On properties of Random Graphs
Graph Theory

Al Kafi Muhtasin

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In this report, we are going to prove propositions 10.2 and 10.3 from the lecture notes which mention some simple properties of random graphs. The proofs are inspired by the lemmas 11.1.2 and 11.1.4 in [1] which are the same as the above propositions.

Proposition 10.2 For any integers $n \geq k \geq 2$ and $\forall G \in \mathcal{G}(n,p)$, the probability that $\alpha(G) \geq k$ is upper estimated by

$$\mathbb{P}(\alpha(G) \geq k) \leq \binom{n}{k}(1-p)^{\binom{k}{2}}$$

Proof. The independence number $\alpha(G)$ is the number of vertices in the largest independent set of G i.e the largest set of vertices with no edges among themselves. Thus, we find the probability of existence of an independent set containing k vertices at first. For these k vertices, the probability that no edges exist between any pairs is $(1-p)^{\binom{k}{2}}$ i.e. $\binom{k}{2}$ edges are absent. Moreover, we can choose these k vertices in $\binom{n}{k}$ different ways. Thus, the probability of existence of a k-independent set is

$$\mathbb{P}(k) = \binom{n}{k}(1-p)^{\binom{k}{2}}$$

Now, increasing k will impose the absence of more edges between other pairs of vertices resulting in decrease of its probability. So, if $\alpha(G) > k$, then the probability of existence of the maximal independent set will be lower than our previous expression. Thus, we have the upper bound of

$$\mathbb{P}(\alpha(G) \geq k) \leq \binom{n}{k}(1-p)^{\binom{k}{2}}$$

□

Proposition 10.3 For any integers $n \geq k \geq 3$ one has

$$\mathbb{E}(k - cycles) = \frac{(n)_k}{2k}p^k$$

Proof. If we have a discrete random variable X, then its expectation is given by

$$\mathbb{E}(X) = \sum_{x \in im(X)} x \cdot \mathbb{P}(X = x)$$

For random graphs, any graph invariant can be treated as a discrete random variable. Thus, we define a random variable X which maps any graph $G \in \mathcal{G}(n,p)$ to its number of $k - cycles$. Then, we are required to calculate $\mathbb{E}(X)$.
Now, with $\mathcal{C}_k$ being the set of all $k - cycle$ in G, $\forall C \in \mathcal{C}_k$ we define another random variable

$$X_c : G \mapsto \begin{cases} 1 & \text{if } C \subseteq G \\ 0 & \text{otherwise} \end{cases}$$

Thus, the map indicates the existenc of the cycle C. Here,

$$\mathbb{E}(X_c) = 0 \cdot \mathbb{P}(X_c = 0) + 1 \cdot \mathbb{P}(X_c = 1) = \mathbb{P}(C \subseteq G) = p^k$$

where the last equality comes from the fact that for the $k - cycle$ to exist, all it's k edges need to exist. We have for every G,

$$X(G) = \sum_{C \in \mathcal{C}_k} X_c(G)$$

as we are getting the number of existent $k - cycles$ in G out of all the possible ones by the sum above. Moreover, due to linearity of expectation, we have

$$\mathbb{E}(X) = \sum_{C \in \mathcal{C}_k} \mathbb{E}(X_c) = \sum_{C \in \mathcal{C}_k} p^k$$

Now, we need to find out the number of elements in $\mathcal{C}_k$ i.e. the number of possible $k - cycles$. To have a $k - cycle$ we need to choose k vertices where we can choose the first one out of n vertices, the second one out of $n - 1$ vertices and so on up to the k_{th} one out of $n - k + 1$ vertices. Thus, the total number of choices we have is

$$(n)_k = n(n-1)\dots(n-k+1)$$

where the vertices are ordered from 1 to k. However, we notice that when making a cycle with these k vertices, we can start with any one of these k vertices and have the identical cycle. Again, starting from one vertex, we have two paths to complete the cycle starting in opposite directions. Thus, the total number $k - cycles$ we have is

$$|\mathcal{C}_k| = \frac{(n)_k}{2k}$$

Finally,

$$\mathbb{E}(X) = \sum_{C \in \mathcal{C}_k} p^k = \frac{(n)_k}{2k} p^k$$

□

References

[1] R. Diestel, *Graph theory*, Fifth edition, Springer, 2017.

[2] Serge Richard, *Graph Theory lecture notes*, 2024.