Proof of 2-connectivity using disjoint paths

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Theorem Statement

A connected graph G with at least 3 vertices is 2-connected if and only if, for every pair of vertices x and y in G, there exist two internally disjoint paths between x and y.

Proof

Assumption 1: G is 2-connected

Definition 5.2

A graph is 2-connected if it remains connected upon the removal of any single vertex. This implies $k(G) \geq 2$, where $k(G)$ is the vertex connectivity of G.

Proof

Given G is 2-connected. Hence, for any vertex v not equal to x or y, the graph $G - \{v\}$ remains connected. Therefore we need to prove that for any two vertices x and y in G, there exist two internally disjoint paths between them.

Since G remains connected upon the removal of any single vertex, there must be a path from x to y that does not pass through any particular vertex $v \neq x, y$.

Additionally, because G is 2-connected, no single vertex w can exist such that all x-y paths pass through w, as this would contradict the definition of 2-connectivity, since the removal of w would disconnect G.

Consider two cases:

1. **Vertex Removal from a Path:** Suppose a vertex v is part of one x-y path but absent in another, and no other vertices except x and y are shared between the paths (since G is 2-connected). By removing v, the graph remains connected due to the second path. This scenario suggests the existence of two paths that do not share vertex v.

2. **General Connectivity:** Since G stays connected regardless of which vertex is removed, there exist alternative paths that bypass any specific

vertex. This guarantees that there can be two paths from x to y that do not share any vertex other than x and y themselves.

Therefore, if G is 2-connected, for any two vertices x and y, it can be shown that there exist two internally disjoint paths in G by leveraging Menger's Theorem, which states that the maximum number of internally disjoint x-y paths is equal to the minimum number of vertices separating x and y. Since G is 2-connected, there are at least 2 such paths.

Assumption 2: For every pair of vertices x and y in G, there exist two internally disjoint paths.

From our assumption we can say that for every pair of vertices x and y in G, there exist two internally disjoint paths.

Proof

Assume for the sake of contradiction that G is not 2-connected. Then there exists some vertex v whose removal disconnects G. This disconnection implies that at least for one pair of vertices x and y, all paths from x to y must pass through v, contradicting our assumption that there exist two internally disjoint paths that do not share any internal vertices including v.

Thus, if, for every pair of vertices x and y in G, there exist two internally disjoint paths, G must be 2-connected.

Conclusion

This proof establishes that a graph G is 2-connected if and only if every pair of vertices x and y has two internally disjoint paths.

Examples of 2-Connected Graphs

Cycle Graph C_4 Complete Graph K_4

Figure 1: Examples of 2-connected graphs