The Bipartite Graph Recognition Theorem

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I、 Theorm Content

A graph G is a bipartite graph if and only if G contains no odd cycles.

II、 Necessity Proof

Using a constructive proof, for each connected component of the graph G:
Start with an arbitrary vertex v and color it red. Then, color all vertices adjacent to v blue. Next, color all vertices adjacent to these blue vertices red, and continue this alternating process until all vertices are colored. If, during this process, we find a vertex that needs to be colored twice with different colors, it indicates the existence of a cycle, and this cycle must be of odd length (because alternating colors will cause a conflict). However, by assumption, the graph G contains no odd cycles, so this situation will not occur.
Therefore, after coloring in this manner, the vertices of the graph G can be divided into two groups such that vertices in each group are only adjacent to vertices in the other group. This is exactly the definition of a bipartite graph.
Since each connected component of G is a bipartite graph, G is also a bipartite graph.

III、 Sufficiency Proof

Proof by contradiction method:
Assume graph G is a bipartite graph and G contains an odd cycle.
We can know graph G can be divided into two disjoint vertex sets V_1 and V_2, such that vertices in V_1 are only adjacent to vertices in V_2, and vice versa.
Suppose there exists an odd cycle C with vertices in the sequence $v_1, v_2, v_3, \ldots, v_{2k+1}$, where k is a positive integer. According to the definition of a bipartite graph, any two adjacent vertices must belong to V_1 and V_2 respectively.
We can assume $v_1 \in V_1$, so $v_2 \in V_2$, then $v_3 \in V_1$, and so on. Since the cycle's length is odd, when we traverse the entire cycle back to v_1, the vertex v_{2k+1} should also belong to V_1. However, v_{2k+1} is adjacent to v_1, and according to the definition of a bipartite graph, they should belong to different vertex sets. This creates a contradiction.
Therefore, a bipartite graph G does not contain odd cycles.