Graceful graphs

SML Spring 2024: Graph Theory

Oleh Dmytruk

July 26, 2024

1 Introduction

This report defines graceful labeling, proves several simple results about gracefulness, and mentions two of the deep results in this field. In general, a labeling of a graph is a map from a set of graph elements to numbers, usually non-negative integers. The most common choices for domain are the set of all vertices and edges, the vertex-set alone, or the edge-set alone. More precisely,

Definition 1.1 (Labeling)**.** *Let Γ be a subset of elements of G, and N be a set of numbers. A* labeling *of a graph G is a map $f : \Gamma \to N$.*

If not specified, definitions are from Professor Richard's Lecture notes [3]. Also, for a graph G, v is the number of vertices, and e is the number of edges.

2 Graceful labeling

The definitions and the following two theorems are adapted from Wallis' *"A Beginner's Guide to Graph Theory"* [5], Chapter 9.1. In 1967, Rosa [4] introduced β-labelings, now known as graceful labelings. We shall first study some of their properties and afterwards showcase two of the related conjectures.

Definition 2.1 (Graceful labeling)**.** *A labeling of a graph G is called a* graceful labeling *if there exists an injective map $\gamma : V \to \{0, ..., e\}$ such that for any $x, y \in V$ values $|\gamma(x) - \gamma(y)|$ are distinct over all edges.*

In other words, if one were to extend the definition to edges $\gamma(xy) := |\gamma(x) - \gamma(y)|$, then every member of $\{1, ..., e\}$ would arise (exactly) once among the edge labels. A graph is called graceful if it has a graceful labeling.

Consider two examples of graceful labeling for a K_4 graph in Figure 1.

If γ is a graceful labeling on G, define another labeling by $\gamma^*(x) := e - \gamma(x)$. Then γ^* is also graceful; in fact, γ and γ^* induce the same edge labeling. For this reason, γ^* is called the complementary labeling or dual labeling of γ. We formally define two labelings of G, γ and σ, to be equivalent if σ can be transformed into γ or γ^* by a graph automorphism. The blue K_4 in Figure 1 is dual to one in green.

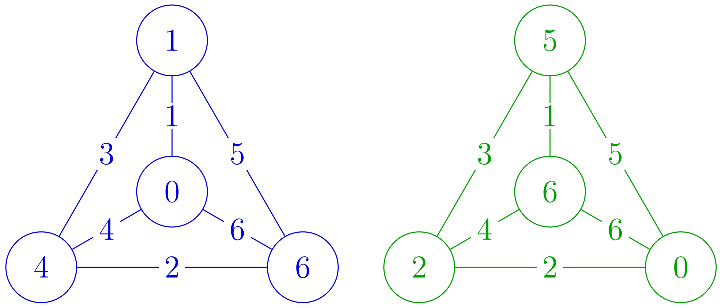

Fig. 1

While constructing graceful labeling for examples with a few vertices, one notices that labels e and 0 have to be included and there has to exist an edge in between. Let us formalize this with a proposition.

Proposition 2.2. *For any graceful labeling there is a vertex labeled e, 0, and either $e - 1$ or 1.*

Proof. Consider a graceful labeling γ. Recall that graceful labeling has to be such that all numbers from $\{1, ..., e\}$ would arise once as a difference between some vertices' labels. In particular, for e to be $\gamma(xy)$ for some vertices x, y, there has to be an edge $e \sim 0$ (between vertices labeled e and 0). Thus, one always includes labels 0 and e. Similarly, there has to be an edge either $(e - 1) \sim 0$, or $e \sim 1$. □

After seeing Figure 1, one may wonder if all K_n graphs are graceful. The answer in no, and we formalize it in the following theorem.

Theorem 2.3. *There is no graceful labeling of K_v for $v > 4$.*

Proof. We shall use x for a vertex labeled x and $x \sim y$ for an edge valued $|x - y|$ between x and y.

Assume there exists a graceful labeling for K_v, $v \geq 5$. By Proposition 2.2, both e and 0 has to be used for labeling vertices. The choice of $e - 1$ or 1 leads to dual labeling, so take 1 without loss of generality. Then we have labels $\{0, 1, e\}$ which induce values $\{1, e-1, e\}$. Note that by the *"set of (vertex) labels L_V induce values* *(edge labels) L_E"* we mean that the set L_E is obtained from the set L_V by taking all possible absolute values of differences between elements of L_V.

Currently, we need to assign $v-3$ more labels and get all numbers from $\{2, ..., e-2\}$ as differences of labels strictly once. There are two ways to achieve value $e-2$. One is to take label 2, other is to take $e-2$. Yet we are not allowed to include label 2, as (due to K_v being fully-connected) this leads to an edge with value 1 for both $1 \sim 0$ and $2 \sim 1$. Thus, we are obliged to choose $e-2$ as the next label. Then we have labels $\{0, 1, e-2, e\}$ which induce values $\{1, 2, e-3, e-2, e-1, e\}$.

Now we have to assign $v-4$ more labels and get all numbers from $\{3, ..., e-4\}$ as differences of labels strictly once. We have already induced 6 values, but by assumption $v \geq 5 \Leftrightarrow e = v(v-1)/2 \geq 10$, so $e > 6$, and thus value $e-4$ is still needed. There are two ways to achieve $e-4$. One is to take label 4, other is to take $e-4$. Yet we are not allowed to include label $e-4$, as this leads to an edge with value 2 for both $e \sim (e-2)$ and $(e-2) \sim (e-4)$. Thus, we are obliged to choose 4 as the next label. Then we have labels $\{0, 1, 4, e-2, e\}$ which induce values $\{1, 2, 3, 4, e-6, e-4, e-3, e-2, e-1, e\}$.

As mentioned, $e \geq 10$. Then we have to have $e-5$ on our list. There are 5 ways to achieve value $e-5$. It is to include one of the labels $3, 5, e-5, e-4$, or $e-1$. But one easily checks that label 3 leads to duplicate of value 3, label 5 leads to duplicate of value 1, $e-5$ leads to duplicate of value 3, and lastly $e-1$ leads to duplicate of value 1. This means there is no way to achieve all values $\{1, ..., e\}$ for $v \geq 5$. Hence, we arrive at contradiction to our initial assumption. □

Having mentioned this important result, we proceed with a less famous one. Note that a caterpillar is a tree for which if all leaves and their associated edges were removed, the result is a path.

Theorem 2.4 (Rosa, 1967)**.** *All caterpillars are graceful.*

Proof. Suppose C is a caterpillar with P denoting a path if all "legs" (vertices of degree 1) are deleted. Select an endpoint of P and name it bone b_0; the vertex adjacent to it is called bone b_1 and so on along P. Let $i \in \{1, ..., v\}$. Divide all vertices v_i of C in two categories: vertex $v_i \in X$ if distance from v_i to b_0 is even (note that X includes b_0), and vertex $v_i \in Y$ otherwise. Observe that every edge connects two vertices, one in X and the other in Y.

Assign the label $v-1$ to bone b_0, and labels $0, 1, ..., n_0-3, n_0-2, n_0-1$ to all n_0 neighbours of b_0 such that bone b_1 (the neighbour of b_0 in the path P) receives the biggest label n_0-1. Proceed by assigning labels $v-2, v-3, ..., v-n_1+1, v-$

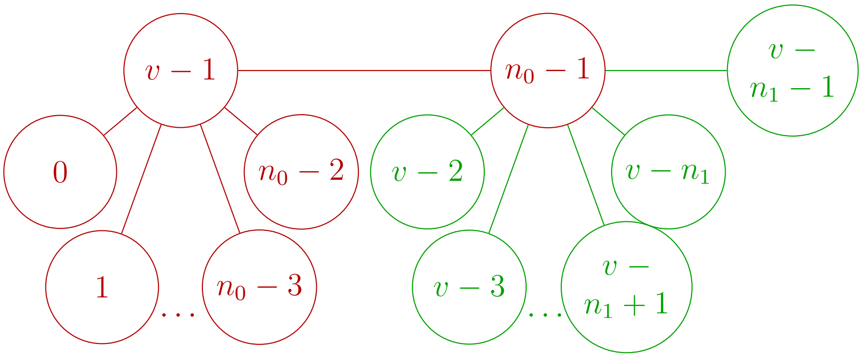

Fig. 2: The first two steps (red and green) in labeling a caterpillar. Notice how n_0-1 is the biggest out of n_0 red labels (except for $v-1$), while $v-n_1-1$ is the smallest out of n_1 green labels (except for n_0-1). Color of a vertex indicate the step at which it was labeled

$n_1, v-n_1-1$ to all n_1 unlabeled neighbours of b_1 such that bone b_2 receives the smallest label $v-n_1-1$. See Figure 2 for some intuition.

Generally, after labeling the bone b_{2i}, label all of its neighbours with increasing integers starting from the smallest unused (greater or equal to 0) and ending with bone b_{2i+1}. Proceed by labeling all neighbours of b_{2i+1} with decreasing integers starting from the largest unused (less or equal to $v-1$) and ending with bone b_{2i+2}.

The result will be a labeling where vertices in X received labels $v-1, v-2, v-3, ...$ and vertices in Y received labels $0, 1, 2, ...$ One readily observes that this labeling is in fact graceful. □

Let us walk through a labeling of a caterpillar with $v = 30$. See Figure 3 as a road-map for this example. Start by locating all of the legs, the rest of the vertices are part of a path P with vertices called bones $b_0, b_1, b_2, ..., b_{N-1}$, with $N = 4$ in this example.

Begin the first step by labeling the left-most bone in the path P, namely bone b_0, by $v-1 = 29$. Now label all of its $n_0 = 8$ neighbours by an increasing integers $0, 1, ..., n_0-3, n_0-2, n_0-1$, or $0, 1, ..., 5, 6, 7$, so that bone b_1 would receive the biggest label $n_0-1 = 7$. This concludes the first step.

Now begin the second step. Label all $n_1 = 7$ unlabeled neighbours of bone b_1 by a decreasing integers $v-2, v-3, ..., v-n_1+1, v-n_1, v-n_1-1$, or $28, 27, ..., 24, 23, 22$, so that bone b_2 would receive the smallest label $v-n_1-1 = 22$. This concludes the second step.

Now begin the third step. Label all $n_2 = 7$ unlabeled neighbours of bone b_2 by an increasing integers starting from the biggest unused, which in this case is $8, 9, 10, 11, 12, 13, 14$, so that bone b_3 would receive the biggest label 14. This concludes the third step.

The process can easily be continued for further elements in the path P, bones $b_0, b_1, b_2, ..., b_{N-1}$. See the last part of Figure 3 for the completed labeling of this particular example.

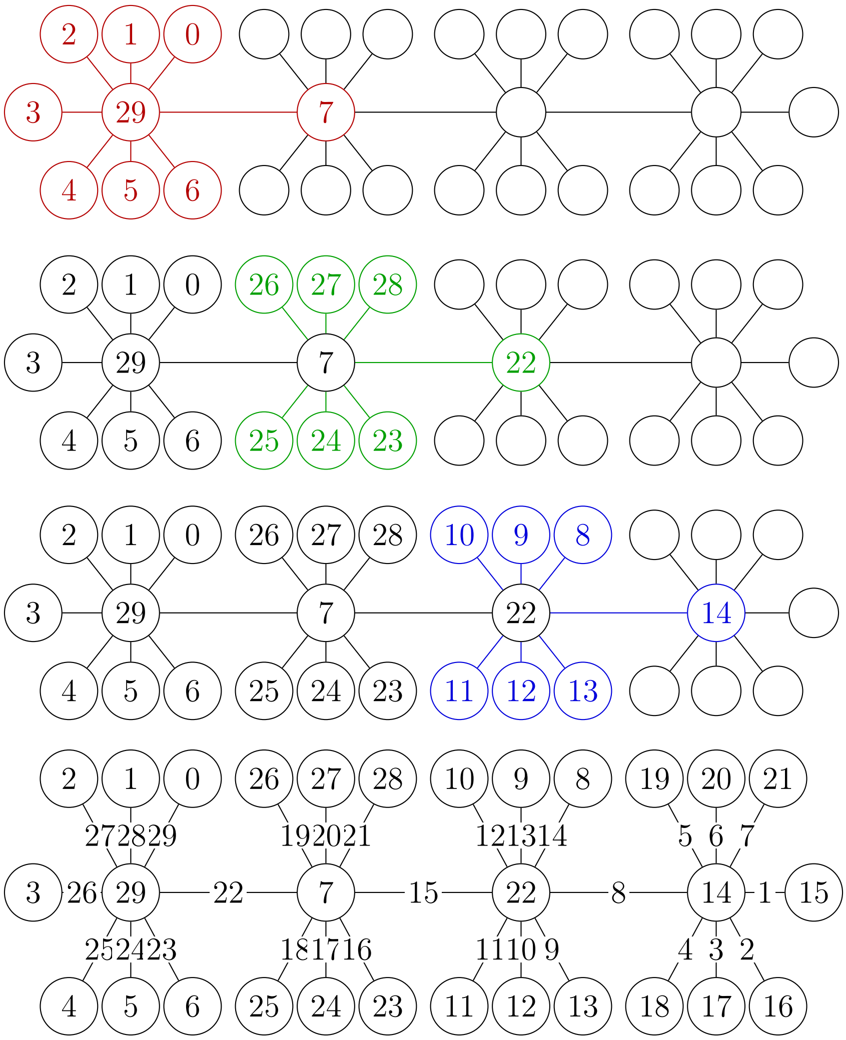

Fig. 3: The first three steps (first, second and third) at graceful labeling of a caterpillar on 30 vertices, and the completed labeling with edge labels

Let us provide solutions for a few simple exercises.

Exercise 2.5 (Nontrivial forests are not graceful [5], p. 126, 9.1.6)**.** *Show that no nontrivial forest (that is, a forest containing at least two trees) is graceful.*

Proof. For an n-vertex tree, there are $n-1$ edges (assuming $n > 1$). Thus, two disjoint trees with a and b vertices (a nontrivial forest) have at most $(a-1)+(b-1) = a+b-2$ edges, while consisting of $a+b$ vertices. One observes that it is not possible to create an injective map (graceful labeling) from a set with cardinality $a+b$ into a set with cardinality $a+b-2$. □

Exercise 2.6 (Stars are graceful [5], p. 126, 9.1.7+9.1.8)**.** *Show that the star $K_{1,n}$ and the path P_n is graceful for every n.*

Proof. For a star, label a central vertex 0 and put other labels arbitraly. One observes that this labeling is graceful. A path is a caterpillar without legs, so statement holds by Theorem 2.4. Refer to an example of both labelings in Figure 4. □

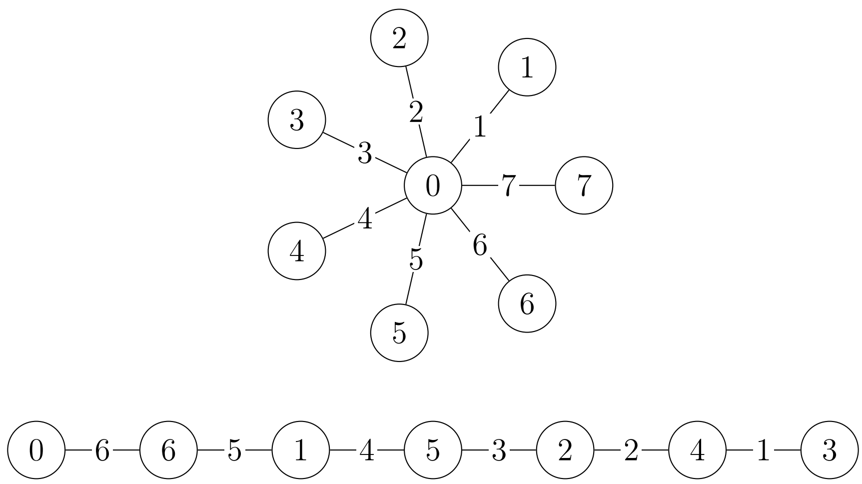

Fig. 4: A graceful $K_{1,7}$ star and a graceful 7-vertex path

Exercise 2.7 (Edges from even to odd vertices [5], p. 126, 9.1.9)**.** *Suppose G is a graceful graph with e edges. Write X and Y for the sets of vertices with even and odd labels respectively. Show that the set $[X, Y]$* (author's note: set of edges with one endpoint in set X and one in set Y) *contains precisely $(e+1)/2$ edges.*

Proof. The statement is not correct as it is; consider example in Figure 5. Let $p \in \{0, 1\}$ be the parity of e, $p \equiv e \pmod 2$. In fact, one has to say that $[X, Y]$ contains precisely $(e+p)/2$ edges. This can be proven by recalling that a graceful labeling implies all edges have different values from $\{1, ..., e\}$. This contains $(e+p)/2$ odd

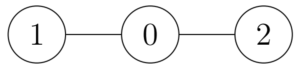

Fig. 5: An example with 2 edges but clearly not 3/2 edges in $[X, Y]$ as defined in Exercise 2.7

and $(e-p)/2$ even numbers. Difference of odd or even numbers is always even, which means we have to have $(e+p)/2$ edges between even and odd vertices, as required. □

We now deviate slightly from the main topic to mention one application of graceful labeling. A typical decomposition question asks whether the edges of some graph G can be partitioned into disjoint copies of another graph H. One of the oldest and best known conjectures in this area, posed by Ringel in 1963, concerns the decomposition of complete graphs into edge-disjoint copies of a tree.

Conjecture 2.8 (Ringel's conjecture)**.** *The complete graph K_{2n+1} can be decomposed into copies of any tree with n edges.*

In fact, graceful labelings of trees were first studied in an attempt to prove it ([5], p. 125). Ringel's conjecture appears to have been proven for large n by Montgomery et al. [2] in a recent 30-page paper published in 2020. Let us also mention an even stronger conjecture.

Conjecture 2.9 (Graceful tree conjecture)**.** *Every tree is graceful.*

This conjecture has attracted a lot of attention in the last 50 years but has only been proved for some special classes of tree. The connection between two conjectures is shown in the following theorem.

Theorem 2.10. *Suppose T is a graceful tree on $n+1$ vertices. Then K_{2n+1} is a union of $2n+1$ edge disjoint copies of T.*

Proof. We shall use a K_{2n+1} with vertices $\{0, ..., 2n\}$ modulo $2n+1$. By this we mean that whenever we refer to a vertex labeled by an integer not from $\{0, ..., 2n\}$, we mean that number modulo $2n+1$. For instance, $2n+1$ refers to 0, and $2n+2$ refers to 1. Let also γ be a graceful labeling of T. We associate x vertex of T with vertex $\gamma(x)$ in K_{2n+1}. In this sense, T is a subgraph of K_{2n+1} with vertices $\{0, ..., n\}$.

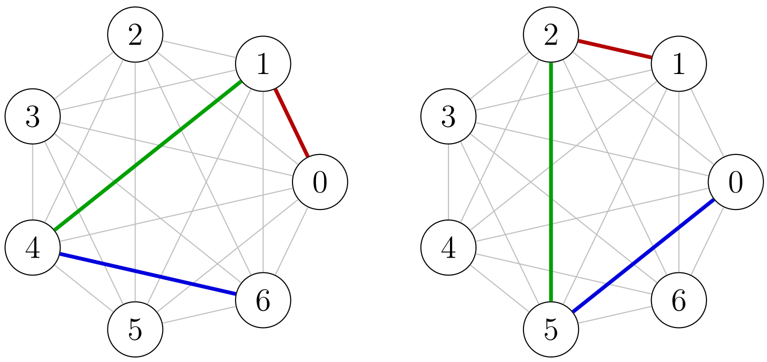

Fig. 6: An example of rotation for $n = 3$. The left $K_{2n+1} = K_7$ shows a graceful tree on $n + 1 = 4$ vertices. The right K_7 shows T_1, a tree obtained from T by rotation by $s/(2n + 1) = s/7$ of a full rotation

We proceed to construct $2n + 1$ such trees T so that they would use all edges of K_{2n+1}, which is the desired decomposition. Let T_i contain an edge $x \sim y$ if and only if $(x - i) \sim (y - i)$ is an edge of T. Observe that $T_0 = T$, and T_{i+1} is obtained by rotating T_0 through $s/(2n+1)$ of a full revolution. See Figure 6 for an example of a rotation with $n = 3$, see also [1] for a more comprehensive visualization of the mentioned rotation.

Each edge $x \sim y$ belongs precisely to one tree because our trees are not intersecting and we have created $2n + 1$ such trees each with n edges, thus using $n(2n + 1)$ edges in total, which is the number of edges of K_{2n+1}. □

In other words, from this theorem one infers that if graceful tree conjecture is true, then Ringel's conjecture is also true. However, unlike the latter, the former is far from settled at the time of this report.

References

[1] K. Hartnett. Mathematicians prove ringel's graph theory conjecture. https://www.quantamagazine.org/mathematicians-prove-ringels-graph-theory-conjecture-20200219/, 2020. [Online; accessed 21-July-2024].

[2] R. Montgomery, A. Pokrovskiy, and B. Sudakov. A proof of ringel's conjecture. https://arxiv.org/abs/2001.02665, 2020. [Online; accessed 21-July-2024].

[3] S. Richard. Graph theory: Cumulative notes. http://www.math.nagoya-u.ac.jp/~richard/teaching/s2024/Graph.pdf, 2024. [Online; accessed 21-July-2024].

[4] A. Rosa. On certain valuations of the vertices of a graph. *Journal of Graph Theory - JGT*, 01 1967.

[5] W. Wallis. *A Beginner's Guide to Graph Theory*. A Beginner's Guide to Graph Theory. Birkhäuser Boston, 2007.