Self-financing Trading Strategy

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Trading Strategy

An investor may decide to invest an amount of money by acquiring shares of the $m+1$ assets of the market. So, a *trading strategy* (or a portfolio) over the trading interval $[0,T]$ is a progressively measurable $(m+1)$-dimensional process $H(t) = (H_0(t), H_1(t), \ldots, H_m(t))$ whose general component $H_i(t)$ stands for the number of units of the i-th security held by an investor at time t.

The value of the portfolio or simply the portfolio $V(H)$ of the strategy H is the corresponding wealth process:

$$V_t(H) = \langle H, S \rangle_t = \sum_{i=0}^{m} H_i(t) S_i(t), \quad t \in [0,T] \tag{1}$$

Discounted Price Process and Portfolio Value

The initial value of the portfolio $V_0(H)$ is the initial investment of the strategy H. The portfolio $V(H)$ associated with this strategy is an Itô process.

Discounted Price Process

The discounted price of an asset i at time t is given by $\tilde{S}_i(t)$, which is the current price $S_i(t)$ divided by the price at time 0, adjusted for the risk-free rate r_s:

$$\tilde{S}_i(t) = \frac{S_i(t)}{S_0(t)} = e^{-\int_0^t r_s \, ds} S_i(t), \quad i = 1, \ldots, m$$

Discounted Portfolio Value

Similarly, the discounted portfolio $\tilde{V}_t(H)$ is the current value $V_t(H)$ divided by the value at time 0, adjusted for the risk-free rate:

$$\tilde{V}_t(H) = \frac{V_t(H)}{S_0(t)} = e^{-\int_0^t r_s \, ds} V_t(H)$$

Applying Itô's Formula

Itô's formula is used to find the differential of the discounted price process, taking the differential of $\tilde{S}_i(t)$ and adjusting it for the risk-free rate:

$$d\tilde{S}_i(t) = -r_t e^{-\int_0^t r_s\,ds} S_i(t)\,dt + e^{-\int_0^t r_s\,ds}\,dS_i(t)$$

Simplifying as $t \to \int_0^t r_s ds$, we get:

$$d\tilde{S}_i(t) = -r_t \tilde{S}_i(t)\,dt + e^{-\int_0^t r_s\,ds}\,dS_i(t) \tag{2}$$

Proof of Proposition 7.2.2

We need to show that a trading strategy H is self-financing if and only if for any $t \in [0, T]$,

$$\tilde{V}_t(H) = V_0(H) + \sum_{i=1}^{m} \int_0^t H_i(u)\,d\tilde{S}_i(u).$$

Proof. Suppose H is self-financing. This means the changes in the value of the portfolio are due to capital losses or gains and not to increase or decrease of the invested funds.

Then, with Itô's formula and equation (2),

$$\begin{aligned}
d\tilde{V}_t(H) &= d\left(e^{-\int_0^t r_s\,ds} V_t(H)\right) \\
&= -r_t e^{-\int_0^t r_s\,ds} V_t(H)\,dt + e^{-\int_0^t r_s\,ds}\,dV_t(H) \\
&= e^{-\int_0^t r_s\,ds}\left(-r_t \sum_{i=0}^{m} H_i(t) S_i(t)\,dt + \sum_{i=0}^{m} H_i(t)\,dS_i(t)\right).
\end{aligned}$$

Using equation (2) for the discounted asset prices,

$$d\tilde{S}_i(t) = -r_t \tilde{S}_i(t)\,dt + e^{-\int_0^t r_s\,ds}\,dS_i(t),$$

and as $H_0(t) dS_0(t) = r_t H_0(t) S_0(t)$, notice we can cancel the term with index $i = 0$ to obtain:

$$\begin{aligned}
d\tilde{V}_t(H) &= e^{-\int_0^t r_s\,ds}\left(-r_t \sum_{i=1}^{m} H_i(t) S_i(t)\,dt + \sum_{i=1}^{m} H_i(t)\,dS_i(t)\right) \\
&= \sum_{i=1}^{m} H_i(t)\left(-r_t \tilde{S}_i(t)\,dt + e^{-\int_0^t r_s\,ds}\,dS_i(t)\right) \\
&= \sum_{i=1}^{m} H_i(t) d\tilde{S}_i(t).
\end{aligned}$$

Conversely, if

$$\tilde{V}_t(H) = V_0(H) + \sum_{i=1}^{m} \int_0^t H_i(u)\, d\tilde{S}_i(u).$$

Applying equation (2) and Itô's formula,

$$\begin{aligned} dV_t(H) &= d\left(e^{\int_0^t r_s\, ds}\tilde{V}_t(H)\right) \\ &= r_t e^{\int_0^t r_s\, ds}\tilde{V}_t(H)\, dt + e^{\int_0^t r_s\, ds}\, d\tilde{V}_t(H). \end{aligned}$$

Given $\tilde{V}_t(H) = V_0(H) + \sum_{i=1}^{m} \int_0^t H_i(u)\, d\tilde{S}_i(u)$, we substitute to get:

$$dV_t(H) = r_t V_t(H)\, dt + e^{\int_0^t r_s\, ds} \sum_{i=1}^{m} H_i(t)\, d\tilde{S}_i(t).$$

Using the relation for the discounted prices in equation (2),

$$d\tilde{S}_i(t) = -r_t \tilde{S}_i(t)\, dt + e^{-\int_0^t r_s\, ds}\, dS_i(t),$$

we have:

$$\begin{aligned} dV_t(H) &= r_t V_t(H)\, dt + e^{-\int_0^t r_s\, ds} \sum_{i=1}^{m} H_i(t) \left(-r_t \tilde{S}_i(t)\, dt + e^{-\int_0^t r_s\, ds}\, dS_i(t)\right) \\ &= r_t V_t(H)\, dt - r_t \sum_{i=1}^{m} H_i(t) S_i(t)\, dt + \sum_{i=1}^{m} H_i(t)\, dS_i(t) \end{aligned}$$

Noting that $H_0(t) dS_0(t) = r_t H_0(t) S_0(t)$, and $\sum_{i=1}^{m} H_i(t)\tilde{S}_i(t) = V_t(H)$, one gets:

$$\begin{aligned} dV_t(H) &= r_t V_t(H)\, dt - r_t V_t(H)\, dt + \sum_{i=0}^{m} H_i(t)\, dS_i(t) \\ &= \sum_{i=0}^{m} H_i(t)\, dS_i(t). \end{aligned}$$

Thus, H is self-financing as we complete the proof. □