The number of odd degree vertices is even

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May 2024

<u>Claim:</u> The number of vertices with odd degree is even in any undirected graph.
<u>Proof:</u> According to the handshaking lemma,

$$\sum_{v \in V} deg(v) = 2|E| \tag{1}$$

Let us divide our vertices into two sets: those with odd degree and those with an even degree. Formally:

$$\begin{aligned} V_{even} &:= \{v \in V | deg(v) = 2k, k \in \mathbb{N}\} \\ V_{odd} &:= \{v \in V | deg(v) \text{ is odd}\} \end{aligned} \tag{2}$$

Because of (1) and (2):

$$\sum_{v \in V} deg(v) = \sum_{v \in V_{even}} deg(v) + \sum_{v \in V_{odd}} deg(v) = 2|E| \tag{3}$$

The sum over V_{even} consists only of even terms, hence it is even. Therefore,

$$\sum_{v \in V_{odd}} deg(v) = 2m, \text{ where } m \in \mathbb{N} \tag{4}$$

By the observation that a finite integer series where each term is odd, is only even when there are an even number of terms,

$$(4) \implies |V_{odd}| = 2n, \text{ where } n \in \mathbb{N} \tag{5}$$

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Examples

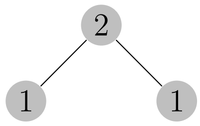
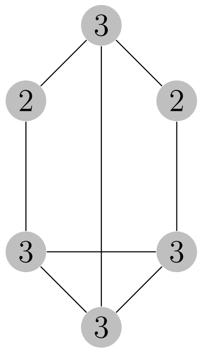
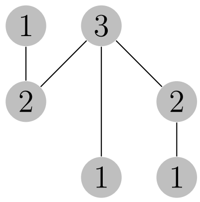
Graph	$\lvert V_{odd}\rvert$ (even)	$\lvert V_{even}\rvert$	$\sum_{v\in V_{odd}} deg(v)$	$\sum_{v\in V_{even}} deg(v)$
	2	1	1+1 =2	2
	4	2	3 + 3 + 3 + 3 = 12	2 + 2 = 4
	4	2	1 + 3 + 1 + 1 = 6	2 +2 = 4

Table 1: The number represents the degree of the vertex