Gambler's Ruin

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Gambler's Ruin with no Drift

The gambler's ruin problem is a well-known problem that can be posed in various forms. Essentially, it deals with calculating the probability that a gambler will reach a specific wealth level before going broke. In the context of Brownian motion, this translates into the following questions:

Let $(B_t, t \geq 0)$ be a standard Brownian motion starting at $B_0 = 0$ with thresholds $a, b > 0$.

1. What is the probability that a Brownian path reaches a before $-b$?
2. What is the expected waiting time for the path to reach either a or $-b$?

To answer the first question, we apply stopping time and martingale properties. Define the hitting time $\tau(\omega)$ as:

$$\tau(\omega) = \min\{t \geq 0 : B_t(\omega) \geq a \text{ or } B_t(\omega) \leq -b\}.$$

Here, τ represents the minimum time when the Brownian path first reaches either a or $-b$.

First, we show that $\tau < \infty$ with probability 1, meaning all Brownian paths will eventually reach a or $-b$. To demonstrate this, let's consider the event E_n where the n-th increment exceeds $a + b$:

$$E_n = \{|B_n - B_{n-1}| > a + b\}, \quad n \geq 1.$$

If E_n occurs, the Brownian path must exit the interval $[-b, a]$. The probability $\mathbb{P}(E_n)$ is constant for all n, denoted by p, where $0 < p < 1$. Since E_n events are independent, we have:

$$\mathbb{P}(E_1^c \cap \cdots \cap E_n^c) = (1 - p)^n.$$

As $n \to \infty$, $\mathbb{P}(E_1^c \cap \cdots \cap E_n^c) \to 0$. We call the event, $F_n = E_1^c \cap \cdots \cap E_n^c$ which is decreasing in n. By the continuity of probability, we conclude:

$$\mathbb{P}\left(\bigcap_{n \geq 1} F_n\right) = 0.$$

Thus, some E_n must occur, indicating that all paths will eventually hit a or $-b$.

Since $\tau < \infty$ with probability 1, B_τ is well-defined and can take the values a or $-b$. The first question translates to finding $\mathbb{P}(B_\tau = a)$.

On one hand, we have:

$$\mathbb{E}[B_\tau] = a\mathbb{P}(B_\tau = a) - b(1 - \mathbb{P}(B_\tau = a)).$$

Then, let's recall Doob's optional stopping theorem. If $(M_t, t \geq 0)$ is a continuous martingale for the filtration $(\mathcal{F}_t, t \geq 0)$ and τ is a stopping time such that $\tau < \infty$ a.s., and the stopped process $(M_{t\wedge\tau}, t \geq 0)$ is bounded, then

$$\mathbb{E}[M_\tau] = \mathbb{E}[M_0].$$

Therefore,

$$\mathbb{E}[B_\tau] = a\mathbb{P}(B_\tau = a) - b(1 - \mathbb{P}(B_\tau = a)) = 0 = \mathbb{E}[B_0].$$

Solving this, we find:

$$\mathbb{P}(B_\tau = a) = \frac{b}{a+b}.$$

This gives a simple solution to the problem, which is nice. ☺

Gambler's Ruin with Drift

Now we can extend the gambler's ruin problem to include a drift. Consider the Brownian motion with drift

$$X_t = \sigma B_t + \mu t,$$

where $(B_t, t \geq 0)$ is a standard Brownian motion. We assume that $\mu > 0$, indicating an upward drift.

We focus on the first passage time τ to levels a or $-b$:

$$\tau = \min\{t \geq 0 : X_t > a \text{ or } X_t < -b\}.$$

The goal is to compute the probability $\mathbb{P}(X_\tau = a)$. In the absence of drift, this probability is $\frac{b}{a+b}$ as shown in the first section. With drift, we shall look a martingale of X_t and apply Doob's optional stopping theorem.

We assume the martingale is of the simplest form, a function of X_t:

$$M_t = g(X_t),$$

where g is a function to be determined. This function must satisfy two properties:

- The process $(g(X_t), t \geq 0)$ is a martingale for the Brownian filtration.
- Boundary conditions $g(a) = 1$ and $g(-b) = 0$.

By Doob's optional stopping theorem, the first condition implies:

$$\mathbb{E}[g(X_\tau)] = g(0).$$

The second condition can be simplified to:

$$\mathbb{E}[g(X_\tau)] = g(a)\mathbb{P}(X_\tau = a) + g(-b)\mathbb{P}(X_\tau = -b) = \mathbb{P}(X_\tau = a).$$

Combining these, we reduce the problem to finding $g(0)$:

$$g(0) = \mathbb{E}[g(X_\tau)] = \mathbb{P}(X_\tau = a).$$

For $g(X_t)$ to be a martingale, consider $g(X_t) = g(\sigma B_t + \mu t)$, and $f(t, x) = g(\sigma x + \mu t)$. Using the chain rule:

$$\begin{aligned}
\partial_t f(t, x) &= \mu g'(\sigma x + \mu t), \\
\partial_x f(t, x) &= \sigma g'(\sigma x + \mu t), \\
\partial_x^2 f(t, x) &= \sigma^2 g''(\sigma x + \mu t).
\end{aligned}$$

In the one variable case, we can get a corollary to construct martingales:

(Brownian martingales) Let $(B_t, t \leq T)$ be a standard Brownian motion. Consider $f \in C^{1,2}([0, T] \times \mathbb{R})$ such that the process $(\partial_t f(t, B_t), t \leq T)$. Then the process is

$$\left(f(t, B_t) - \int_0^t \left\{ \partial_s f(s, B_s) + \frac{1}{2}\partial_x^2 f(s, B_s) \right\} ds, t \leq T \right)$$

is a martingale for the Brownian filtration. In particular, if $f(t, x)$ satisfies the partial differential equation $\partial_t f = -\frac{1}{2}\partial_x^2 f$, then the process $(f(t, B_t), t \leq T)$ is itself a martingale.

Here, we can see that *the study of martingales is closely related to the study of differential equations.* This connection is profound because it allows techniques from one area (martingale theory) to solve problems in another (partial differential equations), providing deep insights and powerful tools for analysis.

Thus by the corollary of Brownian martingales, for $g(X_t)$ to be a martingale, g must satisfy:

$$\mu g' = \frac{\sigma^2}{2} g''.$$

Solving this differential equation, we obtain:

$$g(y) = Ce^{-2\mu y/\sigma^2} + C'.$$

Applying the boundary conditions $g(a) = 1$ and $g(-b) = 0$, we determine the constants:

$$g(y) = \frac{1 - e^{-2\mu(y+b)/\sigma^2}}{1 - e^{-2\mu(a+b)/\sigma^2}}.$$

Thus, the probability is:

$$\mathbb{P}(X_\tau = a) = g(0) = \frac{1 - e^{-2\mu b/\sigma^2}}{1 - e^{-2\mu(a+b)/\sigma^2}}.$$

Key observations:

- For $\mu = \sigma = 1$ and $a = b = 1$, the probability is:

$$\mathbb{P}(X_\tau = a) = \frac{1 - e^{-2}}{1 - e^{-4}} \approx 0.881\ldots.$$

  Compare this to $\mu = 0$, where the probability is $\frac{1}{2}$.

- We reduced the problem to solving a differential equation with boundary conditions.

- The solution is general; starting the process at $y \in [-b, a]$ instead of 0 yields $g(y)$ as given in equation,

$$g(y) = \frac{1 - e^{-2\mu(y+b)/\sigma^2}}{1 - e^{-2\mu(a+b)/\sigma^2}}.$$

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- The identity:

$$\mathbb{E}[g(X_\tau)] = \mathbb{P}(X_\tau = a)$$

  is intuitive. Since $g(a) = 1$ and $g(-b) = 0$, only paths hitting a contribute to the expectation, while paths hitting $-b$ do not. Hence, the probability is given by averaging the Bernoulli variable $g(X_\tau)$ over all paths.

- In the limit as $b \to \infty$:

$$\mathbb{P}(X_\tau = a) \to 1, \quad b \to \infty,$$

  which makes sense given the upward drift. If $a \to \infty$, then:

$$\mathbb{P}(X_\tau = -b) \to e^{-2\mu b/\sigma^2}.$$

  This shows that some paths will never hit $-b$, regardless of how small the drift is.