Whitney's Connectivity Inequalities

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Relevant Concepts

Minimum Degree: The smallest degree of vertices in the undirected graph G, denoted as $\delta(G)$.

Edge Connectivity: The minimum number of edges that need to be removed to make G disconnected or a trivial graph, denoted as $\lambda(G)$.

Vertex Connectivity: The minimum number of vertices that need to be removed to make G disconnected or a trivial graph, denoted as $\kappa(G)$.

Content of the Theorem

For any graph G:

$$\kappa(G) \leq \lambda(G) \leq \delta(G)$$

Proof

Step 1: Prove $\lambda(G) \leq \delta(G)$

- If G is disconnected, then:

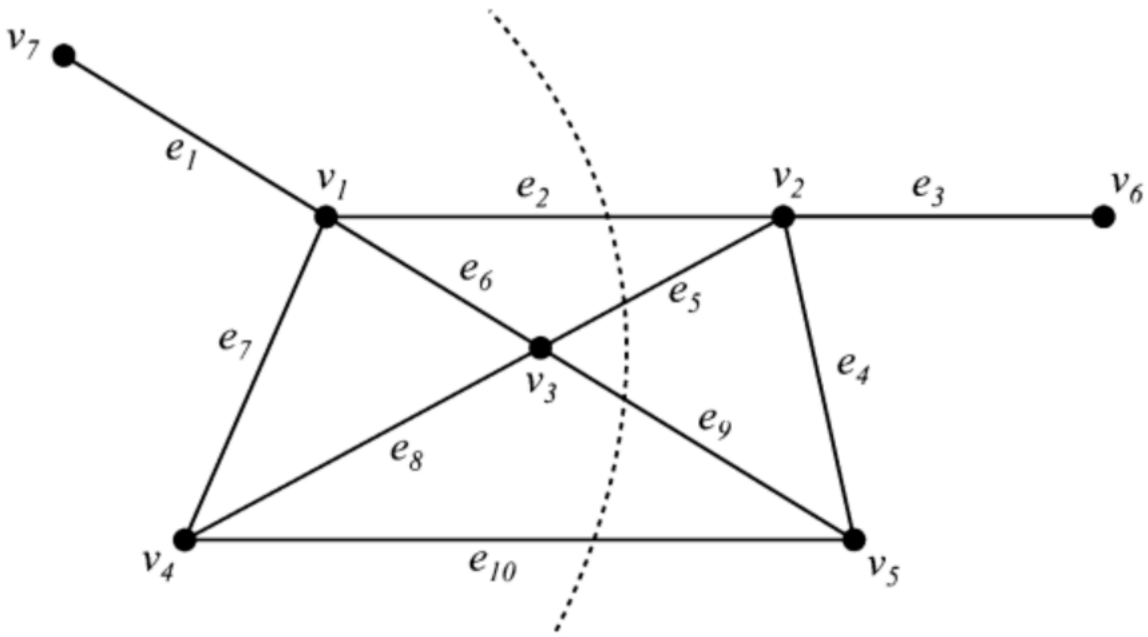
$$\lambda(G) = 0 \leq \delta(G)$$

- If G is connected, then there exists a vertex v with $\deg(v) = \delta(G)$. Removing all edges incident to v will make G disconnected.

Therefore:

$$\lambda(G) \leq \delta(G)$$

Step 2: Prove $\kappa(G) \leq \lambda(G)$



Consider the smallest edge cut set F of the graph G. It suffices to show that there exists a vertex cut set S in G such that $|S| \leq |F|$. For F, let it divide G into two connected components C_1 and C_2, where C_1 is the connected component and C_2 is the other connected component. Define $T = \{v | v \in C_1 \wedge (v, v') \in F, v' \in C_2\}$. Then $|T| \leq |F|$. We will show T is a vertex cut set. For any edge $(v, v') \in F$, by definition, we have $v \in C_1, v' \in C_2$ or $v \in C_2, v' \in C_1$. WOLG let $v \in C_1$, then $v \in T$. Thus all edges in F are covered by T, and to cut T means to cut at least all edges in F then make the graph disconnected.

Therefore:

$$\kappa(G) \leq |T| \leq |F| = \lambda(G)$$

Step3: Conclusion

Combining the above results, we have:

$$\kappa(G) \leq \lambda(G) \leq \delta(G)$$

Reference

Connectivity, R. Diestel, Graph theory, Springer.