BROWNIAN MOTION IN FREE PROBABILITY THEORY

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1. Classical Brownian motion

Recall Brownian motion in probability theory based on [R24, Chapter 2, sections 2.3, 2.4]. We denote by Ω and $\mathcal{F}$ a sample space and a σ-algebra over Ω, respectively.

Definition 1.1. Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space and $\mathcal{T}$ be a subset of $\mathbb{R}$. A *filtration* $(\mathcal{F}_t)_{t\in\mathcal{T}}$ is a family of σ-subalgebras of $\mathcal{F}$ such that $\mathcal{F}_s \subset \mathcal{F}_t$.

Remark 1.2. If $(\mathcal{F}_t)_{t\in\mathcal{T}}$ is a filtration, then $L^\infty(\Omega, \mathcal{F}_s, \mathbb{P}) \subset L^\infty(\Omega, \mathcal{F}_t, \mathbb{P})$ for any $s, t \in \mathcal{T}$ with $s \leq t$. Since $L^\infty(\Omega, \mathcal{F}_t, \mathbb{P})$ contains every characteristic function 1_S of measurable set $S \in \mathcal{F}_t$, the family $(L^\infty(\Omega, \mathcal{F}_t, \mathbb{P}))_{t\in\mathcal{T}}$ is the translation of the notion of filtration in terms of function space. This viewpoint is important to generalize the notion of filtration to some "non-commutative" framework as seen later.

Definition 1.3. A tuple $(\Omega, \mathcal{F}, \mathbb{P}, (\mathcal{F}_t)_{t\in\mathcal{T}}, (X_t)_{t\in\mathcal{T}})$ is a *stochastic process* if $(\Omega, \mathcal{F}, \mathbb{P})$ is a probability space, $(\mathcal{F}_t)_{t\in\mathcal{T}}$ is a filtration on $(\Omega, \mathcal{F}, \mathbb{P})$ and $(X_t)_{t\in\mathcal{T}}$ is a family of random variables on $(\Omega, \mathcal{F})$ (taking values in another measurable space $(\Lambda, \mathcal{E})$) such that X_t is $(\mathcal{F}_t, \mathcal{E})$-measurable for any $t \in \mathcal{F}_t$.

We are now in a position to give the definition of Brownian motion.

Definition 1.4. A stochastic process $(\Omega, \mathcal{F}, \mathbb{P}, (\mathcal{F}_t)_{t\geq 0}, (B_t)_{t\geq 0})$ is a *(1-dimensional) Brownian motion* if the process satisfies the following properties:

(1) $B_0 = 0$ a.s.,
(2) the random variable $B_t - B_s$ is independent of $\mathcal{F}_s$ for any $t \geq s \geq 0$,
(3) $B_t - B_s$ is a Gaussian random variable with mean 0 and variance $t-s$ for any $t \geq s \geq 0$.

In the sequel, we will give the free probabilistic analogue of Brownian motion, which is called *free Brownian motion*.

2. Basics of free probability theory

We will briefly introduce basics of free probability theory. *Free probability theory* is initiated by Voiculescu, who was motivated by the study of operator algebra [V85]. Precisely, Voiculescu found that a kind of probabilistic structure appears in some "free product" of operator algebras. ("Free" of free probability theory comes from that of free product.) Firstly, let us see this (see [VDN92] for more details and various topics).

2.1. C^*-probability space and W^*-probability space.

Definition 2.1. Let $\mathcal{H}$ be a Hilbert space over $\mathbb{C}$. We say that $\mathcal{A}$ is a *unital C^*-algebra* (resp. unital von Neumann algebra) over $\mathcal{H}$ if $\mathcal{A}$ is a unital $*$-subalgebra of $B(\mathcal{H})$ and closed with respect to operator norm topology (resp. weak operator topology (abbreviated by WOT)).

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It is clear that every von Neumann algebra is a C^*-algebra, since norm-convergence implies WOT-convergence.

Definition 2.2. Let $\mathcal{A}$ be a unital C^*-algebra over some Hilbert space $\mathcal{H}$. A linear functional $\tau : \mathcal{A} \to \mathbb{C}$ is *positive* if $\tau(a^*a) \geq 0$ for any $a \in \mathcal{A}$, and τ is a *state* if τ is positive and bounded with norm one. We say that τ is *faithful* if $\tau(a^*a) = 0$ implies $a = 0$ for any $a \in \mathcal{A}$ and *tracial* if $\tau(ab) = \tau(ba)$ for any $a, b \in \mathcal{A}$. When $\mathcal{A}$ is a unital von Neumann algebra, τ is *normal* if τ is WOT-continuous on the unit ball of $\mathcal{A}$.

A pair $(\mathcal{A}, \tau)$ consisting of a unital C^*-algebra (resp. unital von Neumann algebra) $\mathcal{A}$ and a faithful tracial state (resp. a faithful normal tracaial state) τ is called a *C^*-probability space* (resp. *W^*-probability space*).

Remark 2.3. If τ is positive, then it is known that $\varphi(1) = \|\varphi\|$. Hence, every state on unital C^*-algebra is automatically unital, that is, $\varphi(1) = 1$. This property means that a state is a generalization of an integration by probability measure in non-commutative setting.

Thus, we have prepared the basic framework of free probability theory corresponding to a pair of a space of random variables and an integration by probability measure. Next, we will introduce a central concept in free probability theory, called *free independence*.

2.2. **Free independence.**

Definition 2.4. Let $(\mathcal{A}, \tau)$ be a C^*-probability space (resp. W^*-probability space). Let $(S_\iota)_{\iota \in I}$ be a family of subset of $\mathcal{A}$. We say that S_ι $(\iota \in I)$ are *freely independent* if for any $n \in \mathbb{N}$, any $\iota_1, \ldots, \iota_n \in I$ with $\iota_j \neq \iota_{j+1}$ and any $\xi_j \in \mathrm{alg}^*(S_{\iota_j})$, we then have $\tau(\xi_1 \xi_2 \cdots \xi_n) = 0$ if $\tau(\xi_j) = 0$, where $\mathrm{alg}(S_\iota)$ denotes the (algebraic) unital $*$-subalgebra of $\mathcal{A}$ generated by S_ι.

Remark 2.5. Let $(\mathcal{A}, \tau)$ be a W^*-probability space. If subsets S_ι $(\iota \in I)$ of $\mathcal{A}$ are freely independent (that is, unital $*$-subalgebras $\mathrm{alg}^*(S_\iota)$ $(\iota \in I)$ are freely independent), then their WOT-closures $W^*(S_\iota)$ $(\iota \in I)$ are also freely independent by Kaplansky's density theorem, which allows us to approximate an element of unital C^*-algebra by norm-uniformly-bounded nets in strong operator topology.

2.3. **The reduced free product of operator algebras.**

Let $(\mathcal{A}_1, \tau_1)$ and $(\mathcal{A}_2, \tau_2)$ be C^*-probability spaces (resp. W^*-probability space).

Definition 2.6. (Rough definition) The *reduced free product* C^*-algebra (resp. von Neumann algebra) $C^*_{\mathrm{red}}(\mathcal{A}_1 * \mathcal{A}_2)$ (resp. $W^*_{\mathrm{red}}(\mathcal{A}_1 * \mathcal{A}_2)$) of $\mathcal{A}_1$ and $\mathcal{A}_2$ is a unital C^*-algebra (resp. a unital von Neumann algebra), in which the algebraic free product $\mathcal{A}_1 *_{\mathrm{alg}} \mathcal{A}_2$ (see Appendix) is dense, with a unique faithful tracial state (resp. faithful normal tracial state) $\tau_1 * \tau_2$ on $C^*_{\mathrm{red}}(\mathcal{A}_1 * \mathcal{A}_2)$ (resp. $W^*_{\mathrm{red}}(\mathcal{A}_1 * \mathcal{A}_2)$) such that $(\tau_1 * \tau_2)|_{\mathcal{A}_i} = \tau_i$ for any $i \in \{1, 2\}$.

It is enough to understand the reduced free product C^*-algebra (resp. von Neumann algebra) in the above way for the following discussion. If the reader is interested in the detail, see [VDN92, Chapter 1].

2.4. **Semi-circular distribution.**

In classical probability theory, *Gaussian (normal) distribution* is one of important and natural objects as known by e.g. central limit theorem. The following fact was proved by Voiculescu [V85], which implies that *semi-circular distribution* may play a role of Gaussian distribution in free probability theory.

Theorem 2.7. *(Free central limit theorem) Let $(\mathcal{A}, \tau)$ be a C^*-probability space and $(a_n)_{n=1}^{\infty}$ be a family of elements of $\mathcal{A}$ such that $\tau(a_i^k) = \tau(a_j^k)$ for any $i \neq j$ and any $k \in \mathbb{N}$ and* *that $\tau(a_1) = 0$ and $\tau(a_1^2) = 1$. Assume that $(a_n)_{n=1}^{\infty}$ are freely independent (that is, a family $(\{a_n\})_{n=1}^{\infty}$ of singletons are freely independent). Then, the following holds: For any $k \in \mathbb{N}$,*

$$\tau\left(\left(\frac{a_1 + \cdots + a_n}{\sqrt{n}}\right)^k\right) \to \int_{-2}^{2} t^k \, d\sigma(t)$$

as $n \to \infty$, where $d\sigma(t) = \sqrt{4 - t^2}\, \frac{dt}{2\pi}$, which is called semi-circular distribution.

Definition 2.8. Let $(\mathcal{A}, \tau)$ be a C^*-probability space. A self-adjoint element $a \in \mathcal{A}$ is called a *semi-circular element* with mean 0 and variance $v \neq 0$ if we have

$$\tau(a^k) = \int_{-2}^{2} t^k \sqrt{4v - t^2}\, \frac{dt}{2\pi v}$$

for any $k \in \mathbb{N}$. In particular, when $v = 1$, we say that a is a *standard* semi-circular element. A family $(a_\iota)_{\iota \in I}$ of freely independent standard semi-circular elements of $\mathcal{A}$ is called a *free standard semi-circular system*.

Remark 2.9. It is easy to see that every odd moment of semi-circular element is equal to 0. Moreover, it is well known that the $2k$-th moment of semi-circular element with mean 0 and variance v is equal to $v^{2k} C_k$, where $C_k = \frac{1}{k+1}\binom{2k}{k}$, which is so-called the k-th *Catalan number*.

3. Free Brownian motion

In the sequel, set $[k] = \{1, 2, \ldots, k\}$ for any $k \in \mathbb{N}$. Let $(\mathcal{A}, \tau)$ be a W^*-probability space, that is, $\mathcal{A}$ is a von Neumann algebra and τ is a faithful normal tracial state on $\mathcal{A}$.

Definition 3.1. We say that a von Neumann algebra $\mathcal{A}$ is *filtered* if there exists a family $(\mathcal{A}_t)_{t \geq 0}$ of unital von Neumann subalgebras of $\mathcal{A}$ indexed by non-negative real numbers t such that $\mathcal{A}_s \subset \mathcal{A}_t$ for any $0 \leq s \leq t$.

This is a generalization of *filtration* in classical probability theory (see Definition 1.1) in terms of function space.

We are now in a position to give the definition of free Brownian motion.

Definition 3.2. Let $(\mathcal{A}, \tau)$ be a W^*-probability space such that $\mathcal{A}$ is filtrated, that is, there exists a family $(\mathcal{A}_t)_{t \geq 0}$ of unital von Neumann subalgebras of $\mathcal{A}$ indexed by non-negative real numbers t such that $\mathcal{A}_s \subset \mathcal{A}_t$ for any $0 \leq s \leq t$. A family $(X_t)_{t \geq 0}$ of elements of $\mathcal{A}$ is an $(\mathcal{A}_t)_{t \geq 0}$*-free Brownian motion* if the following conditions are satisfied:

(1) each X_t is a semi-circular element with mean 0 and variance t in $(\mathcal{A}, \tau)$,
(2) $X_t \in \mathcal{A}_t$ for any $t \geq 0$,
(3) For any $t \geq s \geq 0$, $X_t - X_s$ is freely independent with $\mathcal{A}_s$ and a semi-circular element with mean 0 and variance $t - s$.

Example 3.3. ([BS98]) Let $\mathcal{F}\big(L^2([0, \infty))\big)$ be the full Fock space over $L^2([0, \infty))$, that is,

$$\mathcal{F}\big(L^2([0, \infty))\big) := \mathbb{C}\Omega \oplus \bigoplus_{n \geq 1} L^2([0, \infty))^{\otimes n}$$

with the naturally induced inner product, where Ω is a formal unit vector, which is also called the *vacuum vector*. Let $\ell_t = \ell_{1_{[0,t]}}$ be a creation operator with respect to $1_{[0,t]}$ defined by

$$\ell_t \Omega = 1_{[0,t]}, \quad \ell_t f_1 \otimes \cdots \otimes f_n = 1_{[0,t]} \otimes f_1 \otimes \cdots \otimes f_n$$

for any $n \in \mathbb{N}$ and any $f_1, \dots, f_n \in L^2([0,\infty))$. It is easy to see that ℓ_t is bounded, and its adjoint operator ℓ_t^* is given by

$$\ell_t^*\Omega = 0, \quad \ell_t^* f = \langle f, 1_{[0,t]}\rangle_{L^2}\Omega, \quad \ell_t^* f_1 \otimes \cdots \otimes f_n = \langle f_1, 1_{[0,t]}\rangle_{L^2} f_2 \otimes \cdots \otimes f_n$$

for any $n \in \mathbb{N}$ and any $f, f_1, \dots, f_n \in L^2([0,\infty))$.

Set $X_t = \ell_t + \ell_t^*$, and let $\mathcal{W}$ be the unital von Neumann algebra over $\mathcal{F}\big(L^2([0,\infty))\big)$ generated by $(X_t)_{t\geq 0}$ with the vacuum state $\rho(w) = \langle w\Omega, \Omega\rangle_{\mathcal{F}\left(L^2([0,\infty))\right)}$ for any $w \in \mathcal{W}$. Let $(\mathcal{A}, \tau)$ be the reduced free product von Neumann algebra $(W^*_{red}(\mathcal{B} * \mathcal{W}), \rho * \omega)$ of $(\mathcal{B}, \omega)$ and $(\mathcal{W}, \rho)$, and $\mathcal{A}_t$ be a unital von Neumann subalgebra of $\mathcal{A}$ generated by $\mathcal{B} \cup \{X_s \mid s \leq t\}\big(\subset \mathcal{A}\big)$. Then, $(X_t)_{t\geq 0}$ is an $(\mathcal{A}_t)$-free Brownian motion.

In the paper [BS98], the proof of that $(X_t)_{t\geq 0}$ is an $(\mathcal{A}_t)$-free Brownian motion was not given. Hence, in the sequel, let us prove it.

Lemma 3.4. *For any $t \geq s \geq 0$, X_t and $X_t - X_s$ are semi-circular elements of $\mathcal{A}$ with mean 0 and variance t and with mean 0 and variance $t - s$, respectively. In particular, $X_t \in \mathcal{A}_t$ for any $t \geq 0$.*

Proof. By construction, it is clear that $X_t \in \mathcal{A}_t$ for any $t \geq 0$.

For any $t \geq 0$ and $k \in \mathbb{N}$, observe that

$$X_t^k = (\ell_t + \ell_t^*)^k = \sum_{\epsilon_1,\dots,\epsilon_k \in \{\pm 1\}} \ell_t^{(\epsilon_k)} \cdots \ell_t^{(\epsilon_2)} \ell_t^{(\epsilon_1)},$$

where $\ell_t^{(+1)} = \ell_t$ and $\ell_t^{(-1)} = \ell_t^*$. By the definitions of the inner product of the full Fock space $\mathcal{F}\big(L^2([0,\infty))\big)$, the vacuum state $\rho(w) = \langle w\Omega, \Omega\rangle_{\mathcal{F}(L^2([0,\infty)))}$ and the creation and annihilation operators ℓ_t, ℓ_t^*, it is easy to see that $\rho\big(\ell_t^{(\epsilon_k)} \cdots \ell_t^{(\epsilon_2)} \ell_t^{(\epsilon_1)}\big) = 0$ whenever $\epsilon_1 + \cdots + \epsilon_k \neq 0$. Thus,

$$\rho(X_t^k) = \sum_{\substack{\epsilon_1,\dots,\epsilon_k \in \{\pm 1\} \\ \epsilon_1 + \cdots + \epsilon_k = 0}} \rho\big(\ell_t^{(\epsilon_k)} \cdots \ell_t^{(\epsilon_2)} \ell_t^{(\epsilon_1)}\big).$$

If k is odd, then $\epsilon_1 + \cdots + \epsilon_k \neq 0$ for any $(\epsilon_1, \dots, \epsilon_k) \in \{\pm 1\}$. Hence, every odd moment of X_t is equal to 0.

In the sequel, assume that k is even. Since $\ell_t^*\Omega = 0$, we can rewrite the expression of $\rho(X_t^k)$ as follows:

$$\begin{aligned}\rho(X_t^k) = &\sum_{\substack{m_1, m_2, \dots, m_{2n} \geq 1,\, 2n \leq k \\ \sum_{i=1}^n m_{2i-1} - \sum_{i=1}^n m_{2i} = 0}} \rho\big((\ell_t^*)^{m_{2n}} \ell_t^{m_{2n-1}} \cdots (\ell_t^*)^{m_2} \ell_t^{m_1}\big) \\ &+ \sum_{\substack{m_1, m_2, \dots, m_{2n+1} \geq 1,\, 2n+1 \leq k \\ \sum_{i=1}^{n+1} m_{2i-1} - \sum_{i=1}^n m_{2i} = 0}} \rho\big(\ell_t^{m_{2n+1}} (\ell_t^*)^{m_{2n}} \ell_t^{m_{2n-1}} \cdots (\ell_t^*)^{m_2} \ell_t^{m_1}\big).\end{aligned}$$

However, it is easy to see that (each term of) the second sum is equal to 0, since

$$\ell_t^{m_{2n+1}} (\ell_t^*)^{m_{2n}} \ell_t^{m_{2n-1}} \cdots (\ell_t^*)^{m_2} \ell_t^{m_1}\Omega \in \mathcal{F}\big(L^2([0,\infty))\big) \ominus \mathbb{C}\Omega$$

if $(\ell_t^*)^{m_{2n}} \ell_t^{m_{2n-1}} \cdots (\ell_t^*)^{m_2} \ell_t^{m_1}\Omega \neq 0$. Also, if there exists a $j \in [n]$ such that $\sum_{i=1}^j m_{2i} > \sum_{i=1}^j m_{2i-1}$, then $\rho\big(\ell_t^{m_{2n+1}} (\ell_t^*)^{m_{2n}} \ell_t^{m_{2n-1}} \cdots (\ell_t^*)^{m_2} \ell_t^{m_1}\big) = 0$ because

$$(\ell_t^*)^{m_{2j}} \ell_t^{m_{2j-1}} \cdots (\ell_t^*)^{m_2} \ell_t^{m_1}\Omega = 0$$

whenever $\sum_{i=1}^{j} m_{2i} > \sum_{i=1}^{j} m_{2i-1}$. Hence,

$$\rho(X_t^k) = \sum_{\substack{m_1, m_2, \ldots, m_{2n} \geq 1,\, 2n \leq k \\ \sum_{i=1}^{n} m_{2i-1} - \sum_{i=1}^{n} m_{2i} = 0 \\ \sum_{i=1}^{j} m_{2i} \leq \sum_{i=1}^{j} m_{2i-1} \text{ for any } j \in [n-1]}} \rho\big((\ell_t^*)^{m_{2n}} \ell_t^{m_{2n-1}} \cdots (\ell_t^*)^{m_2} \ell_t^{m_1}\big)$$

Here, note that $m_1, \ldots, m_{2n}$ in each term above satisfies that $m_1 + \cdots + m_n = k$ and that

$$(\ell_t^*)^{m_2} \ell_t^{m_1} \Omega = t^{m_2} 1_{[0,t]}^{\otimes(m_1 - m_2)}$$

$$(\ell_t^*)^{m_4} \ell_t^{m_3} (\ell_t^*)^{m_2} \ell_t^{m_1} \Omega = t^{m_2} (\ell_t^*)^{m_4} 1_{[0,t]}^{\otimes(m_1 + m_3 - m_2)} = t^{m_2 + m_4} 1_{[0,t]}^{\otimes(m_1 + m_3 - m_2 - m_4)}$$

$$\vdots$$

$$(\ell_t^*)^{m_{2n}} \ell_t^{m_{2n-1}} \cdots (\ell_t^*)^{m_2} \ell_t^{m_1} \Omega = t^{\sum_{i=1}^{n} m_{2i}} 1_{[0,t]}^{\otimes(\sum_{i=1}^{n} m_{2i-1} - \sum_{i=1}^{n} m_{2i})} = t^{\frac{k}{2}} \Omega,$$

where $\sum_{i=1}^{n} m_{2i} \,(= \sum_{i=1}^{n} m_{2i-1}) = \frac{k}{2}$ and $1_{[0,t]}^{\otimes 0} = \Omega$. Thus, we can see that

$$\rho\big((\ell_t^*)^{m_{2n}} \ell_t^{m_{2n-1}} \cdots (\ell_t^*)^{m_2} \ell_t^{m_1}\big) = t^{\frac{k}{2}} \cdot 1$$

for any $m_1, \ldots, m_{2n} \geq 1$ with $2n \leq k$ such that $\sum_{i=1}^{n} m_{2i-1} - \sum_{i=1}^{n} m_{2i} = 0$ and that $\sum_{i=1}^{j} m_{2i} \leq \sum_{i=1}^{j} m_{2i-1}$ for any $j \in [n-1]$. By the way, it is known that the number of tuples of positive number $(m_1, \ldots, m_{2n})$ with $2n \leq k$ such that $\sum_{i=1}^{n} m_{2i-1} - \sum_{i=1}^{n} m_{2i} = 0$ and that $\sum_{i=1}^{j} m_{2i} \leq \sum_{i=1}^{j} m_{2i-1}$ for any $j \in [n-1]$ is equal to the Catalan number $C_{\frac{k}{2}}$. Therefore, $\rho(X_t^k) = t^{\frac{k}{2}} C_{\frac{k}{2}}$ whenever k is even, as desired.

The proof for $X_t - X_s$ is similar to the above argument, since $X_t - X_s = \ell_{1_{(s,t]}} + \ell^*_{1_{(s,t]}}$. □

Next, we show the free independence of $X_t - X_s$ and $\mathcal{A}_s$ for any $t \geq s \geq 0$. For that, we introduce a special polynomial, whose behavior with respect to semi-circular distribution is very good.

Definition 3.5. *Chebyshev polynomials of the second kind* are polynomials $(U_n)_{n=0}^{\infty}$ such that each U_n is a polynomial of degree n with $U_{n+1}(X) = XU_n(X) - U_{n-1}(X)$ and $U_0(X) = 1$, $U_1(X) = X$.

Remark 3.6. Chebyshev polynomials of the second kind $(U_n)_{n=0}^{\infty}$ is a basis of one-variable $\mathbb{C}$-coefficients polynomial ring $\mathbb{C}\langle X \rangle$.

Lemma 3.7. *Let $(U_n)_{n=0}^{\infty}$ be Chebyshev polynomials of the second kind. Let also $(I_j)_{j=1}^{k}$ be a family of interval of $[0, \infty)$ such that $I_j \cap I_{j+1} = \emptyset$, and set $X_{I_j} = \ell_{1_{I_j}} + \ell^*_{1_{I_j}}$ for any $j \in [k]$. Then, we have*

$$U_{n_k}\left(\frac{X_{I_k}}{\sqrt{|I_k|}}\right) \cdots U_{n_2}\left(\frac{X_{I_2}}{\sqrt{|I_2|}}\right) U_{n_1}\left(\frac{X_{I_1}}{\sqrt{|I_1|}}\right) \Omega$$

$$= \left(\frac{1_{I_k}}{\sqrt{|I_k|}}\right)^{\otimes n_k} \otimes \cdots \otimes \left(\frac{1_{I_2}}{\sqrt{|I_2|}}\right)^{\otimes n_2} \otimes \left(\frac{1_{I_1}}{\sqrt{|I_1|}}\right)^{\otimes n_1}$$

for any $n_1, \ldots, n_k \in \mathbb{N}$ (with $n_j \neq 0$ for any $j \in [k]$), where $|I_j|$ denotes the length of the interval I_j.

Proof. Note that each $\frac{X_{I_j}}{\sqrt{|I_j|}}$ is a standard semi-circular element, denoted by $\overline{X}_{I_j}$ bellow. We also set $\overline{1}_{I_j} = \frac{1_{I_j}}{\sqrt{|I_j|}}$. It is clear that $U_1(\overline{X}_{I_1})\Omega = \overline{1}_{I_1}$ by definition. If we can show that $U_n(\overline{X}_{I_1})\Omega = \overline{1}_{I_1}^{\otimes n}$ for any $n \le n_1$ ($n_1 \in \mathbb{N}$), then observe that

$$\begin{aligned} U_{n_1+1}(\overline{X}_{I_1})\Omega &= \overline{X}_{I_1} U_{n_1}(\overline{X}_{I_1})\Omega - U_{n_1-1}(\overline{X}_{I_1})\Omega \\ &= \frac{\ell_{I_1} + \ell_{I_1}^*}{\sqrt{|I_1|}} \overline{1}_{I_1}^{\otimes n_1} - \overline{1}_{I_1}^{\otimes(n_1-1)} \\ &= \overline{1}_{I_1}^{\otimes(n_1+1)} + \overline{1}_{I_1}^{\otimes(n_1-1)} - \overline{1}_{I_1}^{\otimes(n_1-1)} = \overline{1}_{I_1}^{\otimes(n_1+1)}. \end{aligned}$$

Assume that we have shown that

$$U_{n_j}(\overline{X}_{I_j}) \cdots U_{n_2}(\overline{X}_{I_2}) U_{n_1}(\overline{X}_{I_1})\Omega = \overline{1}_{I_j}^{\otimes n_j} \otimes \cdots \otimes \overline{1}_{I_2}^{\otimes n_2} \otimes \overline{1}_{I_1}^{\otimes n_1}$$

for any $n_1, \ldots, n_j \in \mathbb{N}$ ($1 \le j < k$). Then, observe that

$$\begin{aligned} &U_1(\overline{X}_{I_{j+1}}) U_{n_j}(\overline{X}_{I_j}) \cdots U_{n_2}(\overline{X}_{I_2}) U_{n_1}(\overline{X}_{I_1})\Omega \\ &= \overline{X}_{I_{j+1}} \overline{1}_{I_j}^{\otimes n_j} \otimes \cdots \otimes \overline{1}_{I_2}^{\otimes n_2} \otimes \overline{1}_{I_1}^{\otimes n_1} = \overline{1}_{I_{j+1}} \otimes \overline{1}_{I_j}^{\otimes n_j} \otimes \cdots \otimes \overline{1}_{I_2}^{\otimes n_2} \otimes \overline{1}_{I_1}^{\otimes n_1}. \end{aligned}$$

If we can show that

$$U_n(\overline{X}_{I_{j+1}}) U_{n_j}(\overline{X}_{I_j}) \cdots U_{n_2}(\overline{X}_{I_2}) U_{n_1}(\overline{X}_{I_1})\Omega = \overline{1}_{I_{j+1}}^{\otimes n} \otimes \overline{1}_{I_j}^{\otimes n_j} \otimes \cdots \otimes \overline{1}_{I_2}^{\otimes n_2} \otimes \overline{1}_{I_1}^{\otimes n_1}$$

for any $n_1, \ldots, n_j \in \mathbb{N}$ and $n \le n_{j+1}$, then

$$\begin{aligned} &U_{n_{j+1}+1}(\overline{X}_{I_{j+1}}) U_{n_j}(\overline{X}_{I_j}) \cdots U_{n_2}(\overline{X}_{I_2}) U_{n_1}(\overline{X}_{I_1})\Omega \\ &= \overline{X}_{I_{j+1}} U_{n_{j+1}}(\overline{X}_{I_{j+1}}) U_{n_j}(\overline{X}_{I_j}) \cdots U_{n_2}(\overline{X}_{I_2}) U_{n_1}(\overline{X}_{I_1})\Omega \\ &\quad - U_{n_{j+1}-1}(\overline{X}_{I_{j+1}}) U_{n_j}(\overline{X}_{I_j}) \cdots U_{n_2}(\overline{X}_{I_2}) U_{n_1}(\overline{X}_{I_1})\Omega \\ &= \overline{X}_{I_{j+1}} \overline{1}_{I_{j+1}}^{\otimes n_{j+1}} \otimes \overline{1}_{I_j}^{\otimes n_j} \otimes \cdots \otimes \overline{1}_{I_2}^{\otimes n_2} \otimes \overline{1}_{I_1}^{\otimes n_1} - \overline{1}_{I_{j+1}}^{\otimes n_{j+1}-1} \otimes \overline{1}_{I_j}^{\otimes n_j} \otimes \cdots \otimes \overline{1}_{I_2}^{\otimes n_2} \otimes \overline{1}_{I_1}^{\otimes n_1} \\ &= \overline{1}_{I_{j+1}}^{\otimes n_{j+1}+1} \otimes \overline{1}_{I_j}^{\otimes n_j} \otimes \cdots \otimes \overline{1}_{I_2}^{\otimes n_2} \otimes \overline{1}_{I_1}^{\otimes n_1} \end{aligned}$$

as desired. □

Using this lemma, we show the following:

Lemma 3.8. *For any $t \ge s \ge 0$, $X_t - X_s$ is freely independent with $\mathcal{A}_s$ in the W^*-probability space $(\mathcal{W}, \tau)$.*

Proof. The cases when $t = s$ or $s = 0$ are obvious. Assume that $t > s > 0$. Recall that $\mathcal{A}_s = W^*\big(\mathcal{B} \cup \{X_u \,|\, s \ge u \ge 0\}\big)$ is a unital von Neumann subalgebra of $(\mathcal{A}, \tau)$ with $\mathcal{A} = W^*_{\mathrm{red}}(\mathcal{B} * \mathcal{W})$. By Remark 2.5, it suffices to show that $\mathcal{B}$, $\{X_u \,|\, s \ge u > 0\}$ and $\{X_t - X_s\}$ are freely independent in $(\mathcal{A}, \tau)$. By the way, $(\mathcal{A}, \tau)$ is the reduced free product von Neumann algebra of $(\mathcal{B}, \omega)$ and $(\mathcal{W}, \rho)$. Thus, $\mathcal{B}$ and $\{X_u \,|\, s \ge u > 0\}$ (resp. $\{X_t - X_s\}$) are freely independent in $(\mathcal{A}, \tau)$. Hence, the remainder is to see that $\{X_u \,|\, s \ge u \ge 0\}$ and $\{X_t - X_s\}$ are freely independent in $(\mathcal{W}, \rho)$.

Recall that $X_t - X_s = X_{1_{(s,t]}}$. Let $u_1, \ldots, u_k > 0$ be chosen so that $s \ge u_1, \ldots, u_k$. Let also $P_1, \ldots, P_k, Q_1, \ldots, Q_k \in \mathbb{C}\langle X \rangle$ be arbitrarily chosen so that $\rho(P_j(X_{1_{(s,t]}})) = 0$ and $\rho(Q_j(X_{u_j})) = 0$. By Remark 3.6, for any $j \in [k]$, we can write

$$P_j(X_{1_{(s,t]}}) = \sum_{n \ge 0}^{\text{finite}} c(P_j; n) U_n\left(\frac{X_{1_{(s,t]}}}{\sqrt{t-s}}\right), \quad Q_j(X_{u_j}) = \sum_{n \ge 0}^{\text{finite}} d(Q_j; n) U_n\left(\frac{X_{u_j}}{\sqrt{u_j}}\right),$$

where $c(P_j;n)$ and $d(Q_j;n)$ are some complex numbers. By Lemma 3.7, we have

$$U_{n_k}\left(\frac{X_{1_{(s,t]}}}{\sqrt{t-s}}\right)U_{m_k}\left(\frac{X_{u_k}}{\sqrt{u_k}}\right)\cdots U_{n_1}\left(\frac{X_{1_{(s,t]}}}{\sqrt{t-s}}\right)U_{m_1}\left(\frac{X_{u_1}}{\sqrt{u_1}}\right)\Omega$$
$$=\left(\frac{1_{(s,t]}}{\sqrt{t-s}}\right)^{\otimes n_k}\otimes\left(\frac{1_{[0,u_k]}}{\sqrt{u_k}}\right)^{\otimes m_k}\otimes\cdots\otimes\left(\frac{1_{(s,t]}}{\sqrt{t-s}}\right)^{\otimes n_1}\otimes\left(\frac{1_{[0,u_1]}}{\sqrt{u_1}}\right)^{\otimes m_1}$$

for any $n_1,\dots,n_k,m_1,\dots,m_k\in\mathbb{N}$. Hence, for any $n_1,\dots,n_k,m_1,\dots,m_k\in\mathbb{N}$, we have

$$\rho\left(U_{n_k}\left(\frac{X_{1_{(s,t]}}}{\sqrt{t-s}}\right)U_{m_k}\left(\frac{X_{u_k}}{\sqrt{u_k}}\right)\cdots U_{n_1}\left(\frac{X_{1_{(s,t]}}}{\sqrt{t-s}}\right)U_{m_1}\left(\frac{X_{u_1}}{\sqrt{u_1}}\right)\right)=0$$

by the definition of the vacuum state ρ. Thus, we have

$$\begin{aligned}
&\rho\left(P_k(X_{1_{(s,t]}})Q_k(X_{u_k})\cdots P_1(X_{1_{(s,t]}})Q_1(X_{u_1})\right)\\
&=\sum_{n_1,\dots,n_k,m_1,\dots,m_k\geq 1} c(P_1;n_1)\cdots c(P_k;n_k)d(Q_1;m_1)\cdots d(Q_k;m_k)\\
&\qquad\rho\left(U_{n_k}\left(\frac{X_{1_{(s,t]}}}{\sqrt{t-s}}\right)U_{m_k}\left(\frac{X_{u_k}}{\sqrt{u_k}}\right)\cdots U_{n_1}\left(\frac{X_{1_{(s,t]}}}{\sqrt{t-s}}\right)U_{m_1}\left(\frac{X_{u_1}}{\sqrt{u_1}}\right)\right)\\
&+\sum_{\substack{\exists j\in[k-1]:n_j=0\\ \text{or }\exists j\in[k]\setminus\{1\}:m_j=0}} c(P_1;n_1)\cdots c(P_k;n_k)d(Q_1;m_1)\cdots d(Q_k;m_k)\\
&\qquad\rho\left(U_{n_k}\left(\frac{X_{1_{(s,t]}}}{\sqrt{t-s}}\right)U_{m_k}\left(\frac{X_{u_k}}{\sqrt{u_k}}\right)\cdots U_{n_1}\left(\frac{X_{1_{(s,t]}}}{\sqrt{t-s}}\right)U_{m_1}\left(\frac{X_{u_1}}{\sqrt{u_1}}\right)\right)\\
&=\sum_{\substack{\exists j\in[k-1]:n_j=0\\ \text{or }\exists j\in[k]\setminus\{1\}:m_j=0}} c(P_1;n_1)\cdots c(P_k;n_k)d(Q_1;m_1)\cdots d(Q_k;m_k)\\
&\qquad\rho\left(U_{n_k}\left(\frac{X_{1_{(s,t]}}}{\sqrt{t-s}}\right)U_{m_k}\left(\frac{X_{u_k}}{\sqrt{u_k}}\right)\cdots U_{n_1}\left(\frac{X_{1_{(s,t]}}}{\sqrt{t-s}}\right)U_{m_1}\left(\frac{X_{u_1}}{\sqrt{u_1}}\right)\right).
\end{aligned}$$

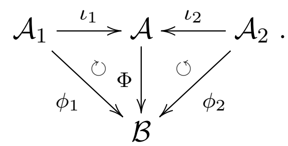
By the way, according to the assumption of $\rho(P_j(X_{1_{(s,t]}}))=0$ and $\rho(Q_j(X_{u_j}))=0$, we can see that $c(P_j;0)=0$ and $d(Q_j;0)=0$ for any $j\in[k]$ by Lemma 3.7. This implies the most right-hand side above is equal to 0, and hence $\rho\left(P_k(X_{1_{(s,t]}})Q_k(X_{u_k})\cdots P_1(X_{1_{(s,t]}})Q_1(X_{u_1})\right)=0$. Other cases follow similarly to the above discussion. Therefore, the proof is completed. □

By Lemmas 3.4 and 3.8, $(X_t)_{t\geq 0}$ is actually an $(\mathcal{A}_t)_{t\geq 0}$-free Brownian motion.

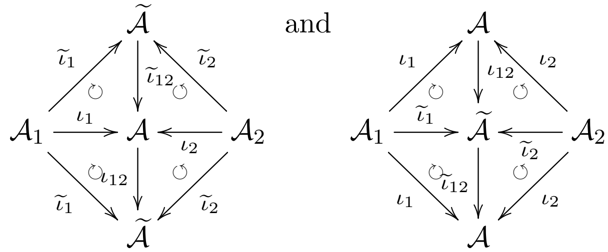
Appendix A. Free product of unital algebras

For the reader's convenience, we give a short explanation for free products of unital algebras. A main reference about this subject is [BMM95].

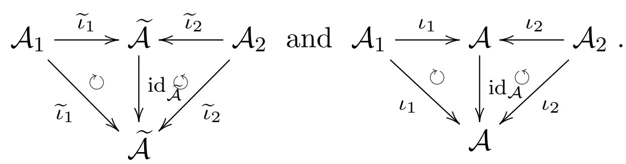
A.1. **Universality.** Let $\mathcal{A}_1$ and $\mathcal{A}_2$ be unital algebras over $\mathbb{C}$. A *free product unital algebra* of $\mathcal{A}_1$ and $\mathcal{A}_2$ is defined to be a triple $(\mathcal{A},\iota_1,\iota_2)$ of unital algebra $\mathcal{A}$ and unital homomorphisms $\iota_j:\mathcal{A}_j\to\mathcal{A}$ ($j\in[2]$) with the following universality: For any unital algebra $\mathcal{B}$ and any unital homomorphisms $\phi_j:\mathcal{A}_j\to\mathcal{B}$, there exists a unique unital homomorphism $\Phi:\mathcal{A}\to\mathcal{B}$ such that $\phi_j = \Phi \circ \iota_j$ for each $j \in [2]$, that is,



By this universality, we can see that a free product of $\mathcal{A}_1$ and $\mathcal{A}_2$ is unique up to isomorphism. Indeed, if $(\widetilde{\mathcal{A}}, \widetilde{\iota}_1, \widetilde{\iota}_2)$ also satisfies the condition to be a free product of $\mathcal{A}_1$ and $\mathcal{A}_2$, then, by universality, there exist uniquely unital homomorphisms $\iota_{12} : \mathcal{A} \to \widetilde{\mathcal{A}}$ and $\widetilde{\iota}_{12} : \widetilde{\mathcal{A}} \to \mathcal{A}$ such that



Also, we also have the following trivial commuting diagrams:



By the uniqueness, these mean that $\iota_{12} \circ \widetilde{\iota}_{12} = \mathrm{id}_{\widetilde{\mathcal{A}}}$ and $\widetilde{\iota_{12}} \circ \iota_{12} = \mathrm{id}_{\mathcal{A}}$. Therefore, $\mathcal{A}$ and $\widetilde{\mathcal{A}}$ are isomorphism as algebras.

Hence, we may denote by $\mathcal{A}_1 *_{\mathrm{alg}} \mathcal{A}_2$ a free product unital algebra of $\mathcal{A}_1$ and $\mathcal{A}_2$.

Remark A.1. By [BMM95, Remark 1.4.1], if $(\mathcal{A}, \iota_1, \iota_2)$ is a free product of $\mathcal{A}_1$ and $\mathcal{A}_2$, then two homomorphisms $\iota_j : \mathcal{A}_j \to \mathcal{A}$ $(j \in [2])$ are injective. Thus, we can identify $\mathcal{A}_1$ and $\mathcal{A}_2$ as subalgebras of $\mathcal{A}$.

Remark A.2. If $\mathcal{A}_1$ and $\mathcal{A}_2$ are $*$-algebras, then their free product is naturally equipped with an involution $*$ induced from $\mathcal{A}_1$ and $\mathcal{A}_2$, and unital homomorphisms in the definition of free product are replaced by unital $*$-homomorphisms.

A.2. **Realization.** Finally, we see one concrete realization, because the definition of free product is very abstract and the existence is non-trivial. The following construction appears in [BMM95].

Let $\mathcal{A}_1$ and $\mathcal{A}_2$ be unital algebras. Let $\mathcal{T}$ be the tensor algebra generated by $\mathcal{A}_1$ and $\mathcal{A}_2$, that is,

$$\mathcal{T} = \bigoplus_{n \geq 1}^{\mathrm{alg}} \bigoplus_{i_1, \ldots, i_n \in \{1,2\}}^{\mathrm{alg}} \mathcal{A}_{i_1} \otimes \mathcal{A}_{i_2} \otimes \cdots \otimes \mathcal{A}_{i_n}.$$

Let $\mathcal{I}$ be the two-sided ideal of $\mathcal{T}$ generated by the following elements:

$$a_1 \otimes a_1 - a_1 a_1', \quad a_2 \otimes a_2' - a_2 a_2', \quad 1_{\mathcal{A}_1} - 1_{\mathcal{A}_2}$$

for all $a_1, a_1' \in \mathcal{A}_1$ and $a_2, a_2' \in \mathcal{A}_2$. We denote by $\mathcal{A}$ the quotient algebra $\mathcal{T}/\mathcal{I}$. Let also α_i be the natural homomorphism $\mathcal{A}_i \to \mathcal{A}$, that is,

$$\begin{array}{ccccc} & & & q & \\ \mathcal{A}_i & \to & \mathcal{T} & \to & \mathcal{A} \\ a & \mapsto & a & \mapsto & q(a), \end{array}$$

where $q : \mathcal{T} \to \mathcal{A}$ is the quotient map. Then, the triple $(\mathcal{A}, \alpha_1, \alpha_2)$ is a free product unital algebra of $\mathcal{A}_1$ and $\mathcal{A}_2$. This follows essentially from the universality of tensor product.

References

[BMM95] K. I. Beidar, W. S. Martindale III & A. V. Mikhalev, *Rings with Generalized Identities*, CRC press, 1995.

[BS98] P. Biane and R. Speicher, Stochastic calculus with respect to free Brownian motion and analysis on Wigner space, *Probab. Theory Relat. Fields* **112**, 373-409 (1998).

[R24] S. Richard, Lecture note of "Introduction to Stochastic Calculus", 2024. http://www.math.nagoya-u.ac.jp/~richard/teaching/s2024/Stochastic.pdf

[V85] D. Voiculescu, Symmetries of some reduced free product C*-algebras, *Operator Algebras and Their Connections with Topology and Ergodic Theory*, Lecture Notes in Math., vol. **1132** (Springer-Verlag,1985), pp.556–588.

[VDN92] D. Voiculescu, K. Dykema & A. Nica, *Free Random Variables*, CRM Monograph Series **1**, American Mathematical Society, Providence, Rhode Island, 1992.

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