Probability Theory (Stochastic Calculus)

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Exercise 1.1.2. Prove this statement: if $\mathcal{F}$ is a collection of subsets, which is closed under complement and countable unions, then it is closed under countable intersections.

Proof. Let $\{A_n\}_{n\in\mathbb{N}} \subset \mathcal{F}$. Then, by De Morgan's law we have,

$$\bigcap_{n\in\mathbb{N}} A_n = \left(\left(\bigcap_{n\in\mathbb{N}} A_n\right)^C\right)^C = \left(\bigcup_{n\in\mathbb{N}} A_n^C\right)^C \in \mathcal{F}$$

which follows from the fact that $\mathcal{F}$ is closed under complement and countable unions. □

Exercise 1.1.6. If $(\Omega, \mathcal{F}, \mathbb{P})$ is a probability space and if $A, B \in \mathcal{F}$, check that

1) $\mathbb{P}(A^C) = 1 - \mathbb{P}(A)$, where $A^C := \Omega \setminus A$.
2) $\mathbb{P}(A \cup B) = \mathbb{P}(A) + \mathbb{P}(B) - \mathbb{P}(A \cap B)$,
3) If $A \subset B$, then $\mathbb{P}(A) \leq \mathbb{P}(B)$.

Proof. 1) It follows from

$$1 = \mathbb{P}(\Omega) = \mathbb{P}(A \cup A^C) = \mathbb{P}(A) + \mathbb{P}(A^C).$$

2) We have

$$\mathbb{P}(A) + \mathbb{P}(B) = \mathbb{P}((A \cap B) \cup (A \setminus (A \cap B))) + P(B) = \mathbb{P}(A \cap B) + \mathbb{P}((A \setminus (A \cap B)) \cup B) = \mathbb{P}(A \cap B) + \mathbb{P}(A \cup B).$$

3) Suppose that $A \subset B$. Then,

$$\mathbb{P}(B) = \mathbb{P}((B \setminus A) \cup A) = \mathbb{P}(B \setminus A) + \mathbb{P}(A) \geq \mathbb{P}(A).$$

□

Lemma 1.1.7 (Continuity of probability). Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space and consider $\{A_j\}_{j\in\mathbb{N}} \subset \mathcal{F}$. If $A_j \subset A_{j+1}$ for any j, then

$$\mathbb{P}\left(\bigcup_{j\in\mathbb{N}} A_j\right) = \lim_{j\to\infty} \mathbb{P}(A_j),$$

while if $A_j \supset A_{j+1}$ for any j, then

$$\mathbb{P}\left(\bigcap_{j\in\mathbb{N}} A_j\right) = \lim_{j\to\infty} \mathbb{P}(A_j).$$

Proof. Because $A_j \subset A_{j+1}$ for any j, we have

$$\bigcup_{1\leq j\leq n} A_j = A_n$$

for any $n \in \mathbb{N}$.

By above, we have

$$\left\{\mathbb{P}\left(\bigcup_{1\le j\le n} A_j\right)\right\}_{n\in\mathbb{N}} = \{\mathbb{P}(A_n)\}_{n\in\mathbb{N}}.$$

From exercise 1.1.6, we know that the sequence above is monotonically nondecreasing. Moreover, it is bounded from above by 1. Therefore, by the monotone convergence theorem, the following limit exists

$$\mathbb{P}\left(\bigcup_{j\in\mathbb{N}} A_j\right) = \lim_{n\to\infty}\mathbb{P}\left(\bigcup_{1\le j\le n} A_j\right) = \lim_{n\to\infty}\mathbb{P}(A_n).$$

The proof for the second part is similar. □

Exercise 1.1.15 (Rephrased). Let X be a random variable on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ taking values in $(\mathbb{R}, \sigma_B)$ and F_X be the cumulative distribution. Prove that

1) $\lim_{x\to-\infty} F_X(x) = 0$,
2) $\lim_{x\to\infty} F_X(x) = 1$,
3) The function F_X is non-decreasing and right-continuous.

And provide an example of a cumulative distribution function which is not left-continuous.

Proof. 1) Consider the sequence $(-n)_{n\in\mathbb{N}}$. Let

$$A_n = \{\omega\in\Omega \mid X(\omega) \le -n\}.$$

By Lemma 1.1.7,

$$\lim_{n\to\infty}\mathbb{P}(A_n) = \mathbb{P}(\bigcap_{n\in\mathbb{N}} A_n)$$

because $A_{n+1}\subset A_n$ for all $n\in\mathbb{N}$. Because $\bigcap_{n\in\mathbb{N}} A_n = \emptyset$, the limit is 0.

Now, let $\epsilon > 0$. Because $\lim_{n\to\infty}\mathbb{P}(A_n) = 0$, there must be $N > 0$ such that $\mathbb{P}(A_N) < \epsilon$. But, for all $x < -N$, we have

$$\{\omega\in\Omega \mid X(\omega)\le x\} \subset \{\omega\in\Omega \mid X(\omega) \le -N\}.$$

Therefore, for all $x < -N$, we have

$$F_X(x) \le P(A_N) < \epsilon.$$

2) The proof is similar to 1).

3) For any $x < y$, we have

$$\{\omega\in\Omega \mid X(\omega)\le x\} \subset \{\omega\in\Omega \mid X(\omega)\le y\}.$$

Therefore, by Exercise 1.1.6 F_X is non-decreasing because

$$F_X(x) = \mathbb{P}(\{\omega\in\Omega \mid X(\omega)\le x\} \le \mathbb{P}(\{\omega\in\Omega \mid X(\omega)\le y\}) = F_X(y).$$

Now, we will prove the right-continuity. Let $x\in\mathbb{R}$ and $A_n = \{\omega\in\Omega \mid X(\omega)\le x+\frac{1}{n}\}$. Then, $A_{n+1}\subset A_n$. Therefore, by Lemma 1.1.7,

$$\lim_{n\to\infty}\mathbb{P}(A_n) = \mathbb{P}\left(\bigcap_{n\in\mathbb{N}} A_n\right) = \mathbb{P}(\{\omega\in\Omega \mid X(\omega)\le x\}) = F_X(x).$$

Let $\epsilon > 0$. By above, we can find $N > 0$ such that

$$|\mathbb{P}(A_N) - F_X(x)| < \epsilon.$$

Let $x < y < x + \frac{1}{N}$. Then,

$$|F_X(y) - F_X(x)| = F_X(y) - F_X(x) < \mathbb{P}(A_N) - F_X(x) = |\mathbb{P}(A_N) - F_X(x)| < \epsilon.$$

The first and last equalities are from the non-decreasing property proven previously, while the inequality follows from Exercise 1.1.6 because

$$\{\omega\in\Omega \mid X(\omega)\le y\} \subset A_N.$$

□

For an example of a cumulative distribution function that is not left-continuous, consider the random variable $X : \Omega \to \mathbb{R}$ such that for all $\omega \in \Omega$, $X(\omega) = 1$.

Then, $F_X(1) = 1$ but $F_X(1-\epsilon) = 0$ for all $\epsilon > 0$.

Exercise 1.2.5. For $\sigma > 0$ and $\bar{x} \in \mathbb{R}$ set $\Pi : \mathbb{R} \to \mathbb{R}_+$ by

$$\Pi(x) := \frac{1}{\sqrt{2\pi}\sigma} \exp\left(-\frac{1}{2\sigma^2}(x-\bar{x})^2\right).$$

Check that $\int \Pi(x)dx = 1$. In the framework of Reminder 1.1.12 we write $X = N(\bar{x}, \sigma^2)$ for the corresponding univariate random variable, called Gaussian random variable. Check that $\mathbb{E}(X) = \bar{x}$, and $\text{Var}(X) = \sigma^2$.

More generally, for $\bar{x} \in \mathbb{R}^N$ and $P \in M_{N\times N}(\mathbb{R})$ with $P > 0$, set $\Pi : \mathbb{R}^N \to \mathbb{R}_+$ with

$$\Pi(x) := \frac{1}{(2\pi)^{N/2}|P|^{1/2}} \exp\left(-\frac{1}{2}(x-\bar{x})^T P^{-1}(x-\bar{x})\right),$$

with $|P| := \det(P)$. Check that $\int \Pi(x)dx = 1$. We write $X = N(\bar{x}, P)$ for the corresponding multivariate random variable, called N-dim Gaussian random variable or Gaussian vector. Check that $\mathbb{E}(X) = \bar{x}$, and that $P = \mathbb{E}\left((X-\bar{x})(X-\bar{x})^T\right)$. Here, P is called the covariance matrix.

Proof. First, we do the substitution $u = \frac{1}{\sqrt{2}\sigma}(x-\bar{x})$ and $du = \frac{dx}{\sqrt{2}\sigma}$.

$$\int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}\sigma} \exp\left(-\frac{1}{2\sigma^2}(x-\bar{x})^2\right) dx = \frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} \exp\left(-u^2\right) du = 1.$$

The last equality is from the well known integral for $\int_{-\infty}^{\infty} \exp(-u^2)du = \sqrt{\pi}$ (for example, problem 1.4.4 in Problem-Solving Through Problems by Loren C. Larson).

Next, we will evaluate $\mathbb{E}(X)$ and $\text{Var}(X)$ by using the Law of Unconcscious Statistician (LOTUS) as follow and the same integral substitution as before,

$$\begin{aligned}
\mathbb{E}(X) &= \int_{-\infty}^{\infty} x\Pi(x)dx && \text{(LOTUS)} \\
&= \int_{-\infty}^{\infty} \frac{x}{\sqrt{2\pi}\sigma} \exp\left(-\frac{1}{2\sigma^2}(x-\bar{x})^2\right) dx \\
&= \int_{-\infty}^{\infty} \frac{\sqrt{2}\sigma u + \bar{x}}{\sqrt{\pi}} \exp(-u^2)du && \text{(substitution)} \\
&= \frac{\sqrt{2}\sigma}{\sqrt{\pi}} \int_{-\infty}^{\infty} u\exp(-u^2)du + \frac{\bar{x}}{\sqrt{\pi}} \int_{-\infty}^{\infty} \exp(-u^2)du.
\end{aligned}$$

The first term is 0 because $u\exp(-u^2) + (-u)\exp(-(-u)^2) = 0$ for all $u \in \mathbb{R}$. While, the second term, as mentioned before is just $\frac{\bar{x}}{\sqrt{\pi}}\sqrt{\pi} = \bar{x}$.

As for the variance,

$$\begin{aligned}
\text{Var}(X) &= \mathbb{E}((X-\mathbb{E}(X))^2) \\
&= \int_{-\infty}^{\infty} \frac{(x-\bar{x})^2}{\sqrt{2\pi}\sigma} \exp\left(-\frac{1}{2\sigma^2}(x-\bar{x})^2\right) dx \\
&= \frac{2\sigma^2}{\sqrt{\pi}} \int_{-\infty}^{\infty} u^2 \exp(-u^2)du \\
&= \frac{2\sigma^2}{\sqrt{\pi}} \left(\left[u\frac{\exp(-u^2)}{-2}\right]_{-\infty}^{\infty} - \int_{-\infty}^{\infty} \frac{\exp(-u^2)}{-2}\right) \\
&= \frac{2\sigma^2}{\sqrt{\pi}}(0 - \frac{\sqrt{\pi}}{-2}) \\
&= \sigma^2.
\end{aligned}$$

Next, we will consider the more general case. Because $P > 0$, $P^{-1} > 0$. So, we can decompose it as $P^{-1} = S^T S$ with S being invertible. Then, $|S| = \sqrt{|P^{-1}|}$.

Because S is invertible, the map $x \mapsto \frac{1}{\sqrt{2}}S(x-\bar{x})$ is injective. Moreover, it is differentiable with continuous partial derivatives. The Jacobian matrix of its inverse map is $\sqrt{2}S^{-1}$. The determinant of this Jacobian matrix is $2^{N/2}|P|^{1/2}$. So, by the change of variable $u = \frac{1}{\sqrt{2}}S(x-\bar{x})$, we have

$$\begin{aligned}
\int_{-\infty}^{\infty} \Pi(x)dx &= \int_{-\infty}^{\infty} \frac{1}{(2\pi)^{N/2}|P|^{1/2}} \exp\left(-u^T u\right) \left|\sqrt{2}S^{-1}\right| du \\
&= \int_{-\infty}^{\infty} \frac{1}{\pi^{N/2}} \exp\left(-u^T u\right) du \\
&= \frac{1}{\pi^{N/2}} \prod_{k=1}^{N} \int_{-\infty}^{\infty} \exp(-u_k^2) du_k \\
&= 1.
\end{aligned}$$

Finally, we will evaluate $\mathbb{E}(X)$ and the covariance matrix by using LOTUS and the above substitution. Let e_i be a vector with all entries 0 except the i-th element equals 1. The expected value can be calculated as

$$\begin{aligned}
\mathbb{E}(X) &= \int_{-\infty}^{\infty} \frac{1}{(2\pi)^{N/2}|P|^{1/2}} x \exp\left(-\frac{1}{2}(x-\bar{x})^T P^{-1}(x-\bar{x})\right) dx && \text{(LOTUS)} \\
&= \int_{-\infty}^{\infty} \frac{1}{\pi^{N/2}} (\sqrt{2}S^{-1}u + \bar{x}) \exp(-u^T u) du && \text{(substitution)} \\
&= \frac{\sqrt{2}}{\pi^{N/2}} S^{-1} \int_{-\infty}^{\infty} u \exp(-u^T u) du + \bar{x} \\
&= \frac{\sqrt{2}}{\pi^{N/2}} S^{-1} \sum_{j=1}^{N} e_j \int_{-\infty}^{\infty} u_j \prod_{k=1}^{N} \exp(-u_k^2) du_j + \bar{x} \\
&= \frac{\sqrt{2}}{\pi^{N/2}} S^{-1} \sum_{j=1}^{N} e_j \int_{-\infty}^{\infty} u_j \exp(-u_j^2) du_j \prod_{\substack{1 \le m \le N \\ m \ne j}} \int_{-\infty}^{\infty} \exp(-u_m^2) du_m + \bar{x}.
\end{aligned}$$

But $\int_{-\infty}^{\infty} u_j \exp(-u_j^2) = 0$ as has been shown in the single variable computation above. Therefore, only the last term survived. So, $\mathbb{E}(X) = \bar{x}$.

Let $e_{i,j}$ be $N \times N$ matrix such that all entries are zeros except the entry at row i and column j is 1. As for the covariance matrix, we have

$$
\begin{aligned}
\mathbb{E}((X-\bar{x})(X-\bar{x})^T) &= \int_{-\infty}^{\infty} \frac{1}{(2\pi)^{N/2}|P|^{1/2}}(x-\bar{x})(x-\bar{x})^T \exp\left(-\frac{1}{2}(x-\bar{x})^T P^{-1}(x-\bar{x})\right) dx \\
&= \int_{-\infty}^{\infty} \frac{1}{\pi^{N/2}}(\sqrt{2}S^{-1}u)(\sqrt{2}S^{-1}u)^T \exp(-u^T u) du \\
&= \frac{2}{\pi^{N/2}} S^{-1} \left(\int_{-\infty}^{\infty} uu^T \exp(-u^T u) du\right) (S^{-1})^T \\
&= \frac{2}{\pi^{N/2}} S^{-1} \left(\sum_{j=1}^{N} e_{j,j} \int_{-\infty}^{\infty} u_j^2 \exp(-u_j^2) du_j \prod_{\substack{1\le k\le N \\ k\neq j}} \int_{-\infty}^{\infty} \exp(-u_k^2) du_k\right) (S^{-1})^T \\
&\quad + \frac{2}{\pi^{N/2}} S^{-1} \left(\sum_{\substack{1\le r,s\le N \\ r\neq s}} e_{r,s} \int_{-\infty}^{\infty} u_r \exp(-u_r^2) du_r \int_{-\infty}^{\infty} u_s \exp(-u_s^2) du_s \prod_{\substack{1\le k\le N \\ k\notin\{r,s\}}} \int_{-\infty}^{\infty} \exp(-u_k^2) du_k\right) (S^{-1})^T \\
&= \frac{2}{\pi^{N/2}} S^{-1} \left(\sum_{j=1}^{N} e_{j,j} \frac{\sqrt{\pi}}{2} \pi^{(N/2-1)}\right) (S^{-1})^T + \frac{2}{\pi^{N/2}} S^{-1} \left(\sum_{\substack{1\le r,s\le N \\ r\neq s}} e_{r,s}(0)\right) (S^{-1})^T \\
&= S^{-1}(S^{-1})^T \\
&= S^{-1}(S^T)^{-1} \\
&= (S^T S)^{-1} \\
&= P.
\end{aligned}
$$

□

Exercise 1.2.6. If $X : \Omega \to \mathbb{R}^N$ is absolutely continuous with pdf Π_X and if $\phi : \mathbb{R}^N \to \mathbb{R}^N$ is bijective and C^∞, show that $Y := \phi(X) : \Omega \to \mathbb{R}^N$ is a new absolutely continuous random variable, with pdf Π_Y given by $\Pi_Y(y) = \prod_X \left(\phi^{-1}(y)\right) \left|J_{\phi^{-1}}(y)\right|$. Here, $\left|J_{\phi^{-1}}\right|$ denotes the determinant of the Jacobian matrix of ϕ^{-1}.

Proof. For $A \in \sigma_B$, we have

$$
\begin{aligned}
\int_A \Pi_Y(y)dy = \int_A \Pi_X\left(\phi^{-1}(y)\right)\left|J_{\phi^{-1}}(y)\right| dy &= \int_{\phi^{-1}(A)} \Pi_X(x)dx = \mu_X(\phi^{-1}(A)) \\
&= \mathbb{P}(X^{-1}(\phi^{-1}(A))) = \mathbb{P}((\phi(X))^{-1}(A)) = \mathbb{P}(Y^{-1}(A)) = \mu_Y(A).
\end{aligned}
$$

□