Dirac's Theorem

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I、 Theorem Content

Let G be a simple graph with n vertices ($n \geq 3$). If every vertex in the graph has a degree of at least $\frac{n}{2}$, then G is a Hamiltonian graph.

II、 Connectivity Proof

First, we prove that the graph is connected. Using proof by contradiction, if the graph is not connected, then there must exist multiple connected components in the graph. For the the smallest component, its maximum number of vertices is half the number of vertices in the graph. Denote the number of vertices as $|G| = n$, then the maximum number is $n/2$. Since we are dealing with simple graphs, there are no parallel edges or loops. Therefore, the maximum degree of a vertex in the smallest component is $|C| - 1$, where $|C|$ is the number of vertices in the component. Suppose v_c is any vertex in C, we can derive the following:

$$d(v_c) \leq |C| - 1 < |C| \leq n/2$$

This contradicts the assumption that the degree of any vertex $d(v) \geq n/2$. Hence, the graph is connected.

III、 The Longest "Path" is a Cycle

Let $P = x_0, x_1, ..., x_k$ be the longest path in G. Since P is the longest path, both endpoints x_0 and x_k must have neighbors inside P, otherwise P could be extended to a longer path. Since x_0 has at least $n/2$ neighbors and the same applies to x_k, there are at most $n-2$ positions excluding x_0 and x_k on P. According to the pigeonhole principle, when distributing n points into $n-2$ positions, there must be a neighbor of x_0 that is adjacent to a neighbor of x_k at the next position in P. Because there are at most $n-2$ positions, placing $n/2$ neighbors of x_0 and another $n/2 - 2 = n/2 - 2$ neighbors of x_k, there will be an overlap. Therefore, the situation where there are two adjacent vertices x_{i-1} and x_i on P must occur, where x_i is a neighbor of x_0 and $x_{i\text{-}1}$ is a neighbor of x_k.

Therefore, we can drawing a graph below and we can get a cycle x_0, ..., x_{i-1}, x_k, ...,x_i, x_0

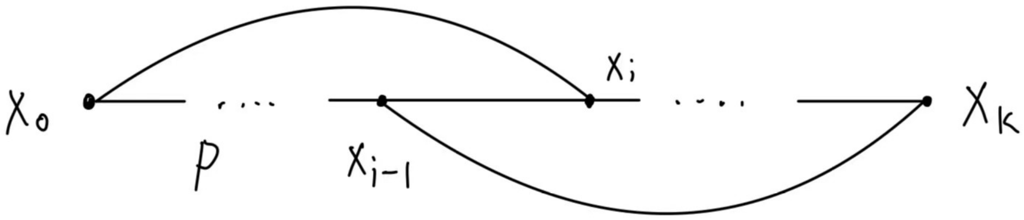

图 1: P must be a cycle

And by contradiction, if there is a longer cycle than x_0, ..., x_{i-1}, x_k, ...,x_i, x_0, it must have a longer path P' (delete one edge from that cycle) than P, which is a contradiction. So we can say, x_0, ..., x_{i-1}, x_k, ...,x_i, x_0 is the longest cycle in G.

IV、 P Contains All Vertices

Since we have previously proved that G is a connected graph, all vertices are connected. If the aforementioned cycle C has a neighbor x such that extending from x forms a path longer than C, this would contradict our assumption that P is the longest path. Therefore, the cycle P has no neighbors. If there is a vertex not in the cycle C, by the connectivity of the graph, this vertex must be connected to some vertex in P. This contradicts the result that C has no neighbors. Hence, P contains all vertices of the graph G.

V、 Conclusion

In summary, we have found a cycle that contains all the vertices of G. By definition, G is a Hamiltonian graph.