Report 4: Stochastic Calculus Exercises

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Exercise 4.4.7 (The Itô exponential). Consider the function $f(t,x) = e^{x-\frac{1}{2}t}$ and show that

$$\int_0^t f(s, B_s) dBs = f(t, B_t) - f(0,0).$$

Observe that for the usual exponential function one has $\int_0^t e^s ds = e^t - e^0$, and for this reason the function f is sometimes called the Itô exponential.

Proof. We have

$$\partial_t f(t,x) = -\frac{1}{2} e^{x-\frac{1}{2}t} \text{ and } f(x,t) = \partial_x f(x,t) = \partial_x^2 f(t,x) = e^{x-\frac{1}{2}t}.$$

Therefore, $\partial_t f(t,x) + \frac{1}{2}\partial_x^2 f(t,x) = 0$ and from Remark 4.4.5 from the lecture notes, we conclude that

$$\int_0^t \partial_x f(s, Bs) dBs = \int_0^t f(s, Bs) dBs = f(t, B_t) - f(0,0).$$

□

Exercise 4.4.9 (Rephrased). Study and report on the gambler's ruin for Brownian motion with drift, as presented in the Section 5.5 of [1].

Gambler's Ruin for Brownian Motion with Drift

This section summarizes and elaborates from [1].

Let $(\Omega, \mathcal{F}, \mathbb{P}, (\mathcal{F}_t)_{t\in\mathbb{Z}_{\geq 0}}, (B_t)_{t\in\mathbb{Z}_{\geq 0}})$ be a one dimensional standard Brownian motion and $\sigma \neq 0, \mu > 0$. We will consider the Brownian motion with drift given by

$$X_t = \sigma B_t + \mu t$$

for all $t \geq 0$.

Let $a, b > 0$. The first passage time of level a or $-b$ is given by

$$\tau = \min\{t \geq 0 : X_t > a \text{ or } X_t < -b\}.$$

First, we will show that $\tau < \infty$ with probability one. Consider the event

$$E_n = \{|X_n - X_{n-1}| > a + b\} = \{|\sigma(B_n - B_{n-1}) - \mu| > a + b\}$$

for $n \geq 1$.

Because $B_n - B_{n-1} = N(0,1)$ from the definition of Brownian motion, it follows that

$$\mathbb{P}(E_n) = \mathbb{P}(E_1)$$

for all $n \geq 1$.

We will show that $X_r - X_{r-1}$ and $X_s - X_{s-1}$ are independent for any $r > s$. We will consider the covariance matrix for $(X_r - X_{r-1}, X_s - X_{s-1})$ for some $r > s$. We have

$$\begin{aligned}
\mathbb{E}[(X_r - X_{r-1})(X_s - X_{s-1})] &= \mathbb{E}[(\sigma(B_r - B_{r-1}) - \mu)(\sigma(B_s - B_{s-1}) - \mu)] \\
&= \sigma^2 \mathbb{E}[(B_r - B_{r-1})(B_s - B_{s-1})] + \mu(\mathbb{E}[B_{r-1}] + \mathbb{E}[B_{s-1}] - \mathbb{E}[B_r] - \mathbb{E}[B_s]) + \mu^2 \\
&= \sigma^2 \mathbb{E}[(B_r - B_{r-1})(B_s - B_{s-1})] + \mu^2 \\
&= \sigma^2 (\mathbb{E}[B_r B_s] - \mathbb{E}[B_r B_{s-1}] - \mathbb{E}[B_{r-1} B_s] + \mathbb{E}[B_{r-1} B_{s-1}]) + \mu^2 \\
&= \sigma^2 (s - (s-1) - s + (s-1)) + \mu^2 \\
&= \mu^2.
\end{aligned}$$

Therefore, the covariance matrix corresponds to the tuple is diagonal. By Proposition 2.10 in [1], $(X_r - X_{r-1})$ and $(X_s - X_{s-1})$ are independent. Therefore, for $n \geq 1$,

$$\mathbb{P}(\Omega \setminus E_1, \ldots, \Omega \setminus E_n) = \mathbb{P}(\Omega \setminus E_1) \cdots \mathbb{P}(\Omega \setminus E_n) = \mathbb{P}(\Omega \setminus E_1)^n.$$

Because $B_1 - B_0 = N(0,1)$, the image of $X_1 - X_0$ is the entire $\mathbb{R}$. Therefore, $0 < \mathbb{P}(E_1) < 1$. Hence,

$$\lim_{n \to \infty} \mathbb{P}(\Omega \setminus E_1, \ldots, \Omega \setminus E_n) = 0.$$

Let $(F_n)_{n \geq 1} = \big((\Omega \setminus E_1) \cap \cdots \cap (\Omega \setminus E_n)\big)$. Clearly, $(F_n)_{n \geq 1}$ gives a non-increasing sequence.

Therefore, as has been shown in Lemma 1.1.7 of the lecture note,

$$\mathbb{P}\left(\bigcap_{n \geq 1} F_n\right) = 0.$$

Therefore, with probability one, there will be an increment of size more than $a + b$ in the Brownian motion with drift. Hence, the Brownian motion with drift will reach somewhere outside the range $[-b, a]$, or in other words, $\tau < \infty$.

We will compute $\mathbb{P}(X_\tau = a)$ by using Doob's optional stopping theorem. First, we need to find a good martingale for the Brownian filtration. Suppose that the martingale M_t is a function of X_t when $t \geq 0$,

$$M_t = g(X_t)$$

for some function $g : \mathbb{R} \to \mathbb{R}$. For convenience, we want to find g with the property that $g(a) = 1$ and $g(-b) = 0$.

Let $f : \mathbb{Z}_{\geq 0} \times \mathbb{R} \to \mathbb{R}$ be such that

$$f(t, x) = g(\sigma x + \mu t).$$

Then, $g(X_t) = f(t, B_t)$. By the chain rule,

$$\partial_t f(t, x) = \mu g'(\sigma x + \mu t), \partial_x f(t, x) = \sigma g'(\sigma x + \mu t), \partial_x^2 f(t, x) = \sigma^2 g''(\sigma x + \mu t).$$

We can find f that satisfies $\frac{\partial f}{\partial t} + \frac{1}{2} \frac{\partial^2 f}{\partial x} = 0$ by solving the differential equation

$$\mu g' = \frac{-\sigma^2}{2} g''.$$

The solution to above is given by $g(y) = C e^{-2\mu y/\sigma^2} + C'$ for two constants C and C'. Because we set the condition that $g(a) = 1$ and $g(-b) = 0$,

$$1 = g(a) = C e^{-2\mu a/\sigma^2} + C' \text{ and } 0 = g(-b) = C e^{2\mu b/\sigma^2} + C'$$

which gives a system of linear equations which we can solve for C and C'. That gives

$$C = \frac{-e^{-2\mu b/\sigma^2}}{1 - e^{-2\mu(a+b)/\sigma^2}} \text{ and } C' = \frac{1}{1 - e^{-2\mu(a+b)/\sigma^2}}$$

which gives

$$g(y) = \frac{1 - e^{-2\mu(y+b)/\sigma^2}}{1 - e^{-2\mu(a+b)/\sigma^2}}.$$

The process described by $(g(X_t))_{t\geq 0}$ belongs to $M^2([0,T])$ because the map

$$t \mapsto \mathbb{E}[|g(X_t)|^p] = \int_0^T |g(x)|^p \Pi_t(x) dx < \infty$$

with $\Pi(x)$ be the density of X_t which is still a normal distribution, is bounded because both $g(x)$ and $\Pi_t(x)$ are continuous and bounded on the interval $[0,T]$. Therefore, by Remark 4.4.5 from the lecture notes, we have

$$f(t, B_t) + f(0,0)$$

is a martingale for the Brownian filtration. Therefore, $f(t, B_t)$ gives a martingale for the Brownian filtration.

By Theorem 3.2.14 in the lecture notes, we have

$$\mathbb{E}[g(X_\tau)|\mathcal{F}_0] = \mathbb{E}[g(0)] = g(0).$$

Moreover, because $B_\tau = B_\tau - B_0$ by definition is independent of $\mathcal{F}_0$, it follows immediately that $X_\tau = X_\tau - X_0$ is also independent of $\mathcal{F}_0$ by how we define X_τ. Therefore,

$$\mathbb{E}[g(X_\tau)] = \mathbb{E}[g(X_\tau)|\mathcal{F}_0] = g(0).$$

Finally, by the convenient choice of $g(a)$ and $g(b)$, we have

$$\mathbb{E}[g(X_\tau)] = g(a)\mathbb{P}(X_\tau = a) + g(-b)\mathbb{P}(X_\tau = -b) = \mathbb{P}(X_\tau = a).$$

Therefore,

$$\mathbb{P}(X_\tau = a) = g(0) = \frac{1 - e^{-2\mu b/\sigma^2}}{1 - e^{-2\mu(a+b)/\sigma^2}}.$$

Exercise 5.2.10 (Rephrased). Study and report on the general solution (5.2.10), getting inspiration from Section 5.3 of [2].

Below is summarized and elaborated from [2].

We will find a solution to a general linear SDE in one dimension

$$dX_t = (\alpha_t + \beta_t X_t)dt + (\gamma_t + \delta_t X_t)dB_t \tag{1}$$

where $\alpha, \beta, \gamma, \delta$ are given adapted processes and are continuous functions of t.

First, we will find solutions for when $\alpha(t) = \gamma(t) = 0$. Then, we have

$$dU_t = \beta_t U_t dt + \delta_t U_t dB_t = U_t\big(\beta_t dt + \delta_t dB_t\big). \tag{2}$$

We define Y_t for the Itô process defined by the expression in the parentheses above,

$$dY_t = \beta_t dt + \delta_t dB_t.$$

Therefore, by Lemma 5.2.9 from the lecture notes, the equation (2) implies

$$\begin{aligned} U_t &= U_0 e^{Y_t - Y_0 - \frac{1}{2}[Y]_t} \\ &= U_0 e^{\int_0^t dY_s - \frac{1}{2}\int_0^t \delta_s^2 ds} && \text{((4.42) from [2])} \\ &= U_0 e^{\int_0^t (\beta_s - \frac{1}{2}\delta_s^2)ds + \int_0^t \delta_s dB_s}. \end{aligned}$$

We will now find a solution to (1) of the form

$$X_t = U_t V_t \tag{3}$$

where

$$dU_t = U_t(\beta_t dt + \delta_t dB_t)$$

and

$$dV_t = a_t dt + b_t dB_t.$$

We then compute dX_t by using Lemma 5.1.5 from the lecture notes, as mentioned in [3],

$$dX_t = d(U_t V_t) = U_t dV_t + dU_t V_t + \delta_t U_t b_t dt. \tag{4}$$

Hence, it is immediate that a and b that satisfies

$$a_t U_t = \alpha_t - \delta_t \gamma_t \text{ and } b_t U_t = \gamma_t$$

satisfies (1) and (4).

Therefore, by integrating $dV(t)$, and using (3) we have

$$X_t = U_t \left(X_0 + \int_0^t \frac{\alpha_s - \delta_s \gamma_s}{U_s} ds + \int_0^t \frac{\gamma_s}{U_s} dB_s \right).$$

Exercise 6.3.6 (Rephrased). Give a probabilistic representation of the solution of the equation

$$\partial_t f + \frac{1}{2}\sigma^2 x^2 \partial_x^2 f + \mu x \partial_x f = rf, \quad f(T, y) = y^2$$

for $r, \sigma, \mu > 0$, see also Example 6.5 of [2].

The following summarizes and elaborates Example 6.5 of [2].

The SDE corresponding to the above PDE is given by

$$dX_t = \mu X_t dt + \sigma X_t dB_t.$$

By using Exercise 5.2.10, the solution is given by

$$X_t = X_0 e^{\int_0^t \left(\mu - \frac{\sigma^2}{2}\right) ds + \int_0^t \sigma dB_s} = X_0 e^{\left(\mu - \frac{\sigma^2}{2}\right)t + \sigma B_t}$$

which implies

$$X_T = X_t e^{\left(\mu - \frac{\sigma^2}{2}\right)(T-t) + \sigma(B_T - B_t)}.$$

Then, by the Feynmann-Kac formula,

$$\begin{aligned}
f(x,t) &= \mathbb{E}\left[e^{-r(T-t)} X_T^2 \mid X_t = x\right] \\
&= e^{-r(T-t)} \mathbb{E}\left[X_T^2 \mid X_t = x\right] \\
&= x^2 e^{(2\mu - \sigma^2 - r)(T-t)} \mathbb{E}\left[e^{2\sigma(B_T - B_t)}\right] \\
&= x^2 e^{(2\mu - \sigma^2 - r)(T-t)} e^{2\sigma^2(T-t)} \\
&= x^2 e^{(2\mu + \sigma^2 - r)(T-t)}.
\end{aligned}$$

The second to last equation is due to Proposition 2.1.4 from the lecture notes and the fact that $B_T - B_t = N(0, T-t)$.

References

[1] J.L.Arguin, *A first course in stochastic calculus.* Pure and Applied Undergraduate texts 53, American Mathematical Society, 2022.

[2] F.Klebaner, *Introduction to stochastic calculus, with applications*, third edition, Imperial College Press, 2012.

[3] T.Mikosch, *Elementary stochastic calculus with finance in view*, Advances Series on Statistical Science and Applied Probability 6, World Scientific, 1998.