Relationship between matrix and walks

082450140

Ota Narumi

Exercise 2.6.24 of [GYA]

Complete the proof of Proposition 2.6.1 by establishing the inductive step.

Proposition 2.6.1: Let G be a simple graph with adjacency matrix A_G. Then the value of element A^r_G [u, v] of the r^{th} power of matrix A_G equals the number of u-v walks of length r from x_u to x_v.

Proof: Firstly, we define a^r_{uv} as the value of element A^r_G [u, v] of the r^{th} power of matrix A_G.

(i) If $r = 1$,

because of the definition of adjacency matrix that

$$a^1_{uv} = \begin{cases} 1 & \text{if } u \text{ and } v \text{ are adjacent} \\ 0 & \text{otherwise} \end{cases},$$

a^1_{uv} directly equals the number of u-v walks of length 1. Therefore, the proposition is correct.

(ii) Assume that if $r = k \in \mathbb{N}$, the value of element A^r_G [u, v] of the r^{th} power of matrix A_G equals the number of u-v walks of length r.

Then, $a^{k+1}_{uv} = \sum_{l=1}^{n} a^k_{ul}\, a^1_{lv}$

Because of the definition of adjacency matrix, $a_{lv}^{1} = 0\ or\ 1$. If $a_{lv}^{1} = 0$, there is no walks from x_l to x_v. If $a_{lv}^{1} = 1$, there is a walk from x_l to x_v. Also, a_{ul}^{k} means the number of u-l walks of length k. The chart 1.1 below shows all u-v walks of length $k+1$. Hence, $a_{ul}^{k} a_{lv}^{1}$ show all u-v walks of length $k+1$. Therefore, a_{uv}^{k+1} equals the number of u-v walks of length $k+1$.

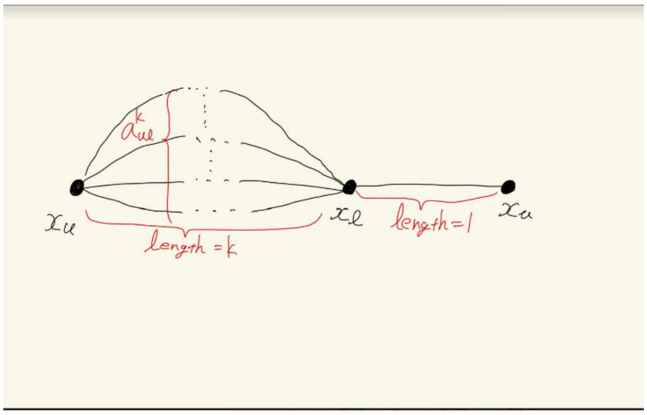

chart 1.1

By (i) and (ii), we can say that the value of element A_G^r [u, v] of the r^{th} power of matrix A_G equals the number of u-v walks of length r from x_u to x_v.

References: [GYA] J.L. Gross, J. Yellen, M. Anderson, Graph theory and its applications, CRC press.