Some Properties of a Line Graph

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1 Introduction

The aim of this report is to display the connection between the line graph of any simple (no loops or multi-edges), connected, undirected graph, and a union of complete graphs. The properties of these complete graphs are much easier to manipulate and in turn will yield results about the line graph. Although we consider no multi-edges, we can extend to these quite naturally (however, not covered in this report).

Definition 1. The line graph of an undirected graph $G = (V, E)$ (without loop) consists in a new graph $L(G) := (V', E')$ with $V' = E$ and two vertices in V' are adjacent if and only if they had a common vertex in G.

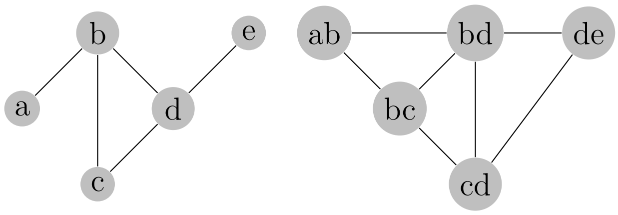

Figure 1: An example of G and its $L(G)$

Theorem 1.1. *If G is connected, $L(G)$ is connected .*

Proof. In a connected graph G, every edge has at least one vertex in common with another edge. Hence $L(G)$ is connected by Definition 1.

□

Definition 2. Let K^n denote the complete undirected graph on n vertices.

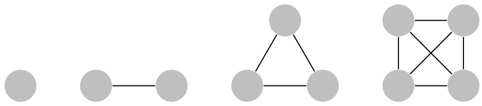

Figure 2: K^1, K^2, K^3, K^4

Theorem 1.2. *The graph K^n has $\binom{n}{2} = \frac{n(n-1)}{2}$ edges.*

Proof. Each vertex in K^n is connected to every other vertex. Hence,

$$\forall v \in K^n,\ deg(v) = n - 1$$

Applying the handshaking lemma,

$$||K^n|| = \frac{1}{2} \sum_{v \in |K^n|} deg(v) = \frac{1}{2} \sum_{k=1}^{n} (n-1) = \frac{1}{2} n(n-1)$$

□

To motivate the following, let us consider the closed graph of neighbours (neighbours and the vertex themselves) of each vertex in Figure 1 (While keeping track of the vertex labels).

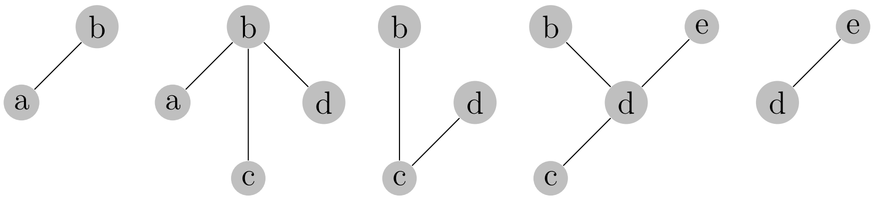

Figure 3: The graphs of closed neighbourhoods, $N[v]$ for each vertex, v in V

$\forall N[v]$, Construct $L(N[v])$, the line graph of each of the neighbourhoods.

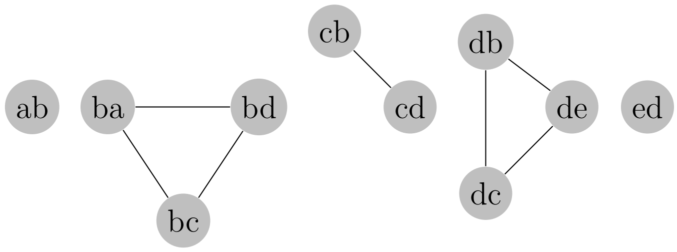

Figure 4: $L(N[v])\ \forall v \in V$

Notice that each $L(N[v])$ is a complete graph on $deg(v)$ vertices.

Claim 1.

$$L(N[v]) \cong K^{deg(v)} \tag{1}$$

In words, the line graph of the closed neighbours at a vertex in G, is isomorphic to a complete graph on $deg(v)$ vertices.

Proof.

The closed neighbourhood graph, $N[v]$ has a vertex set consisting of the vertex, v and its neighbours. Since they are neighbours, v shares an edge to each other vertex in $N[v]$.

$$\begin{aligned} N[v] = (V_N &= \{\{v\} \cup N(v)\}, \\ E_N &= \{(vw) | \forall w \in N(v)\}) \end{aligned}$$

Note that $N(v)$ denotes the set of vertices that are the neighbours of v. (does not contain v). Wheras $N[v]$ is the graph of the Closed Neighbourhood of v. By the definition of the line graph,

$$\begin{aligned} L(N[v]) = (V_L &= E_N, \\ E_L &= \{(vw, vx) | \forall w, x \in N(v); w \neq x\}) \end{aligned}$$

Meanwhile on the RHS,

$$K^{deg(v)} = (V_K, E_K)$$

And with the information of the labelling conserved,

$$V_K = V_L$$

This is reasonable as the cardinality of the vertex set

$$|V_K| = deg(v) = |N(v)| = |E_N| = |V_L|.$$

And by the definition of the complete graph, each vertex is connected to every other.

$$V_K = \{(vw, vx) | \forall w, x \in N(v); w \neq x\} \equiv V_L$$
$$E_K = \{(y, z) | \forall y, z \in V_K \equiv V_L\} \equiv E_L$$

□

Theorem (Whitney's Line Graph Theorem)**.** *If G and G' are connected and their line graphs are isomorphic, G and G' are isomorphic except if G is isomorphic to K_3 ant the G' is isomorphic to $K_{3,1}$ [1]*

The exception to the above is shown below. Note, that if we conserve information on the labelling of the vertices and the edges, that is by implementing a naming scheme such that the names of edges depend on vertices, we can make a clear distinction, even in the exception as shown in the following.

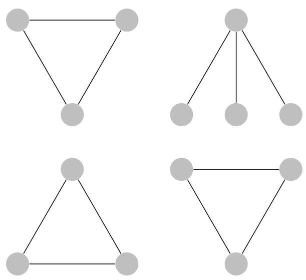

Figure 5: Above: K_3 and $K_{3,1}$ (not isomorphic). Below: Their Corresponding Line Graphs (isomorphic).

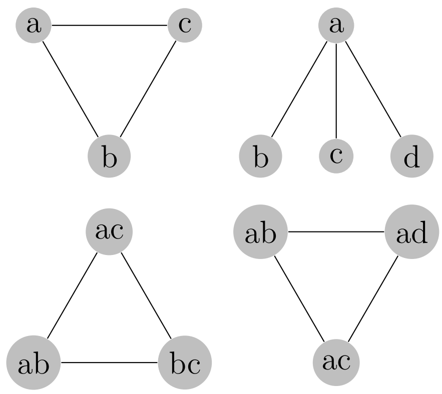

Figure 6: Above : K_3 and $K_{3,1}$ with labelling scheme (not isomorphic),
Below : Corresponding Line Graphs with labelling (not isomorphic)

Claim 2.

$$L(G) \cong \bigcup_{v \in V} L(N[v])$$

We illustrate the above claim for the example from earlier. By the big union sign, we take the disjoint union of the graphs **ensuring that the labelling stays the same** (so there are no pesky exceptions) and if vertices reoccur, we keep only one copy.

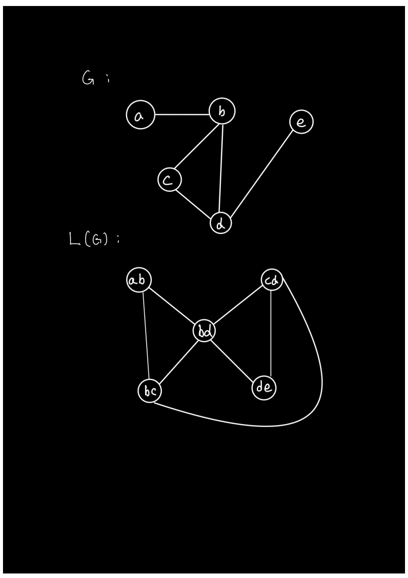

Figure 7: Graph and Line Graph

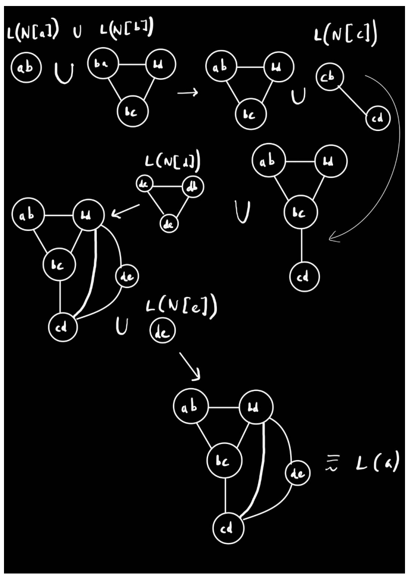

Figure 8: Union of Closed Neighbors

Proof. Let $G = (V, E)$ be a graph and its line graph $L(G) = (V_L, E_L)$. Consider the family of closed neighborhoods $\{N[v]\}_{v \in V}$.
By Claim 1, we know that the line graph of each closed neighborhood $N[v]$ is isomorphic to a complete graph on $\deg(v)$ vertices:

$$L(N[v]) \cong K^{\deg(v)}$$

Now, consider the union of the line graphs of all closed neighborhoods of G:

$$\bigcup_{v \in V} L(N[v])$$

Each edge $e \in E$ appears in the closed neighborhoods of both of its endpoints. Hence, when taking the union of all such closed neighborhoods, we end up with a connected graph where vertices represent edges of G. Let us denote this graph by $G_C = (V_C, E_C)$, where:

$$V_C = \{e \mid e \in E\}$$

$$E_C = \{(e, e') \mid \text{edges } e \text{ and } e' \text{ share a common vertex in } G\}$$

On the other hand, each edge $e \in E$ corresponds to a vertex in $L(G)$. Thus, $V_L = \{e \mid e \in E\}$, and $L(G)$ is connected by Theorem 1.1.

$$E_L = \{(e, e') \mid \text{edges } e \text{ and } e' \text{ share a common vertex in } G\}$$

Since both $L(G)$ and G_C have the same vertex set $V_C = V_L$ and edge set $E_C = E_L$, we conclude that:

$$L(G) \cong \bigcup_{v \in V} L(N[v])$$

□

Size of a Line Graph
Let us conclude by computing the size of the line graph by isomorphism :

$$||L(G)|| = \sum_{v \in V} ||K^{deg(v)}|| = \sum_{v \in V} \frac{(deg(v)(deg(v) - 1)}{2} = \sum_{v \in V} \binom{\deg(v)}{2}$$

References

[1] R. L. Hemminger. "On Whitney's Line Graph Theorem". In: *The American Mathematical Monthly* 79.4 (1972), pp. 374–378. ISSN: 00029890, 19300972. URL: http://www.jstor.org/stable/2978089 (visited on 08/07/2024).