# Triangle Detection in Graphs

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In this report, we will show how to find triangles in graphs using the adjacency matrix. The technique used in this report can be applied to find many other different shapes in graphs, however, for simplicity, we have chosen to stick with triangles.

First, we need to provide a theorem. Let G be a graph and A be its adjacency matrix. $A_{i,j}^n$ is the number of walks of length n from $u_i$ to $u_j$, where $u_i$ and $u_j$ are two vertices of G. We will prove this by induction on n.

proof: For the base case, let n=1. $A_{i,j}^1$ is the number of walks of length 1 from $u_i$ to $u_j$. This is the definition of the adjacency matrix, where 1 means a path exists and 0 means a path doesn't exist.

For the induction hypothesis, let n>=2. $A_{i,j}^n$ is the number of walks of length n from $u_i$ to $u_j$. We will now prove that $A_{i,j}^{n+1}$ is the number of walks of length n+1 from $u_i$ to $u_j$. Let $W_i^n$ be a walk of length n from $u_i$ to $u_l$, where $0 < i < n$. To find $W_i^{n+1}$ from $u_i$ to $u_j$, we must combine the walk from $u_i$ to $u_l$ and the walk from $u_l$ to $u_j$ such that $W_i^{n+1} = W_i^n * u_l u_j$. If we want to find the number of such walks, we will rewrite this as $\sum_l W_i^n * u_l u_j$.

However, this can be rewritten as $A_{i,l}^{n} * A_{l,j} = A_{i,j}^{n+1}$, therefore proving the above statement.

Now that we have shown that the power of the adjacency matrix is related to the length of the paths that connect its elements, we will now propose the following:

If a diagonal element of the adjacency matrix to the third power ($A_{i,i}^{3}$) is greater than or equal to 1, then a triangle with starting point $A_{i,i}^{3}$ exists.

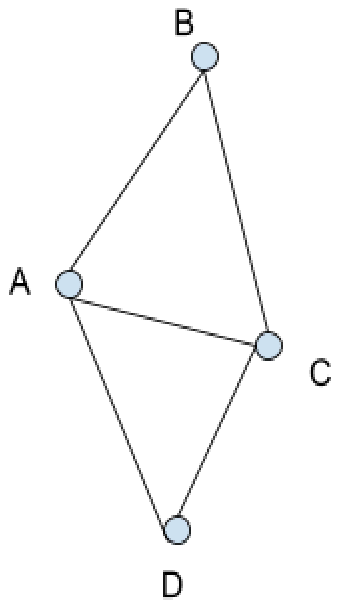
According to the definition of the adjacency matrix, we know that an element on the diagonal matrix must both start from and end on itself. We also know that $A_{i,i}^{3}$ must be a walk of length 3 (proven above). Therefore, $A_{i,i}^{3}$ must be a walk of length 3 that starts from and ends on itself = $A_{i,i}^{3}$ is a triangle with starting vertex $u_i$.

Finally, we will prove an interesting observation of the above result: *For any adjacency matrix A, the diagonal of $A^3$ is always made up of even numbers.*

*Proof*: Let A be the adjacency matrix of graph G, and let n be the number of vertices v in G. We choose an arbitrary starting point for our path $v_0$, which would be represented by the first row, first column of A. To find the triangles of G that start at $v_0$, we take $A^{3}_{0,0} = \sum^{\mathbf{n}}_{\mathbf{i=1}}\sum^{\mathbf{n}}_{\mathbf{j=1}} A_{0,i}A_{i,j}A_{j,0}$. However, the path taken from the starting point $v_0$ has two directions that will lead to the same triangle. In the above summation, the path $v_0$ -> $v_i$

-> $v_j$ -> $v_0$ and the path $v_0$ -> $v_j$ -> $v_i$ -> $v_0$ will always lead to the same triangle, meaning that each triangle has at least two paths. Since the diagonal elements of $A^3$ represent the number of triangles, and each triangle has at least two paths, each diagonal element will be a factor of 2, implying that each element will be even.

In summary, to determine whether there is a triangle in a graph, one must compute the 3rd power of the adjacency matrix and look at the diagonal elements. If any of them are greater than 0, then a triangle must exist in the graph. A figure with an example is shown below.




$$A = \begin{array}{c|cccc} & A & B & C & D \\ \hline A & 0 & 1 & 1 & 1 \\ B & 1 & 0 & 1 & 0 \\ C & 1 & 1 & 0 & 1 \\ D & 1 & 0 & 1 & 0 \end{array}$$

$$A^3 = \begin{array}{c|cccc} & A & B & C & D \\ \hline A & 4 & 5 & 5 & 5 \\ B & 5 & 2 & 5 & 2 \\ C & 5 & 5 & 4 & 5 \\ D & 5 & 2 & 5 & 2 \end{array}$$

As we can see in $A^3$, there are 4 triangles starting from $A_{A,A}$, 2 from $A_{B,B}$, 4 from $A_{C,C}$, and 2 from $A_{D,D}$