<u>Proof of Proposition 2.2</u>

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Definition of Adjacency Matrix

Let $G = (V, E)$ be a finite graph, and set $V = \{x_1, x_2, \ldots, x_n\}$. The adjacency matrix A_G is an $N \times N$ matrix with entries $a_{jk} = \# \{e \in E \mid i(e) = (x_j, x_k)\}$

Proposition 2.2

Let G be a $N \times N$ adjacency matrix. For any $r \in \mathbb{N}$ the entry $(A_G^r)_{jk}$ of the r^{th} power of A_G is equal to the number of walks of length r from x_j to x_k.

Proof by induction

For $r = 1$ follows from the definition since

$(A_G^1)_{jk} = (A_G)_{jk}$ which is the number of walks of length 1 from x_j to x_k.

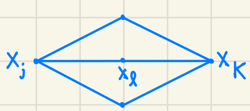
Thus the result holds for $r = 1$

For $r = 2$

$$(A_G^2)_{jk} = \sum_{l=1}^{n} (A_G)_{jl} (A_G)_{lk}$$

$(A_G)_{jl}$ is the number of walks of length 1 from x_j to x_l and $(A_G)_{lk}$ is the number of walks of length 1 from x_l to x_k therefore $(A_G)_{jl}(A_G)_{lk}$ is the number of walks of length 2 consisting of a walk of length 1 from x_j to x_l and a walk of length 1 from x_l to x_k.

Figure 1: A graph which shows a few walks from x_j to x_k with length 2.



Thus $\sum_{l=1}^{n} (A_G)_{jl} (A_G)_{lk}$ indicates the number of all walks of length 2 from x_j to x_k.

Thus the result holds for $r = 2$.

Let's assume that the result holds for r.

For $r+1$

$$(A_G^{r+1})_{jk} = \sum_{l=1}^{n} (A_G^r)_{jl} (A_G)_{lk}$$

$(A_G^r)_{jl}$ is the number of walks of length r from x_j to x_l while $(A_G)_{lk}$ is the number of walks of length 1 from x_l to x_k. Thus $(A_G^r)_{jl} (A_G)_{lk}$ represents the number of walks of length $r+1$ consisting of a walk of length r from x_j to x_l and a walk of length 1 from x_l to x_k.

Thus $\sum_{l=1}^{n} (A_G)_{jl} (A_G)_{lk}$ indicates the number of all walks of length $r+1$ from x_j to x_k.

Henceforth proposition 2.2 is true for any $r \in \mathbb{N}$.