{n}(-1)^n 2^{2n+1} = \frac{1}{n+1}\binom{2n}{n}.$$

□

References

[1] Anderson Gross, Yellen. *Graph Theory and Its Applications.* 2019.