Menger's Theorem

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I、 Theorem Content

In a finite, undirected simple graph G, for any two non-adjacent vertices u and v, the size of the smallest vertex cut (a set of vertices whose removal disconnects u from v) is equal to the maximum number of pairwise internally vertex-disjoint u-v paths.

II、 Theorem Proof

Prove by mathematical induction:
Perform induction on $m = |E|$. Suppose the graph G has a minimum vertex cut S of size k. It is clear that the number of internally disjoint $u - v$ paths is $\leq k$. We only need to prove that this number is equal to k.
- When $m = 0$, the statement is obviously true for the empty graph.
- If the statement holds for all graphs with $|E| \leq m - 1$, then for a graph G with $|E| = m$, we prove the above proposition. When $k \leq 1$, the conclusion is obviously true. We only need to consider $k \geq 2$. The cases are as follows:
1. If there exists an $x \in S$ such that x is adjacent to both u and v, then $G - x$ has a vertex cut set $S \setminus \{x\}$ of size $k - 1$, and $|E - x| < m$. By the induction hypothesis, $G - x$ has $k - 1$ internally disjoint $u - v$ paths. Therefore, in G, adding $\langle u, x, v\rangle$, there are k internally disjoint $u - v$ paths.

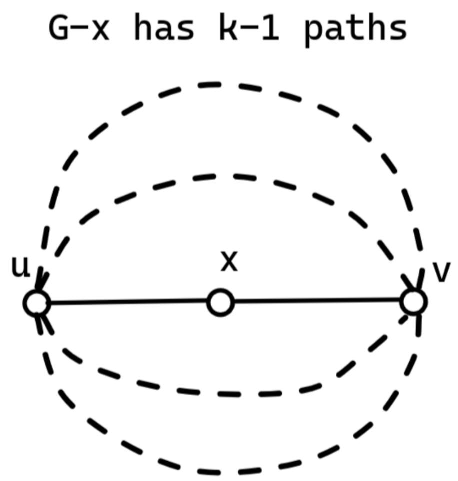

图 1: Case 1

2. If there exists an S such that there is a vertex v' in S that is not adjacent to u, Let $W = \{w_1, w_2, \ldots, w_k\}$ Construct G_u as the subgraph of G, including vertices $\{u, w_1, w_2, \ldots, w_k\}$ and all $u - w_i$ paths. Then construct G'_u: based on G_u, add a new vertex v', which is directly connected to all w_i. Similarly, from the perspective of v, construct G_v and G'_v.
Since there exist vertices in W that are not adjacent to u and v, the edge count of G'_u and G'_v must be less than m. (This is the core of the induction: in G'_u, we essentially replace all paths from w_i to v with a single edge, and since there exists some w_j not adjacent to v, there must be a reduction in the number of edges). The separator sets of G'_u and G'_v are both W, and $|W| = k$, and since their edge counts are less than m, we can apply the induction hypothesis to obtain that the maximum number of internally disjoint paths in both graphs is k.
Combining these two graphs, it is clear that the original graph G has k internally disjoint $u - v$ paths.

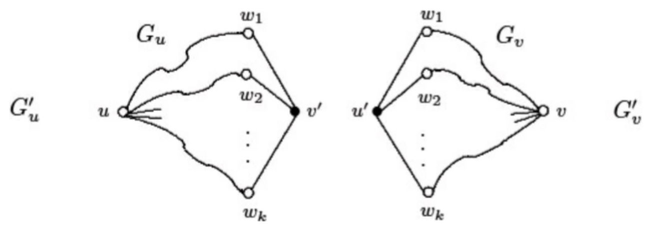

图 2: Case 2

3. The remaining case is that for ALL S, all vertices are either ONLY adjacent to u or only adjacent to v. Suppose the shortest path from u to v is $\langle u, x, y, \ldots, v\rangle$, let $e = (x, y)$.
Let $G' = G - e$ be the graph obtained by removing the edge e.
Let Z be the minimum vertex cut of G', it is clear that $k - 1 \leq |Z| \leq k$(equal to G or 1 less than G). We only need to prove $|Z| = k$ and use the induction hypothesis (since $|E'| < m$).
By contradiction, suppose $|Z| = k - 1$. According to our description, both $Z \cup \{x\}$ and $Z \cup \{y\}$ are minimum vertex cuts in G. Due to our case 3 assumption, all vertices in Z are either only adjacent to U or V. That means x, y is both adjacent to U or both adjacent to V. Therefore, edge e should not exist is the shortest path between <u,v>, which cause a contradiction. Therefore,$|Z| = k$.
Then we can proceed with the induction hypothesis. Clearly, the edge count of $G - e$ is less than m. We have just proven that its minimum separator set size is k. Therefore, the maximum number of internally disjoint $u-v$ paths in $G-e$ is k, so the maximum number of internally disjoint $u-v$ paths in G is k, which completes the proof.

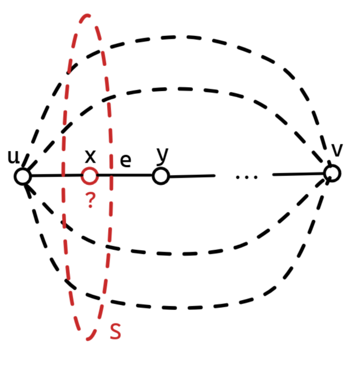

图 3: Case 3