König's Theorem in Graph Theory

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Introduction

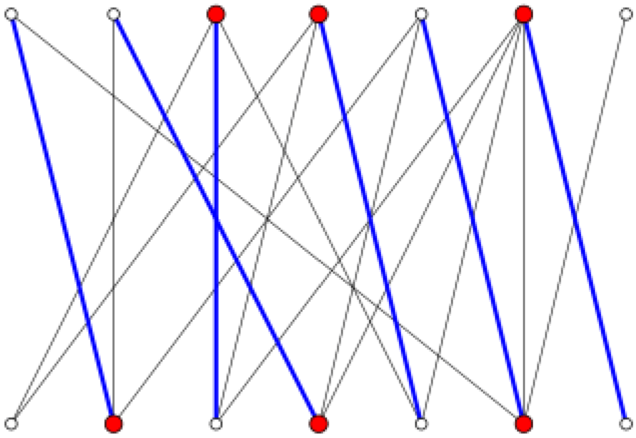
In graph theory, König's theorem states that the maximum matching number and the minimum vertex cover number are equal in a bipartite graph. This theorem is named after the Jewish Hungarian mathematician Dénes Kőnig. In 1931, Hungarian mathematician Jenő Egerváry independently discovered a more general form of this theorem for weighted graphs.

Matching and Covering

For example, in a bipartite graph, the maximum matching (blue edges) and the minimum vertex cover (red vertices) both have a size of 6. A vertex cover of a graph is a set of vertices such that every edge of the graph has at least one endpoint in this set. If there is no vertex cover with fewer vertices, it is called a minimum vertex cover.

A matching in a graph is a set of edges such that no two edges share a common vertex. A matching is called a maximum matching if there is no matching with more edges.

For each edge in a matching, at least one endpoint of this edge is in the vertex cover, so the number of vertices in any vertex cover is greater than or equal to the number of edges in the matching. König's theorem states that in bipartite graphs, the sizes of these two sets are equal.



Content of the Theorem

In any bipartite graph, the number of edges in a maximum matching is equal to the number of vertices in a minimum vertex cover.

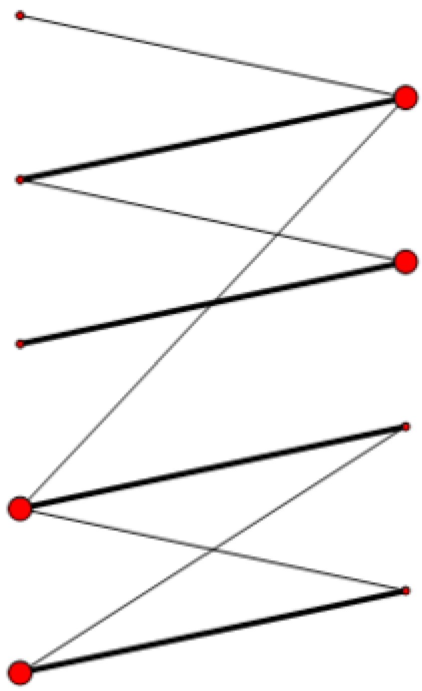
Proof

Let M be a maximum matching. Since the vertex cover is greater than or equal to the size of the matching, we only need to construct a vertex cover of size $|M|$. The construction is as follows:

Let the two parts of the bipartite graph be A and B. An **alternating path** with respect to matching M is a path starting from an unmatched vertex in the graph and alternating between edges not in M and in M. Define the vertex set U as follows: For each edge in M, if there exists an alternating path ending at the vertex in B, then that vertex belongs to U; otherwise, the vertex in A of that edge belongs to U. Since each vertex in U corresponds one-to-one with each edge in M, we have $\mid U \mid = \mid M \mid$. Therefore, we need to prove that U is a vertex cover.

Assume there is an edge ab not covered by U, that is, $a \in A$ and $b \in B$ are not in U. If a is not an endpoint of some edge in M, then ab itself is an alternating path starting from an unmatched vertex, so $b \in U$, which is a contradiction. If a is an endpoint of some edge in M, say ab', then $b' \in U$. Thus, there exist a path ending at b'.

If $b \notin P$, then $P \cup \{(b', a), (a, b)\}$ is an alternating path ending at b. Else, path P stops at b and it is also an alternating path ending at b. Hence $b \in U$, which is a contradiction.



Reference

Matching Covering and Packing, R. Diestel, Graph theory, Springer.