Equivalence of Loops and Bridges in Dual Graph

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Theorem statement

An edge of G is a loop if and only if the associated edge e^* is a bridge in G^*.

Proof

Assumption1: An edge of G is a loop

Let e be a loop in G. This means e is an edge that connects a vertex to itself, thus creating only one face f_2. Therefore, there exist only one face f_1 adjacent to f_2 and separated by e.
In the dual graph G^*, e^* connects the dual vertices $x^*(f_1)$ and $x^*(f_2)$ corresponding to the faces f_1 and f_2 in G.
Since e is a loop, removing e^* from G^* would disconnect $x^*(f_1)$ from $x^*(f_2)$ leaving $x^*(f_2)$ with no edge. Thus e^* is a bridge in G^*.

Assumption2: the associated edge e^* is a bridge in G^*

Let e^* be a bridge edge in G^*. This means that removing e^* would disconnect G^*.
Therefore, e^* must be the only edge connecting two adjacent dual vertices $x^*(f_1)$ and $x^*(f_2)$ in G.
This implied that e in the original graph G must separate the corresponding faces f_1 and f_2 in such a way that it is the only path between these faces.
Hence, e must be the loop in G.

Conclusion

This proof establishes that an edge of G is a loop if and only if the associated edge e^* is a bridge in G^*.

Graph Illustration

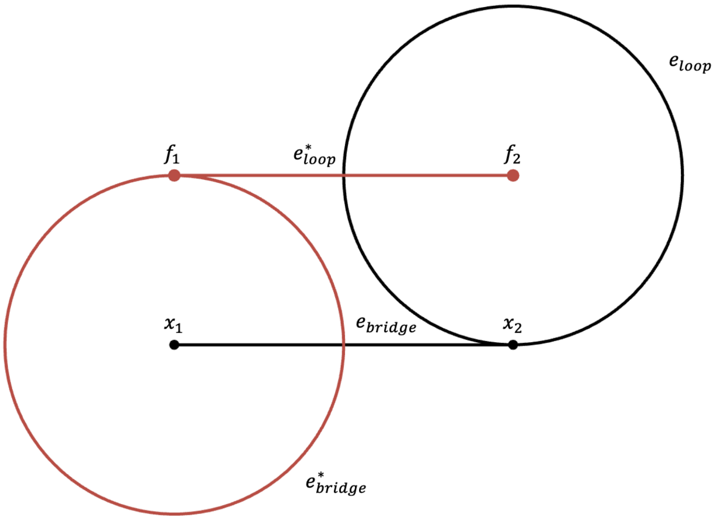

図 1　Graph G with a loop and a bridge, and its dual graph G^*.