# A lemma about trees

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This report contains the proof of a lemma about trees.

## 1 Definitions

Let us start with some definitions which will be useful. These definitions are all from (or based on) the lecture notes [1].

**Definition 1.1 (Graph)** *A graph $G$ consists in a pair $G = (V, E)$ of two sets together with a map $i : E \to V \times V$ assigning to every $e \in E$ a pair of elements of $V$. Elements of $V$ are called vertices, elements of $E$ are called edges. If $i(e) = (x, y)$, the vertices $x$ and $y$ are also called the endpoints of $e$.*

**Definition 1.2 (Walk)** *A walk of length $N$ on a graph $G = (V, E)$ is an alternating sequence*

$$W = (x_0, e_1, x_1, e_2, ..., x_{N-1}, e_N, y)$$

*with the requirement that $i(e_j) = (x_{j-1}, x_j)$ for $j \in \{1, ..., N\}$.*

**Definition 1.3 (Path)** *A path is a walk with no repeated edges and vertices, except possibly for the endpoints $x_0$ and $x_N$.*

**Definition 1.4 (Connected)** *An undirected graph $G = (V, E)$ is connected if for any $x, y \in V$ there exists a path between $x$ and $y$.*

**Definition 1.5 (Cycle)** *A cycle is a non-trivial closed path.*

**Definition 1.6 (Tree)** *A tree is a connected graph whose underlying undirected graph has no cycle.*

**Definition 1.7 (Component)** *A component of a graph $G$ is a maximal connected subgraph of $G$.*

**Definition 1.8 (Cut-edges)** *Let $G = (V, E)$ be a graph. An edge $e$ is a cut-edge if by removing this edge from $G$ the number of components of $G$ increases.*

## 2 The lemma and its proof

**Lemma 2.1** *Let $G = (V, E)$ be an undirected tree with $n$ vertices. Then, the following statements are equivalent.*

1. *$G$ is a tree.*
2. *$G$ is acyclic and has $n - 1$ edges.*
3. *$G$ is connected and has $n - 1$ edges.*
4. *$G$ is connected and all edges are cut-edges.*
5. *$\forall x, y \in V$, $\exists!1$ path from $x$ to $y$.*

6. *G has no cycle and any new edge will create a cycle.*

For the proof I will first prove that some of the statements imply some of the other statements.

- $\underline{5 \Rightarrow 6}$ : I will use indirect proof. Firstly, let us assume that $G$ has a cycle. Let $x, y \in V$ be two points of this cycle. This means that there exists two paths between $x$ and $y$, which contradicts the 5th statement. Hence, if the 5th statement holds, $G$ has no cycle.
Next, let us use that $G$ has no cycle and assume that $\exists x, y \in V$ such that connecting these two points does not create a cycle (let us denote the edge resulting from connecting $x$ and $y$ as $e$). However, the 5th statement says that there already existed exactly one path $P = (x, e_1, ..., e_k, y)$, where $e_1, ..., e_k \in E$, between $x$ and $y$, which means that $P \cup (y, e, x)$ is a cycle. But this is a contradiction, therefore any new edge will create a cycle.

- $\underline{6 \Rightarrow 5}$ : Let us assume that $\exists x, y \in V$ such that there are at least two paths between these two vertecies. But then the union of two of these paths would form a cycle, so $G$ would have a cycle. This contradicts the fact that $G$ has no cycle (6th statement); hence, there is at most one path between any two vertecies of $G$.
Next, let us assume that there is no path connecting two verecies $x$ and $y$. But then connecting these two vertecies would not create a cycle, which contradicts the 6th statement. Therefore, there is at least one path between any two verticies of $G$.
The previous two statements together imply the 5th statement.

- $\underline{5 \Rightarrow 1}$ : Since $\forall x, y \in V$ there is a path connecting $x$ and $y$, the graph is connected. Also, in the $5 \Rightarrow 6$ part we already saw that the 5th statement implies that $G$ has no cycle. Hence, $G$ is a tree.

- $\underline{1 \Rightarrow 5}$ : The fact that $G$ is connected (by the definition of trees) implies that $\forall x, y \in V$, there exists at least one path from $x$ to $y$ (by the definition of connected graphs). Also, we saw in the first half of the $6 \Rightarrow 5$ section that no cycle $\Rightarrow$ at most one path between any two vertices. Hence, there is exactly one path between any two vertices.

- $\underline{5 \Rightarrow 4}$ : We have seen in the previous two parts that $G$ is connected $\iff$ there is at least one path connecting any two vertices of $G$.
Next, let $x, y \in V$ be connected by $e \in E$ and let us assume that there is at most one path between any two vertices. Then, if we remove $e$, then there will be no path between $x$ and $y$. This means that if we consider two subgraphs, one containing only $x$ and the other containing only $y$, then these two subgraphs were connected before removing $e$ and are not connected after $e$ is removed. This means that the removal of $e$ increased the number of components of $G$, so $e$ is a cut-edge. Therefore, if there is at most one path between any two vertices, then all edges are cut-edges.

- $\underline{4 \Rightarrow 5}$ : We have seen that $G$ is connected $\iff$ there is at least one path connecting any two vertices of $G$.
Next, let us assume that $G$ has a cycle. Let $x, y \in V$ be two vertices of this cycle connected by $e$. If we remove $e$, it is still possible to go from $x$ to $y$ (through the remaining part of the cycle), so $e$ is not a cut-edge. But this contradicts the 4th statements, so $G$ has no cycle. Also, we have seen that $G$ has no cycle $\Rightarrow$ there is at most one path between any two vertices of $G$. Hence, the 4th statement implies the 5th.

- $\underline{1 \iff 3}$ : Let $G = (V, E)$ be a connected graph. If $G$ has a cycle, then we can remove any one edge of the cycle and the graph is still connected; therefore, $|E|$ is minimal if $G$ has no cycle. Next, let us prove that if the number of edges in a connected graph with $n$ vertices is minimal, then it has $n-1$ edges and no cycles. To do this mathematical induction will be used. Firstly, when $n = 1$ the graph is connected even when it has no edges, so the minimal number of edges is 0 and obviously it has no cycles. Next, let us assume that a connected graph with $k$ edges has to have a minimum of $k-1$ edges and that it has no cycles when it has $k-1$ edges, and add a new vertex $x$ to this graph. Since currently there is no edges to $x$, $x$ is isolated, so we need at least one edge starting from $x$ so that it is connected to the rest of the graph. However, since the rest of the $k+1$ vertices were already connected, if $x$ is connected to any other vertices, the graph is again connected. Also, one can observe that after adding this edge no cycle is created: $x$ did not become a part of a cycle (because only one edge was added) and no cycles were created among the original $k$ vertices (since one of the endpoints of the new edge is $x$). Hence, the graph with $k+1$ vertices needs at least $k-1+1=k$ edges so that it is connected and it is acyclic when it has only $k$ edges. Hence, if the number of edges in a connected graph with $n$ vertices is minimal, then it has $n-1$ edges and no cycles. Finally, since we have just shown that a connected graph has $n-1$ (the minimum number of) edges $\iff$ it is acyclic, $1 \iff 3$.

- $\underline{2 \iff 3}$ : We have just shown that if a connected graph has $n-1$ edges then it is acyclic. Next, let us prove that if the number of edges is maximal in an acyclic graph, then it is connected. Firstly, let us observe that the fact that the number of edges is maximal means that any new edge creates a cycle. Then, since $6 \Rightarrow 5$ and $5 \Rightarrow 1$, $6 \Rightarrow 1$, which means that if any new edge will create a cycle in an acyclic graph, then the graph is a tree (and therefore it is connected). From these one can infer that if the number of edges is maximal in an acyclic graph, then it is connected.
  Next, let us prove that the maximum number of edges in an acyclic graph with $n$ vertices is $n-1$. For this I will use mathematical induction. Firstly, when $n=1$, then any edge is a loop, which is also a cycle, so the maximum number of edges is indeed $n-1=0$. Next, let us assume that an acyclic graph with $k$ vertices has a maximum of $k-1$ edges, and add a new vertex $x$. Firstly, one can observe that if an edge is added connecting $x$ to any one of the original $k$ vertices no cycle is created. However, the original $k$ vertices were connected, and with the addition of this one edge $x$ is connected to the rest of the graph. This means that this graph with $k+1$ vertices is connected, so it is impossible to add a new edge without creating a cycle. Also, we added only one edge, so the number of edges is $k-1+1=k$. Hence, the maximal number of edges in an acyclic graph with $n$ vertices is $n-1$. Finally, since the maximal number of edges in an acyclic graph with $n$ edges is $n-1$ and an acyclic graph with the maximal number of edges is connected, we can conclude that if the number of edges in an acyclic graph with $n$ vertices is $n-1$, then it is connected.

Finally, consider the graph in Figure 1. In this graph the vertices represent the 6 statements, and two vertices are connected if we have shown that they are equivalent. We know that if $a \iff b$ and $b \iff c$ then $a \iff c$, so if there exists a path between two statements, they are equivalent. Therefore, the lemma is proven $\iff$ the graph is connected. It can be observed that the graph is indeed connected, which completes the proof of the lemma. ∎

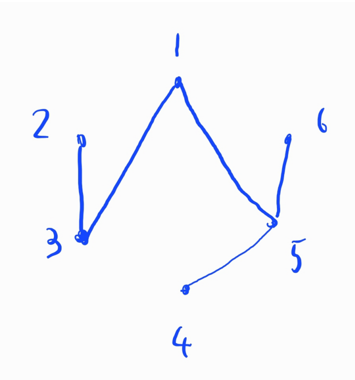


Figure 1: A graph representing the 6 statements. Two vertices are connected if we have proven that the corresponding statements are equivalent.

## 3 References

1. Lecture notes