Six Different Characterizations of a Tree

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This report aims to study and prove a theorem regarding the six different characterizations of an undirected tree. Before delving into the crux of the report, let us write some statements and propositions regarding the basic properties of a tree.

Definition 1. *(Tree) A tree is a connected graph whose underlying undirected graph has no cycle.*

From the definition above, we can also imply that directed trees would be in the form of connected and directed graphs that have no cycle. Directed trees are also called oriented trees, poly-trees, or singly connected networks. In this report, we work on the notion of undirected trees instead of directed trees.

Definition 2. *In an undirected tree, a leaf is a vertex of degree 1.*

If a leaf is deleted from a tree, then the resulting graph is a tree having one vertex fewer, since removing a vertex means removing all edges having the vertex on endpoint. Thus, induction is a natural approach to proving tree properties, provided one can always find a leaf. The following proposition guarantees the existence of such a vertex.

Proposition 1. *Every tree with at least one edge has at least two leaves.*

Proof. Let $\mathcal{P} = (v_1, v_2, \ldots, v_m)$ be a path of maximum length in a tree T. Suppose one of its endpoints, say v_1, has degree greater than 1. Then v_1 is adjacent to vertex v_2 on path $\mathcal{P}$ and also to some other vertex w (see Figure 1). If w is different from all of the vertices v_i, then $\mathcal{P}$ could be extended, contradicting its maximality. On the other hand, if w is one of the vertices v_i on the path, then the acyclic property of T would be contradicted. Thus, both endpoints of path $\mathcal{P}$ must be leaves in tree T. □

Corollary 1. *If the degree of every vertex of a graph is at least 2, then that graph must contain a cycle.*

Proof. By Proposition 1, we know that a leaf has to have a degree of 1 and we have argued that if v_1 in the previous construction has a degree greater than 1, then it contradicts the acyclic property of

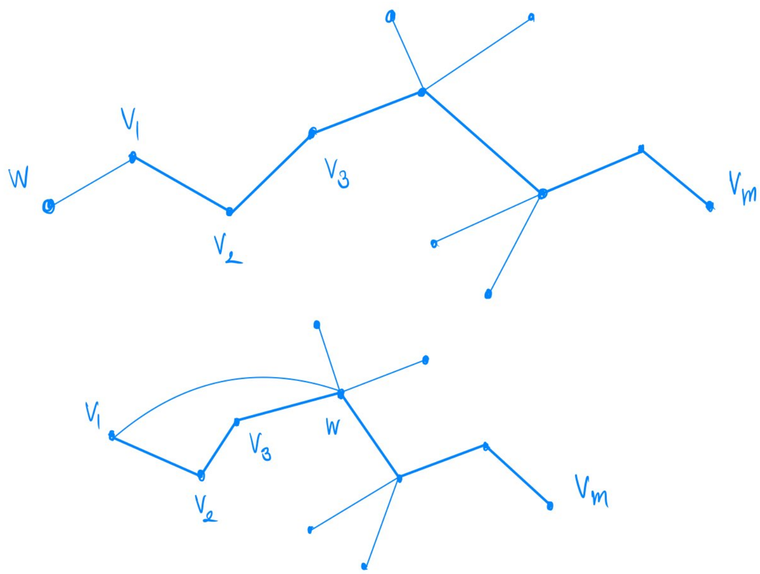

Figure 1: The two cases in the proof of Proposition 1

T, hence giving a cyclic graph. □

Proposition 2. *Let e be an edge of a connected graph G. Then $G - e$ is connected if and only if e is a cycle-edge of G.*

Proof. Assume u and v are the vertices connected by edge e. If the edge e is part of a cycle within graph G, an alternative path that traverses this cycle can be found, connecting vertices u and v, which affirms that the graph $G - e$ remains connected. Conversely, if the removal of edge e from G preserves its connectedness, then there exists a path $\mathcal{P}$ that links vertices u and v without incorporating edge e. Reinserting edge e into this configuration, along with path $\mathcal{P}$, establishes a cycle in G that includes edge e. Hence, $G - e$ is connected if and only if e is a cycle-edge of G. □

Corollary 2. *Let e be any edge of a graph G. Then*

$$c(G - e) = \begin{cases} c(G), & \textit{if } e \textit{ is a cycle-edge} \\ c(G) + 1, & \textit{otherwise} \end{cases}$$

with $c(G)$ denotes the number of components of a graph G.

Proof. If the edge e is a cycle-edge, then by Proposition 2, we know that after the deletion operation from graph G into graph $G - e$, the graph $G - e$ is still connected. However, if the edge e is not a cycle-edge, then the graph $G - e$ is no longer connected again, thus increasing the number of components by one. □

The next proposition establishes a fundamental property of trees. Its proof is the first of several instances that demonstrate the effectiveness of the inductive approach to proving assertions about trees.

Proposition 3. *Every tree on n vertices contains exactly $n-1$ edges.*

Proof. A tree on one vertex is the trivial tree, which has no edges.

Let us presume for a certain integer $k \geq 1$, based on the inductive hypothesis, that each tree with k vertices holds precisely $k-1$ edges. We now turn our attention to a tree T populated with $k+1$ vertices. In accordance with Proposition 1, tree T encompasses a leaf, denoted by v. Consequently, the graph $T-v$ retains its acyclic character, as the removal of a vertex and the corresponding edge from an acyclic graph cannot introduce a cycle. Furthermore, $T-v$ is still connected, owing to the degree one of vertex v within T. Hence, $T-v$ is a tree of k vertices, and by our inductive assumption, it contains $k-1$ edges. Given that the degree of v is one, it ensues that T possesses precisely one more edge than $T-v$, leading to the conclusion that T is comprised of k edges. This completes the verification of our initial claim. □

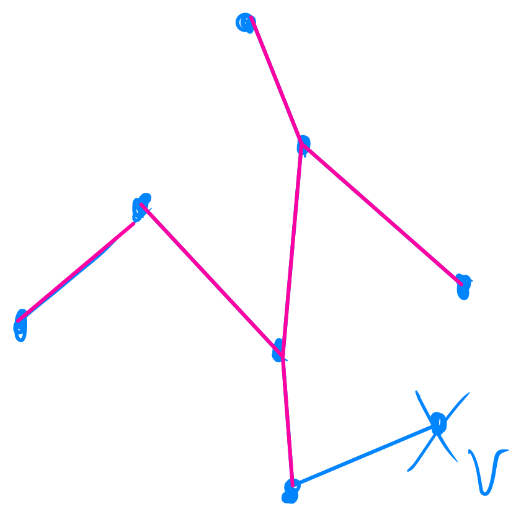

Figure 2: Sketch for the proof of Proposition 3, the removal of a vertex also removes the corresponding edge.

Corollary 3. *A forest G on n vertices has $n - c(G)$ edges.*

Proof. Apply Proposition 3 to each of the components of G, thus giving $n-c(G)$ edges as the number of components of G is given by $c(G)$. □

Corollary 4. *Any graph G on n vertices has at least $n - c(G)$ edges.*

Proof. If G has cycle-edges, then remove them one at a time until the resulting graph $\hat{G}$ is acyclic. Then $\hat{G}$ has $n - c(\hat{G})$ edges, by Corollary 3, and $c(\hat{G}) = c(G)$, by Corollary 2. □

Definition 3. *A **complete graph** is a simple graph such that every pair of vertices is joined by an edge. Any complete graph on n vertices is denoted K_n*

Example 1. *Complete graphs on one, two, three, four, and five vertices are shown in Figure 3*

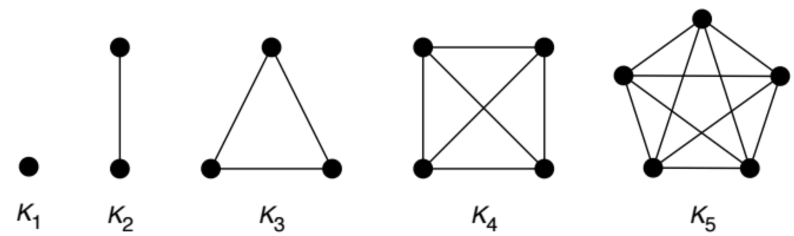

Figure 3: The first five complete graphs ([1], pg. 15).

Proposition 4. *Let G be a simple graph with n vertices and k components. If G has the maximum number of edges among all such graphs, then*

$$|E(G)| = \binom{n-k+1}{2}$$

Proof. Since the number of edges is maximum, each component of G is a complete graph. If $n = k$, then G consists of k isolated vertices, and the result is trivially true. If $n > k$, then G has at least one non-trivial component. We show that G has exactly one nontrivial component. Assume there exist multiple nontrivial components within the graph, with component C_1 being a complete graph of r vertices, denoted $C_1 = K_r$, and another component C_2, also complete, with s vertices, expressed as $C_2 = K_s$, where r and s satisfy $r \geq s \geq 2$. By using combinatorics, the sum of the edges is given by $\binom{r}{2} + \binom{s}{2}$. However, if we reformulate C_1 into K_{r+1} and C_2 into K_{s-1}, the resultant graph has $\binom{r+1}{2} + \binom{s-1}{2}$ edges. Simple arithmetic reveals that this newly configured graph possesses $r - s + 1$ additional edges compared to the previous graph, which is a positive number due to our previous assumption that $r \geq s$. This contradicts the initial graph's edge maximality. Consequently, G is structured as $k - 1$ solitary vertices alongside a complete graph spanning $n - (k - 1)$ vertices, implying that $|E(G)| = \binom{n-(k-1)}{2} = \binom{n-k+1}{2}$, which completes the proof. □

Corollary 5. *A simple n-vertex graph with more than $\binom{n-1}{2}$ edges must be connected.*

Proof. Let us take $k = 2$, according to **Proposition 4** the maximum number of edges of a graph G with n vertices and two components is given by $\binom{n-1}{2}$ edges. If we add one more edge to the graph G, the two components of the graph would be connected. □

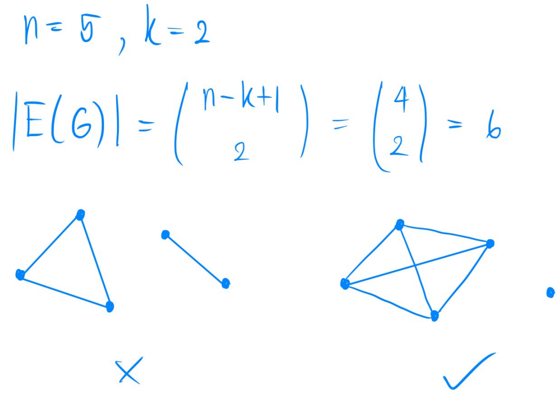

Figure 4: Sketch for the proof of Proposition 4, For $n = 5$ and $k = 2$, the maximum number of edges.

The following theorem refers to **Definition 1 (Undirected Tree)** for the definition of a tree.

Theorem 1 ([1], pg. 124)**.** *Let T be a graph with n vertices. Then the following statements are equivalent.*

1. *T is a tree.*
2. *T contains no cycles and has $n-1$ edges.*
3. *T is connected and has $n-1$ edges.*
4. *T is connected, and every edge is a cut-edge.*
5. *Any two vertices of T are connected by exactly one path.*
6. *T contains no cycles, and for any new edge e, the graph $T+e$ has exactly one cycle.*

Proof. If $n = 1$, then all six statements are trivially true. So assume $n \geq 2$.

($1 \Rightarrow 2$) By Proposition 3, we know that every tree on n vertices contains exactly $n-1$ edges.

($2 \Rightarrow 3$) Suppose that T has k components. Then, by Corollary 3, the forest T has $n-k$ edges. But, we know that T has $n-1$ edges, this gives $k = 1$. Hence, T is connected

($3 \Rightarrow 4$) Let e be an edge of T. Since $T-e$ has $n-2$ edges, Corollary 4 implies that $n-2 \geq n-c(T-e)$. So $T-e$ has at least two components. This implies that every edge is a cut-edge because $\{e\}$ is an edge-cut which gives the tree T more components if the edge e is to be cut.

($4 \Rightarrow 5$) By using contradiction, suppose that Statement 5 is not true, any 2 vertices could be connected with more than one path or with no path which means that they are not connected or that there in cycle inside the graph. Consider the unconnected graph, it contradicts Statement 4 stating that T is connected. In the other case, there exists a cycle inside the graph, the edge in the cycle is not a cut-edge, contradicting Statement 4, and thus any two vertices of T are connected by exactly one path.

($5 \Rightarrow 6$) The absence of cycles within T is established due to the fact that the presence of a cycle would entail two distinct routes between any pair of vertices on the cycle, which contravenes the assertion made in statement 5. Moreover, introducing an edge e into T invariably yields a cycle, attributed to the fact that its vertices, u and v, were previously linked via a path in T as it should be according to statement 5. To argue for the singularity of this resultant cycle, one might hypothesize the formation of two cycles or more. Nevertheless, such an assumption would imply two separate $u - v$ trajectories within T, which would be in direct conflict with Statement 5.

($6 \Rightarrow 1$) Let us prove the statement by contradiction. Now suppose that statement 6 is true that the graph $T + e$ has exactly one cycle given for any new edge e. Then, suppose that there exists a graph T which is not a tree that satisfies the statement above. However, we know that there exists such a graph that would not be connected as the addition of an edge joining a vertex in a different component would not make a cycle. Moreover, we also know that there exists a graph that is connected that satisfies the above conditions, however, this graph is a cyclic graph as for a certain edge e, the graph $T + e$ would have two cycles or more which would contradict Statement 6. Hence, T has to be connected acyclic graph, and thus can be considered as a tree. $\square$

References

[1] Anderson Gross, Yellen. *Graph Theory and Its Applications.* 2019.