Brownian Process (Stochastic Calculus)

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Exercise 2.1.2 (General version). Check that if $X_1, \dots, X_N$ are independent and standard Gaussian random variables, then $(X_1, \dots, X_N)^T$ is a Gaussian vector. Show that the random variable $a_1 X_1 + \dots a_N X_N$ is a Gaussian random variable with mean 0 and variance $a_1^2 + \cdots + a_N^2$.

Proof. This proof is summarized from [1] (and double checked the characteristic function formula with Wikipedia), which I found to be very enlightening without complicated integration calculations.

Let $X : \Omega \to \mathbb{R}^m$ be an m-dimensional random variable. The characteristic function of μ_X is defined for $\theta \in \mathbb{R}^m$ as

$$\hat{\mu}_X(\theta) = \int_{\mathbb{R}^m} e^{i\langle \theta, x \rangle} \mu_X(dx) = \mathbb{E}\left[e^{i\langle \theta, X \rangle}\right].$$

For $m = 1$, let X be a Gaussian random variable with mean $\bar{x}$ and variance σ^2, then by LOTUS, for $\theta \in \mathbb{R}$,

$$\hat{\mu}_X(\theta) = \frac{1}{\sqrt{2\pi}\sigma} \int_{-\infty}^{\infty} e^{i\theta x} e^{-\frac{(x-\bar{x})^2}{2\sigma^2}} dx = \frac{1}{\sqrt{2\pi}\sigma} \int_{-\infty}^{\infty} e^{i\theta(y+\bar{x})} e^{\frac{-y^2}{2\sigma^2}} dy.$$

Let u be such that

$$u(\theta) = \frac{1}{\sqrt{2\pi}\sigma} \int_{-\infty}^{\infty} e^{i\theta x} e^{\frac{-x^2}{2\sigma^2}} dx.$$

By Leibniz integral rule and integration by parts, we have

$$\begin{aligned}
u'(\theta) &= \frac{1}{\sqrt{2\pi}\sigma} \int_{-\infty}^{\infty} ix e^{i\theta x} e^{-\frac{x^2}{2\sigma^2}} dx \\
&= \frac{1}{\sqrt{2\pi}\sigma} \left[-i\sigma^2 e^{i\theta x} \left(e^{-\frac{x^2}{2\sigma^2}}\right)\right]_{-\infty}^{\infty} - \frac{\sigma^2\theta}{\sqrt{2\pi}\sigma} \int_{-\infty}^{\infty} e^{i\theta x} e^{-\frac{x^2}{2\sigma^2}} dx \\
&= -\sigma^2 \theta u(\theta)
\end{aligned}$$

which is a first-order differential equation. Because $u(0) = 1$, we have $u(\theta) = e^{-\frac{1}{2}\sigma^2\theta^2}$. Therefore,

$$\hat{\mu}_X(\theta) = e^{i\theta\bar{x}} e^{-\frac{1}{2}\sigma^2\theta^2}.$$

Suppose that $Y_1, \dots, Y_N$ are independent Gaussian random variables each with means $y_1, \dots, y_N$ and non-zero variances $\sigma_1^2, \dots, \sigma_N^2$. For $\theta \in \mathbb{R}^N$,

$$\begin{aligned}
\hat{\mu}_{Y_1,\dots,Y_N}(\theta) &= \int_{\mathbb{R}^N} e^{i\langle \theta, x \rangle} \mu_{Y_1,\dots,Y_N}(dx) \\
&= \int_{\mathbb{R}} e^{i\theta_1 x_1} \mu_{Y_1}(dx_1) \cdots \int_{\mathbb{R}} e^{i\theta_N x_N} \mu_{Y_N}(dx_N) \\
&= \hat{\mu}_{Y_1}(\theta_1) \cdots \hat{\mu}_{Y_N}(\theta_N) \\
&= e^{i\langle \theta, y \rangle} e^{-\frac{1}{2}\sigma_1^2\theta_1^2} \cdots e^{-\frac{1}{2}\sigma_N^2\theta_N^2}.
\end{aligned}$$

So, for $\theta \in \mathbb{R}$,

$$\hat{\mu}_{a_1Y_1+\cdots+a_NY_N}(\theta) = \mathbb{E}\left[e^{i\langle \theta, a_1Y_1+\cdots+a_NY_N \rangle}\right] = \mathbb{E}\left[e^{i\langle (a_1,\dots,a_N)^T\theta, Y \rangle}\right] = \hat{\mu}_{Y_1,\dots,Y_N}(a_1\theta, \dots, a_N\theta) = e^{i\theta(y_1+\cdots+y_n)} e^{-\frac{(a_1^2+\cdots+a_N^2)\theta^2}{2}}.$$

The above computation is more general than what is asked in this exercise, because we will use them for later exercises.

Then, we will use a fact about characteristic functions that we will take from [1] without proof: if $\hat{u} = \hat{v}$, then $u = v$. And because the above characteristic function is a Gaussian distribution's characteristic function, we conclude that $a_1X_1 + \cdots + a_NX_N$ is a Gaussian random variable with mean 0, because $y_j = 0$ for all j, and variance $a_1^2 + \cdots + a_N^2$. □

Lemma 2.1.3. If X is a N-dimensional Gaussian vector and if $M \in M_{N\times N}(\mathbb{R})$, check that the new vector MX is also a N-dimensional Gaussian vector.

Proof. Each row of MX will be a linear combination of $X_1, \ldots, X_N$, which by definition of Gaussian vector should be a Gaussian random variable. And because each element of MX is a Gaussian random variable, any linear combinations of them will be another Gaussian random variable due to the computation in Exercise 2.1.2. Therefore, MX is a Gaussian vector. □

Exercise 2.4.2 (Rephrased). Prove the following statement: The random variables X^1, X^2 on $(\Omega, \mathcal{F}, \mathbb{P})$ are independent if and only if the σ-algebras $\sigma(X^1), \sigma(X^2)$ they generate are independent, and a random variable X on $(\Omega, \mathcal{F}, \mathbb{P})$ is independent of a σ-subalgebra $\mathcal{G}$ of $\mathcal{F}$ if and only X is independent of every $\mathcal{G}$-measurable random variables.

Proof. We will suppose that X^1, X^2 are two random variables on $(\Omega, \mathcal{F}, \mathbb{P})$ with X^k takes values on $(\Lambda_k, \varepsilon_k)$.

Suppose that X^1 and X^2 are independent. Let $A^1 \in \sigma(X^1), A^2 \in \sigma(X^2)$. Then, by definition, there are $B^1 \in \varepsilon_1, B^2 \in \varepsilon_2$ such that $X^k(A^k) = B^k$. Hence,

$$\mathbb{P}(A^1, A^2) = \mathbb{P}(X^1 \in B^1, X^2 \in B^2) = \mathbb{P}(X^1 \in B^1)\mathbb{P}(X^2 \in B^2) = \mathbb{P}(A^1)\mathbb{P}(A^2).$$

Therefore, $\sigma(X^1)$ and $\sigma(X^2)$ are independent.

Now, suppose that $\sigma(X^1)$ and $\sigma(X^2)$ are independent. Let $B^1 \in \varepsilon_1$ and $B^2 \in \varepsilon_2$. Then, $A^1 = (X^1)^{-1}(B^1) \in \sigma(X^1)$ and $A^2 = (X^2)^{-1}(B^2) \in \sigma(X^2)$. Therefore,

$$\mathbb{P}(X^1 \in B^1, X^2 \in B^2) = \mathbb{P}(A^1, A^2) = \mathbb{P}(A^2)\mathbb{P}(A^2) = \mathbb{P}(X^1 \in B^1)\mathbb{P}(X^2 \in B^2).$$

Now, with the same setup, suppose that X^2 is measurable on a σ-subalgebra $\mathcal{G} \subset \mathcal{F}$.

Suppose that X^1 is independent of $\mathcal{G}$. Let $A^1 \in \varepsilon_1, A^2 \in \varepsilon_2$. By definition, there must be $B^1 \in \sigma(X^1)$ and $B^2 \in \mathcal{G}$ such that $(X^k)^{-1}(B^k) = A^k$. Therefore,

$$\mathbb{P}(X^1 \in A^1, X^2 \in A^2) = \mathbb{P}(B^1, B^2) = \mathbb{P}(B^1)\mathbb{P}(B^2) = \mathbb{P}(X^1 \in A^1)\mathbb{P}(X^2 \in A^2).$$

Finally, suppose that X^1 is independent of every $\mathcal{G}$-measurable random variables. Let $A \in \sigma(X^1)$ and $B \in \mathcal{G}$. Let Y be a $\mathcal{G}$-measurable random variable on $(\Omega, \mathcal{F}, \mathbb{P})$ that maps elements of B to 1 and everything else to 0. Then,

$$\mathbb{P}(A, B) = \mathbb{P}\big(X^1 \in X^1(A), Y \in Y(B)\big) = \mathbb{P}\big(X^1 \in X^1(A)\big)\mathbb{P}\big(Y \in Y(B)\big) = \mathbb{P}(A)\mathbb{P}(B).$$

□

Exercise 2.4.3. Show that the Brownian process is a Gaussian process, see also Remarks 3.1 of [1].

Proof. The following proof is an elaboration of Remarks 3.1 of [1].

First, we know that B_0 is a Gaussian random variable with mean 0 and variance 0. For $t > 0$, we also have $B_t = B_t - B_0$ is a Gaussian random variable with mean 0 and variance t.

By induction on n, we will show that for any $a_1, \ldots, a_n \in \mathbb{R}$ and $0 < t_1 < \cdots < t_n$, we have $a_1B_{t_1} + \cdots + a_nB_{t_n}$ is a Gaussian random variable. When $n = 1$, B_{t_1} is a Gaussian random variable. By the computation in Exercise 2.1.2 above, we now that aB_{t_1} gives a Gaussian random variable.

Let $k > 1$. Suppose that for any $b_1, \ldots, b_{k-1}$ and $0 < t_1 < \cdots < t_{k-1}$, we have $b_1B_{t_1} + \cdots + b_{k-1}B_{t_{k-1}}$ is a Gaussian random variable. Let $a_1, \ldots, a_k \in \mathbb{R}$ and $0 < t_1 < \cdots < t_k$. We have,

$$a_1B_{t_1} + \cdots + a_kB_{t_k} = \left(\sum_{i=1}^{k-2} a_iB_{t_i} + (a_{k-1} + a_k)B_{t_{k-1}}\right) + a_k(B_{t_k} - B_{t_{k-1}}).$$

Let A be the summation of the terms inside the parentheses on the right-hand side. From the induction hypothesis, A is a Gaussian random variable. Furthermore, because a Brownian process by definition is also a Stochastic process, A is measurable with respect to $\mathcal{F}_{t_{k-1}}$. However, $B_{t_k} - B_{t_{k-1}}$ is independent of $\mathcal{F}_{t_{k-1}}$. Therefore, A and $B_{t_k} - B_{t_{k-1}}$ are independent random variables. By the computation in exercise 2.1.2, $B_{t_k} - B_{t_{k-1}}$ is a Gaussian random variable. Therefore, $A + a_k(B_{t_k} - B_{t_{k-1}})$ represents a linear combination of two independent Gaussian random variables. And hence, by the computation in Exercise 2.1.2, it is a Gaussian random variable.

Finally, if $0 = t_1 < \cdots < t_n$ and $a_1, \ldots, a_n \in \mathbb{R}$, then

$$a_1 B_{t_1} + \cdots + a_n B_{t_n} = a_2 B_{t_2} + \cdots + a_n B_{t_n}$$

is either $N(0,0)$ or is also a Gaussian random variable as has been shown above. □

Proposition 2.4.6 Let $(\Omega, \mathcal{F}, \mathbb{P}, (\mathcal{F}_t)_{t\in\mathbb{R}_+}, (B_t)_{t\in\mathbb{R}_+})$ be a 1-dimensional Brownian motion. Then,

1. For any $s \geq 0$, $(\Omega, \mathcal{F}, \mathbb{P}, (\mathcal{F}_{t+s})_{t\in\mathbb{R}_+}, (B_{t+s} - B_s)_{t\in\mathbb{R}_+})$ is a 1-dimensional Brownian motion (time shift),
2. $(\Omega, \mathcal{F}, \mathbb{P}, (\mathcal{F}_t)_{t\in\mathbb{R}_+}, (-B_t)_{t\in\mathbb{R}_+})$ is a 1-dimensional Brownian motion (mirror reflection),
3. For any $c > 0$, $(\Omega, \mathcal{F}, \mathbb{P}, (\mathcal{F}_{t/c^2})_{t\in\mathbb{R}_+}, (cB_{t/c^2})_{t\in\mathbb{R}_+})$ is a 1-dimensional Brownian motion (scaling),
4. The random variables defined by $Z_t := tB_{1/t}$ for $t > 0$ and $Z_0 = 0$ define a natural Brownian motion.

Proof. The proof is summarized and elaborated from the Proposition 3.2 of [1].

For the first three properties, when $t = 0$,

$$B_{t+s} - B_s = -B_t = cB_{t/c^2} = 0.$$

For the time shift property, when $t < u$, we have

$$(B_{u+s} - B_s) - (B_{t+s} - B_s) = B_{u+s} - B_{t+s},$$

which is independent of $\mathcal{F}_{t+s}$ and is a Gaussian random variable $N(0, (u+s) - (t+s))$.

For the scaling property, when $t < u$, we have

$$(cB_{u/c^2} - cB_{t/c^2}) = c(B_{u/c^2} - B_{t/c^2}).$$

Of course, $u/c^2 > t/c^2$, so $B_{u/c^2} - B_{t/c^2}$ is independent of $\mathcal{F}_{t/c^2}$. Let $R = B_{u/c^2} - B_{t/c^2}$ and Y be a random variable measurable on $\mathcal{F}_{t/c^2}$. Then,

$$\mathbb{P}(cR \in A, Y \in C) = \mathbb{P}(R \in A/c, Y \in C) = \mathbb{P}(R \in A/c)\mathbb{P}(Y \in C) = \mathbb{P}(cR \in A)\mathbb{P}(Y \in C).$$

Therefore, by Exercise 2.4.2, $c(B_{u/c^2} - B_{t/c^2})$ is also independent of $\mathcal{F}_{t/c^2}$. Finally, $c(B_{u/c^2} - B_{t/c^2})$ is a Gaussian random variable $N(0, (c^2)(u/c^2 - t/c^2)) = N(0, u - t)$. The mirror reflection property is just a special case when $c = -1$.

Now, we will show that Z_t defines a natural Brownian motion by using Proposition 2.4.4. By definition, $Z_0 = 0$. Let $0 \leq t_1 < \cdots < t_N$ and $a_1, \ldots a_N \in \mathbb{R}$. If $t_1 = 0$, then

$$a_1 Z_1 + \cdots + a_N Z_N = a_2 Z_2 + \cdots + a_N Z_N.$$

So, without loss of generality, suppose that $t_1 > 0$. We have,

$$a_1(t_1 B_{1/t_1}) + \cdots + a_N(t_N B_{1/t_N}) = (a_1 t_1)B_{1/t_1} + \cdots + (a_N t_N)B_{1/t_N}$$

which give a Gaussian random variable because it is a linear combination of some elements of $(B_t)_{t\in\mathbb{R}_+}$.

Let e_i be a vector with all zero entries except one at index i. Clearly, from the integral definition, we have

$$\mathbb{E}[e_i t_i B_{1/t_i}] = t_i \mathbb{E}[e_i B_{1/t_i}] = 0.$$

Therefore,

$$\mathbb{E}[(t_1 B_{1/t_1}, ..., t_k B_{1/t_k})^T] = t_1 \mathbb{E}[e_1 B_{1/t_1}] + \cdots + t_k \mathbb{E}[e_k B_{1/t_k}] = 0.$$

Finally, for $s \leq t$,

$$\mathbb{E}[sB_{1/s} tB_{1/t}] = st\mathbb{E}[B_{1/s} B_{1/t}] = \frac{st}{t} = s.$$

□

Exercise 3.1.8 We will summarize and elaborate example 4.5 of [1]: The position of a Brownian motion at a random time.

Let B be an m-dimensional Brownian motion and ζ be a positive random variable ($\zeta > 0$ almost surely) independent of B. We will compute the density and characteristic function of B_ζ.

Let $A \subset \mathbb{R}^m$ be a Borel set. From the lecture note, we know that

$$\mathbb{P}(B_\zeta \in A) = \mathbb{E}[1_A(B_\zeta)].$$

And by equation (3.1.6) in the lecture, we have

$$\mathbb{P}(B_\zeta \in A) = \mathbb{E}[\mathbb{E}[1_A(B_\zeta)|\sigma(\zeta)]].$$

By Exercise 2.4.2, the independence of B and ζ implies the independence of B and $\sigma(\zeta)$.
To put it in the context of the lecture, we will define things a bit differently from [1]. We can view B_t as a function B of t. Let $\Psi(X^1, X^2) = 1_A(X^2_{X^1})$. So, $1_A(B_\zeta) = \Psi(\zeta, B)$. So, we can apply the freezing lemma,

$$\mathbb{P}(B_\zeta \in A) = \mathbb{E}[\mathbb{E}[1_A(B_\zeta)|\sigma(\zeta)]] = \mathbb{E}[\mathbb{E}[\Psi(\zeta, B)|\sigma(\zeta)]] = \mathbb{E}[\mathbb{E}[\Psi(\cdot, B)](\zeta)] = \mathbb{E}[\Phi(\zeta)]$$

where $\Phi(t) = \mathbb{E}[\Psi(\cdot, B)](t)$.
From the definition of Brownian process, B_t is an m-dimensional Gaussian random variable $N(0, t\,I)$. So,

$$\Phi(t) = \mathbb{E}[\Psi(\cdot, B)](t) = \mathbb{E}[1_A(B.)](t) = \mathbb{P}(B_t \in A) = \frac{1}{(2\pi t)^{m/2}} \int_A e^{-\frac{|x|^2}{2t}} dx.$$

By the definition of expectation,

$$\mathbb{P}(B_\zeta \in A) = \mathbb{E}[\Phi(\zeta)] = \int_0^\infty \Phi(t)\mu_\zeta(dt).$$

Then, by Fubini's theorem,

$$\mathbb{P}(B_\zeta \in A) = \int_0^\infty \mu_\zeta(dt) \int_A \frac{1}{(2\pi t)^{m/2}} e^{-\frac{|x|^2}{2t}} dx = \int_A dx \int_0^\infty \frac{1}{(2\pi t)^{m/2}} e^{-\frac{|x|^2}{2t}} \mu_\zeta(dt).$$

Therefore, by definition of $\mathbb{P}(B_\zeta \in A)$, we have the density g of B_ζ be

$$g(x) = \int_0^\infty \frac{1}{(2\pi t)^{m/2}} e^{-\frac{|x|^2}{2t}} \mu_\zeta(dt).$$

We have used characteristic function in Exercise 2.1.2. Here, we will compute the characteristic function $\hat{\mu}_{B_\zeta}$ with the freezing lemma.

Let $\Psi(X^1, X^2) = e^{i\langle\theta, X^2_{X^1}\rangle}$. Hence, by equation (3.1.6) in the lecture,

$$\mathbb{E}[e^{i\langle\theta, B_\zeta\rangle}] = \mathbb{E}[\mathbb{E}[\Psi(\zeta, B)|\sigma(\zeta)]] = \mathbb{E}[\mathbb{E}[\Psi(\cdot, B)](\zeta)] = \mathbb{E}[\Phi(\zeta)]$$

where $\Phi(t) = \mathbb{E}[\Psi(\cdot, B)](t)$.

But this time, we have

$$\Phi(t) = \mathbb{E}[\Psi(t, B)] = \mathbb{E}[e^{i\langle\theta, B_t\rangle}] = e^{-\frac{t|\theta|^2}{2}}.$$

Hence,

$$\hat{\mu}_{B_\zeta}(\theta) = E[e^{i\langle\theta, B_\zeta\rangle}] = \int_0^\infty e^{-\frac{t|\theta|^2}{2}} \mu_\zeta(dt).$$

As an example application, we will do exercise 4.6 from [1]. In the exercise below, I will also use LOTUS to evaluate the integrals by viewing them as expectation. I think it is a very interesting way to look at some Lebesgue integration from statistical point of view.

Exercise 4.6 [1]. Let B be an m-dimensional Brownian motion and ζ an exponential random variable with parameter λ and independent of B.

(a) Compute the characteristic function of B_ζ (the position of the Brownian motion at the random time ζ).

(b1) Let X be a real random variable with a Laplace density with parameter r (replaced from the original μ to avoid ambiguity), i.e., with density

$$f_X(x) = \frac{r}{2}e^{-r|x|}.$$

Compute the characteristic function of X.

(b2) What is the law of B_ζ for $m = 1$?

Solution (partially use the solution in [1] as solution reference).

(a) From exercise 1.2 of [1], a random variable X has exponential law with parameter λ if it has density

$$f(x) = \lambda e^{-\lambda x} 1_{[0,+\infty[}(x).$$

From above, the characteristic function of B_ζ is given by

$$\hat{\mu}_{B_\zeta} = \int_0^\infty e^{-\frac{t|\theta|^2}{2}} \mu_\zeta(dt) = \mathbb{E}\left[e^{-\frac{\zeta|\theta|^2}{2}}\right].$$

By LOTUS, we can calculate above as

$$\hat{\mu}_{B_\zeta}(\theta) = \int_0^\infty e^{-\frac{t|\theta|^2}{2}} f(t)dt = \int_0^\infty \lambda e^{-\frac{t(|\theta|^2+2\lambda)}{2}} dt = \left[-\frac{2\lambda}{|\theta|^2+2\lambda} e^{-\frac{t(|\theta|^2+2\lambda)}{2}}\right]_0^\infty = \frac{2\lambda}{|\theta|^2+2\lambda}.$$

(b) Then, by LOTUS, the characteristic function of f_X is given by

$$\begin{aligned}
\hat{\mu}_X(\theta) &= \int_{-\infty}^\infty e^{i\langle\theta,x\rangle} \frac{r}{2} e^{-r|x|} dx \\
&= \frac{r}{2}\int_{-\infty}^\infty e^{i\theta x - r|x|} dx \\
&= \frac{r}{2}\int_{-\infty}^\infty (ie^{-r|x|}\sin(\theta x) + e^{-r|x|}\cos(\theta x))dx \\
&= r\int_0^\infty e^{-rx}\cos(\theta x)dx \qquad (e^{-r|x|}\sin(\theta x)\text{: odd, } e^{-r|x|}\cos(\theta x)\text{: even}) \\
&= r\Re\left(\int_0^\infty e^{i\theta x - rx}dx\right) \\
&= r\Re\left(\left[\frac{e^{(i\theta - r)x}}{i\theta - r}\right]_0^\infty\right) \\
&= r\Re\left(\left[\frac{e^{-rx}(i\sin(\theta x)+\cos(\theta x))(i\theta + r)}{-\theta^2 - r^2}\right]_0^\infty\right) \\
&= r\Re\left(\left[\frac{-\theta e^{-rx} sin(\theta x) + re^{-rx}\cos(\theta x)}{-\theta^2 - r^2}\right]_0^\infty\right) \\
&= \frac{r^2}{\theta^2 + r^2}.
\end{aligned}$$

(c) From (a), the characteristic function of B_ζ is given by

$$\mu_{B_\zeta}(\theta) = \frac{2\lambda}{\theta^2 + 2\lambda}$$

which coincides with the characteristic function of the random variable given in (b1). Therefore, as also cited from [1] in Exercise 2.1.2, two random variables have the same density functions if and only if their characteristic functions are the same. Therefore, it follows that the density function is also the Laplace density function with parameter $\sqrt{2\lambda}$.

Exercise 3.2.4 Show that the standard 1-dimensional Brownian motion is a martingale.

Proof. Let $(\Omega, \mathcal{F}, \mathbb{P}, (\mathcal{F}_t)_{t\in\mathcal{T}}, (B_t)_{t\in\mathcal{T}})$ be a standard 1-dimensional Brownian motion. As has been shown before, $B_t = B_t - B_0$ is a Gaussian random variable $N(0,t)$. So, $B_t \in L^1(\Omega, \mathcal{F}, \mathbb{P})$ because

$$\mathbb{E}[|B_t|] = \int_{-\infty}^{\infty} \frac{|x|}{\sqrt{2\pi t}} e^{-\frac{x^2}{2t}}\, dx = \sqrt{\frac{2}{\pi t}} \int_0^{\infty} x e^{-\frac{x^2}{2t}}\, dx = \sqrt{\frac{2}{\pi t}} \left[-t e^{\frac{-x^2}{2t}}\right]_0^{\infty} = \sqrt{\frac{2t}{\pi}}.$$

Because $B_t - B_s$ is independent of $\mathcal{F}_s$, by Proposition 3.1.3, when $s \le t$,

$$\mathbb{E}[B_t|\mathcal{F}_s] = \mathbb{E}[B_t - B_s + B_s|\mathcal{F}_s] = \mathbb{E}[B_t - B_s|\mathcal{F}_s] + \mathbb{E}[B_s|\mathcal{F}_s] = \mathbb{E}[B_t - B_s] + B_s$$

where $\mathbb{E}[B_t - B_s] = 0$ because $B_t - B_s$ is a Gaussian random variable $N(0, t-s)$. □

References

[1] P. Baldi, *Stochastic calculus, and introduction through theory and exercises*, Universitext, Springer, 2017.