Vertex-colorings

082450140

Narumi Ota

Exercise 8.1.24 of [GYA]

Prove that the chromatic number of an interval graph equals its clique number.

[Definition]

Intersection graph: A simple graph G with vertex-set $V_G = \{v_1, v_2, \dots, v_n\}$ is an intersection graph if there exists a family of sets F = $\{S_1, S_2, \dots, S_n\}$ such that vertex v_i is adjacent to v_j if and only if $i \neq j$ and $S_i \cap S_j \neq \emptyset$.

Interval graph: A simple graph is an interval graph if it is an intersection graph corresponding to a family of intervals on the real line.

[Proof] Decide to write the chromatic number as $\chi(G)$ and clique number as $\omega(G)$. To prove $\chi(G) = \omega(G)$, we prove

$\chi(G) \geq \omega(G)$ first and then prove $\chi(G) \leq \omega(G)$.

$(i)\ \chi(G) \geq \omega(G)$

By the definition, every vertex of maximum clique is connected. Thus, if $\omega(G) = k(2 \leq k \in \mathbb{N})$, at least k Colors are needed. Therefore, $\omega(G) \leq \chi(G)$.

$\therefore \chi(G) \geq \omega(G)$

$(ii)\ \chi(G) \leq \omega(G)$

First, we can order the vertices according to the endpoints of their corresponding intervals. In this ordering, an interval e_i precedes another interval e_j if the right endpoint of e_i is less or equal to the left endpoint of e_j. Second, apply the Sequential Coloring Algorithm. The number of colors required at each step of the Sequential Coloring Algorithm is determined by the number of its neighbors that have already been colored. By the definition of interval graph, any vertex can be part of a clique and its size is at most the number of colors used.

Therefore, $\chi(G)$ is bounded by the size of the largest clique because this clique determines the maximum number of overlapping intervals. Because the interval graph can be colored sequentially by using at most the size of the largest clique, $\chi(G) \leq \omega(G)$

By (i) and (ii), we can say that the chromatic number of an interval graph equals its clique number.

[References] [GYA] J.L. Gross, J. Yellen, M. Anderson, Graph theory and its applications, CRC press