A Report on Eulerian Properties

NGO Gia Linh (042001463), HIRAO Momona (082340544)

SML: Introduction to Graph Theory (Spring 2024)

Definition of Eulerian trails

- An **Euleurian trail** is a trail that contains every edge of a graph.
- An **Eulerian tour** is a closed Eulerian trail.
- An **Eulerian graph** is a connected graph which possesses an Eulerian tour.

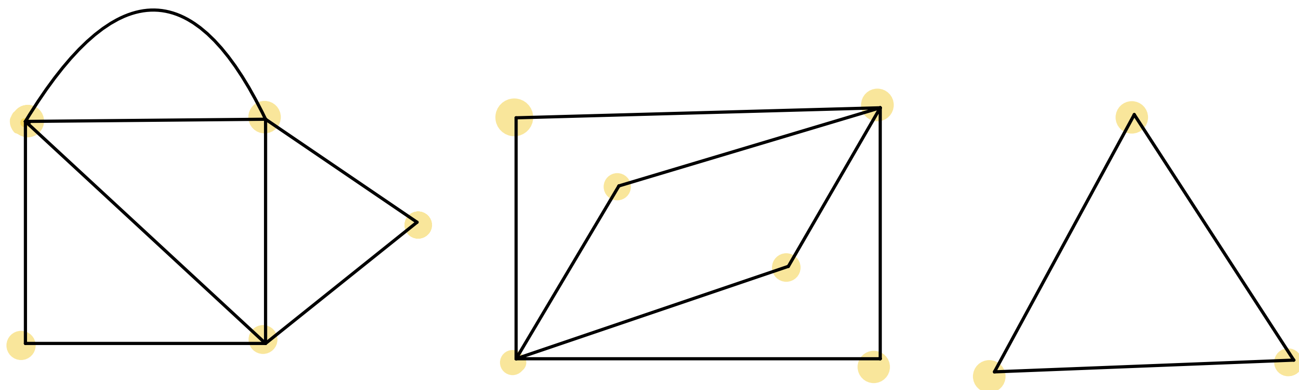
Theorem 1.25.

A connected, undirected, and finite graph is Eulerian if and only if every vertex has an even degree.

Proof

($\Longrightarrow$) Suppose that a graph G is a connected, undirected, and finite Eulerian graph. Consider starting an Eulerian tour C from a vertex v_0 in G. Since G is an Eulerian graph, all vertices reached through C must be passed with two unused edges, one to enter the vertex and one to exit the vertex. And since it is an Eulerian tour, it must end at v_0. Thus, there is an even number of traversed edges incident with each vertex, and since each edge of G is traversed exactly once, this number is the degree of that vertex.

($\Longleftarrow$) Suppose that every vertex of a connected, undirected, and finite graph G has an even degree. Take one vertex in G, let's say v_0, and start a walk such like $v_0 \to v_1 \to v_2 \to \ldots$. Since every vertex of G has an even degree, there must be a trail that starts from v_0 and ends at v_0. Let's call this cycle as C_0, then it is an Eulerian graph. If $C_0 = G$, G is an Eulerian graph. Otherwise, consider G' created by removing all edges in C_0 from G. Since the number of edges in C_0 is even, every vertex of G' still has an even degree. Now, consider v_k such that $v_k \in C_0 \land v_k \in G'$. If v_k has edges in G', it must be a cycle containing it in G', let's say C_k. The graph composed by binding C_0 and C_k is an Eulerian graph. The above operation, the removal of cycles from the graph when it cannot be composed by binding all cycles that had been removed, is repeatable. By repeating this, the number of edges of the graph finally reaches 0. Hence, G can be completely created by binding cycles and therefore G is an Eulerian graph.



Example of Eulerian Graph

Remark

Eulerian trails that are not closed can also be useful. Such trails would correspond to trails visiting all edges of the graph but with an initial point and a final point being different. We call such a trail an *open Eulerian trail*.

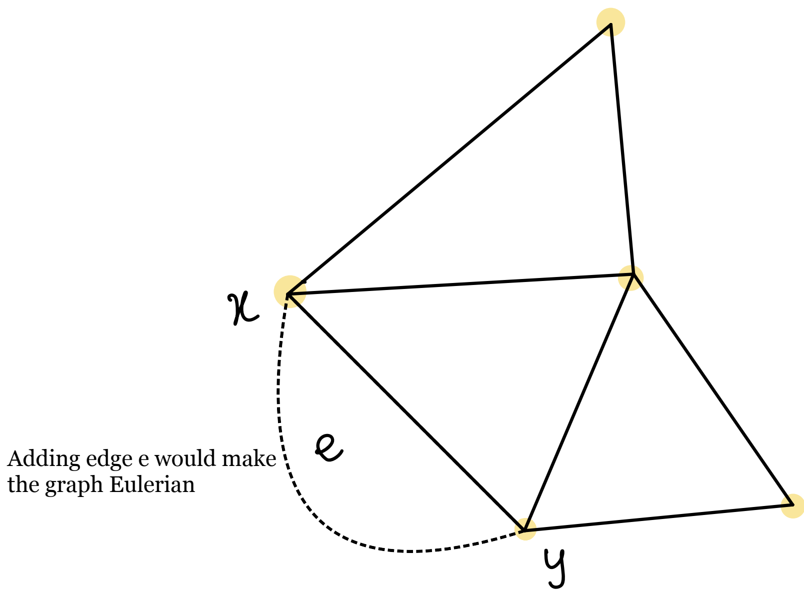
Theorem 6.1

A connected, undirected and finite graph admits an open Eulerian trail if and only if it has exactly two vertices of odd degree. Furthermore, the initial and the final vertices of any Eulerian trail must be the two vertices of odd degree.

Proof

($\Longrightarrow$) Suppose that $(x, e_1, v_1, e_2, ..., e_m, y)$ is an open Eulerian trail in G, with x and y be the initial and the final vertices respectively. Adding a new edge e joining vertices x and y creates a new graph $G_* = G + e$ with an Eulerian tour $(x, e_1, v_1, e_2, ..., e_m, y)$. By the Eulerian-graph characterization (Theorem 1.25), the degree in the Eulerian graph G_* of every vertex must be even. This makes the degree of vertex x and y in graph G, which is the graph G_* without edge e, being odd.

($\Longleftarrow$) Suppose that x and y are the only two vertices of graph G with odd degree. If e is a new edge joining x and y, then all the vertices of the resulting graph G_* have even degree. It follows from the Eulerian-graph characterization that graph G_* has an Eulerian tour T. Hence, the trail $T - e$ obtained by deleting edge e from tour T is an open Eulerian trail of $G_* - e = G$, with x and y being the initial and final vertices.



Example of an open Eulerian trail