# Martingales

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**Abstract**

This report introduces the basic theory of martingale in probability space equipped with filtration. About discrete time martingale, the proof of Doob's decomposition theorem is given. While in continuous time martingale, stopping time and stopping theorem is introduced. As a prototype of stochastic integral, martingale transform is defined.

## 1 Definitions

All the definitions below are based on probability space $(\Omega, \mathcal{F}, P)$ and $\mathcal{T} \subset \mathbb{R}_+$.

**Definition 1.1.** A **filtration** $(\mathcal{F}_t)_{t\in\mathcal{T}}$ is a family of $\sigma$-subalgebras of $\mathcal{F}$ satisfying $\mathcal{F}_s \subset \mathcal{F}_t$ whenever $s \leq t$.

**Definition 1.2.** A **stochastic process** consists of the tuple

$$X := (\Omega, \mathcal{F}, P, (\mathcal{F}_t)_{t\in\mathcal{T}}, (X_t)_{t\in\mathcal{T}})$$

with filtration $(\mathcal{F}_t)_{t\in\mathcal{T}}$ and $(X_t)_{t\in\mathcal{T}}$ a family of random variables on $\Omega$, taking values in measurable space $(S, \mathcal{S})$, and $X_t$ is measurable with respect to $\mathcal{F}_t$.

**Definition 1.3.** For $\mathcal{T} \subset \mathbb{R}_+$, a real-valued stochastic process

$$(\Omega, \mathcal{F}, P, (\mathcal{F}_t)_{t\in\mathcal{T}}, (M_t)_{t\in\mathcal{T}})$$

satisfying $M_t \in L^1(\Omega, \mathcal{F}, P)$ for $\forall\ t \in \mathcal{T}$ is a **martingale** if $E[M_t|\,\mathcal{F}_s] = M_s$ for all $s \leq t$. It is a **supermartingale** if $E[M_t|\,\mathcal{F}_s] \leq M_s$, and is a **submartingale** if $E[M_t|\,\mathcal{F}_s] \geq M_s$.

According to definition, if $(M_t)_{t\in\mathcal{T}}$ is a supermartingale, then $(-M_t)_{t\in\mathcal{T}}$ is a submartingale. To understand martingale, the following easy examples are given.

**Example 1.4** (pp.26 Exercise 3.2.2.). Let $X$ be a univariate random variable on probability space $(\Omega, \mathcal{F}, P)$ with $E[|X|] < \infty$ and $(\mathcal{F}_t)_{t\in\mathcal{T}}$ be a filtration. Set $X_t := E[X|\,\mathcal{F}_t]$, then $(\Omega, \mathcal{F}, P, (\mathcal{F}_t)_{t\in\mathcal{T}}, (X_t)_{t\in\mathcal{T}})$ is a martingale.

*Proof.* For all $s \leq t$,

$$\begin{aligned} E[X_t|\,\mathcal{F}_s] &= E[E[X|\,\mathcal{F}_t]|\,\mathcal{F}_s] \\ &= E[X|\,\mathcal{F}_s] \quad \text{(since } \mathcal{F}_s \subset \mathcal{F}_t \text{ and tower property)} \\ &= X_s \end{aligned}$$

Hence it is a martingale. □

**Example 1.5** (pp.26 Exercise 3.2.3.). Consider $\mathcal{T} = \mathbb{N}$ and a sequence $(X_n)_{n\in\mathbb{N}}$ of independent and real-valued random variables satisfying $E[|X_n|] < \infty$ and $E[X_n] = 0$. Set $Y_n := \sum_{j=1}^{n} X_j$, then $(Y_j)_{j\in\mathbb{N}}$ and the natural filtration define a martingale.

*Proof.* Let $\mathcal{F}_m = \sigma(Y_1, \ldots, Y_m)$, we can also know $\mathcal{F}_m = \sigma(X_1, \ldots, X_m)$. If $n > m$, write $Y_n = Y_m + X_{m+1} + \cdots + X_n$, since random variables $X_{m+1}, \ldots, X_n$ are of mean 0 and independent of $\mathcal{F}_m$,

$$\begin{aligned} E[Y_n|\,\mathcal{F}_m] &= E[Y_m|\,\mathcal{F}_m] + E[X_{m+1} + \cdots + X_n|\,\mathcal{F}_m] \\ &= Y_m + E[X_{m+1} + \cdots + X_n] \\ &= Y_m \end{aligned}$$

for every $n > m$. Hence $(Y_j)_{j\in\mathbb{N}}$ and natural filtration define a martingale. □

**Example 1.6** (pp.27 Exercise 3.2.4.). The standard 1-dimensional Brownian motion is a martingale. Standard Brownian motion means the Brownian motion endowed with the right-continuous filtration generated by the augmented natural filtration.

*Proof.* Let $B = (\Omega, \mathcal{F}, P, (\mathcal{F}_t)_{t\in\mathcal{T}}, (B_t)_{t\in\mathcal{T}})$ be standard one dimensional Brownian motion. For all $s \leq t$,

$$E[B_t|\,\mathcal{F}_s] = E[B_t - B_s|\,\mathcal{F}_s] + E[B_s|\,\mathcal{F}_s] = E[B_t - B_s] + B_s = B_s$$

since $B_t - B_s$ is independent of $\mathcal{F}_s$ and its mean is 0. □

One can understand the statement below as alternative definition of martingale.

**Definition 1.7.** A real-valued stochastic process $(\Omega, \mathcal{F}, P, (\mathcal{F}_t)_{t\in\mathcal{T}}, (M_t)_{t\in\mathcal{T}})$ satisfying $M_t \in L^1(\Omega, \mathcal{F}, P)$ for $\forall\, t \in \mathcal{T}$ is a martingale if $E[(M_t - M_s)|\, \mathcal{F}_s] = 0$ for all $s \leq t$.

To see this, if $E[(M_t - M_s)|\, \mathcal{F}_s] = 0$ for all $s \leq t$, then it is equivalent to $E[M_t|\, \mathcal{F}_s] = E[M_s|\, \mathcal{F}_s]$ under $s \leq t$. Since $M_s$ is $\mathcal{F}_s$-measurable, according to conditional expectation, $E[M_s|\, \mathcal{F}_s] = M_s$. Hence for $s \leq t$, $E[M_t|\, \mathcal{F}_s] = M_s$, meaning this stochastic process is indeed a martingale.

# 2 Doob's decomposition theorem

In this section, we will state an important theorem of discrete time martingales, that is the situation $\mathcal{T} = \mathbb{N}$. A special process needs to be introduced.

**Definition 2.1.** A discrete time stochastic process

$$(\Omega, \mathcal{F}, P, (\mathcal{F}_n)_{n\in\mathbb{N}}, (A_n)_{n\in\mathbb{N}})$$

is said to be an **increasing predictable process** if $A_0 = 0$, $A_n \leq A_{n+1}$, and $A_{n+1}$ is $\mathcal{F}_n$-measurable for all $n \in \mathbb{N}$.

The $\mathcal{F}_n$-measurability of $A_{n+1}$ roughly means that at time $n$, we know the value of the process at time $n+1$.

**Theorem 2.2** (Doob's decomposition theorem)**.** *Let $(\Omega, \mathcal{F}, P, (\mathcal{F}_n)_{n\in\mathbb{N}}, (X_n)_{n\in\mathbb{N}})$ be a submartingale, then there exsits a unique decomposition $X_n = M_n + A_n$ where $(M_n)_{n\in\mathbb{N}}$ a martingale and $(A_n)_{n\in\mathbb{N}}$ an increasing predictable process, which is also called the compensator.*

*Proof.* We can define process $(A_n)_{n\in\mathbb{N}}$ recursively as $A_0 = 0$, $A_{n+1} = A_n + E[X_{n+1}|\, \mathcal{F}_n] - X_n$ for $n \geq 0$. Since $(X_n)_{n\in\mathbb{N}}$ is a submartingale, the term $E[X_{n+1}|\, \mathcal{F}_n] - X_n$ is non-negative. So $(A_n)_{n\in\mathbb{N}}$ is increasing sequence. Note that $A_{n+1}$ is sum of $\mathcal{F}_n$-measurable random variables, by construction it is an increasing predictable process. As $A_{n+1}$ is $\mathcal{F}_n$-measurable, the term $E[X_{n+1}|\, \mathcal{F}_n] - X_n$ can be rewritten as

$$E[X_{n+1} - A_{n+1}|\, \mathcal{F}_n] = E[X_{n+1}|\, \mathcal{F}_n] - A_{n+1} = X_n - A_n$$

Define the right side as $M_n = X_n - A_n$. Then $(M_n)_{n\in\mathbb{N}}$ is a martingale from for every $n \in \mathbb{N}$, $E[M_{n+1}|\, \mathcal{F}_n] = M_n$. And $M_n$ is measurable respect to $\mathcal{F}_n$. Hence for submartingale $(X_n)_{n\in\mathbb{N}}$, it can be decomposed into the sum of a martingale $(M_n)_{n\in\mathbb{N}}$ and an increasing predictable process $(A_n)_{n\in\mathbb{N}}$.

Next show the uniqueness. Suppose $X_n = M'_n + A'_n$ is another composition where $(M'_n)_{n\in\mathbb{N}}$ a martingale and $(A'_n)_{n\in\mathbb{N}}$ an increasing predictable process. We have

$$A'_{n+1} - A'_n = X_{n+1} - X_n - (M'_{n+1} - M'_n)$$

Conditioning with respect to $\mathcal{F}_n$ on both sides, we can see

$$E[(A'_{n+1} - A'_n)|\,\mathcal{F}_n] = E[X_{n+1}|\,\mathcal{F}_n] - E[X_n|\,\mathcal{F}_n] - E[(M'_{n+1} - M'_n)|\,\mathcal{F}_n]$$

Since $(M'_n)_{n\in\mathbb{N}}$ is a martingale and $X_n$ is $\mathcal{F}_n$-measurable, the term $E[(M'_{n+1} - M'_n)|\,\mathcal{F}_n]$ becomes 0, and the term $E[X_n|\,\mathcal{F}_n]$ is $X_n$. Hence,

$$A'_{n+1} - A'_n = E[(A'_{n+1} - A'_n)|\,\mathcal{F}_n] = E[X_{n+1}|\,\mathcal{F}_n] - X_n = A_{n+1} - A_n$$

as $A_0 = A'_0 = 0$, by recurrence it follows that $A'_n = A_n$ and $M'_n = M_n$ for every $n \in \mathbb{N}$. The uniqueness of decomposition is shown as desired. □

# 3 Stopping time

We introduce a notion of random times well suited to the filtration $(\mathcal{F}_t)_{t\in\mathcal{T}}$.

**Definition 3.1.** Let $(\mathcal{F}_t)_{t\in\mathcal{T}}$ be a filtration on probability space $(\Omega, \mathcal{F}, P)$. A random variable $\tau : \Omega \to \mathcal{T} \cup \{+\infty\}$ on this probability space is **stopping time** (or **Markov time**) for this filtration if for $\forall\ t \in \mathcal{T}$,

$$\{\tau \le t\} \equiv \{\omega \in \Omega \mid \tau(\omega) \le t\} \in \mathcal{F}_t$$

For any stopping time $\tau$, we set $\mathcal{F}_\tau$ for the $\sigma$-subalgebra of $\mathcal{F}$ defined by

$$\mathcal{F}_\tau = \{A \in \mathcal{F}_\mathcal{T} \mid (A \cap \{\tau \le t\}) \in \mathcal{F}_t,\ \forall\ t \in \mathcal{T}\}$$

where $\mathcal{F}_\mathcal{T}$ is the smallest $\sigma$-subalgebra of $\mathcal{F}$ containing all $\mathcal{F}_t$ for $t \in \mathcal{T}$.

There are a few general results about stopping time.

**Example 3.2.** (1) $\tau \equiv c$ where $c$ is constant, is a stopping time.

(2) (**First hitting time**) Assume $(X_t)_{t\in\mathcal{T}}$ is a $\mathbb{R}^N$-valued process, for $A \in \mathcal{B}(\mathbb{R}^N)$, set $\tau_A = \inf\{t > 0 \mid X_t \in A\}$ as first hitting time of $A$. It is a stopping time.

In fact, for every $t \in \mathcal{T}$, $\{\tau_A < t\} = \bigcup_{s\in\mathbb{Q}_+, s<t}\{X_s \in A\} \in \mathcal{F}_t$, $\tau_A$ is indeed stopping time. (This will use the fact $\mathbb{Q}_+$ is dense in $\mathbb{R}_+$.)

**Lemma 3.3.** *Let $\tau, \eta$ be two stopping times for the same filtration.*

*(1) $\tau$ is $\mathcal{F}_\tau$-measurable.*

*(2) $\tau \vee \eta := \max\{\tau, \eta\}$ and $\tau \wedge \eta := \min\{\tau, \eta\}$ are stopping times.*

*(3) If $\eta \leq \tau$, then $\mathcal{F}_\eta \subset \mathcal{F}_\tau$.*

*(4) $\mathcal{F}_{\tau\wedge\eta} = \mathcal{F}_\tau \cap \mathcal{F}_\eta$.*

*Proof.* (1) Just needs to show for $\forall\ s \geq 0$, we have $\{\tau \leq s\} \in \mathcal{F}_\tau$, it is obvious that $\{\tau \leq s\} \in \mathcal{F}_s$. If $t \leq s$, then $\{\tau \leq s\} \cap \{\tau \leq t\} = \{\tau \leq t\} \in \mathcal{F}_t$. If $t > s$, then $\{\tau \leq s\} \cap \{\tau \leq t\} = \{\tau \leq s\} \in \mathcal{F}_s \subset \mathcal{F}_t$, leading to what we want to show.

(2) For every $t \in \mathcal{T}$, it is easy to see

$$\{\tau \vee \eta \leq t\} = \{\tau \leq t\} \cap \{\eta \leq t\} \in \mathcal{F}_t$$

and

$$\{\tau \wedge \eta \leq t\} = \{\tau \leq t\} \cup \{\eta \leq t\} \in \mathcal{F}_t$$

So they are both stopping times.

(3) For $A \in \mathcal{F}_\eta$, then for every $t$, $A \cap \{\eta \leq t\} \in \mathcal{F}_t$. As $\{\tau \leq t\} \subset \{\eta \leq t\}$, we have

$$A \cap \{\tau \leq t\} = A \cap \{\eta \leq t\} \cap \{\tau \leq t\}$$

Since $A \cap \{\eta \leq t\} \in \mathcal{F}_t$ and $\{\tau \leq t\} \in \mathcal{F}_t$, their intersection $A \cap \{\tau \leq t\}$ is also belongs to $\mathcal{F}_t$. $A$ is arbitrary in $\mathcal{F}_\eta$, then $\mathcal{F}_\eta \subset \mathcal{F}_\tau$.

(4) By $\min\{\tau, \eta\}$ not greater than $\tau$ or $\eta$ and (3), we have $\mathcal{F}_{\tau\wedge\eta} \subset \mathcal{F}_\tau$ and $\mathcal{F}_{\tau\wedge\eta} \subset \mathcal{F}_\eta$, hence $\mathcal{F}_{\tau\wedge\eta} \subset \mathcal{F}_\tau \cap \mathcal{F}_\eta$. To show its converse, suppose $A \in \mathcal{F}_\tau \cap \mathcal{F}_\eta$, in fact $A$ is in $\mathcal{F}_\mathcal{T}$ and $A \cap \{\tau \leq t\} \in \mathcal{F}_t$, $A \cap \{\eta \leq t\} \in \mathcal{F}_t$, we have

$$A \cap \{\tau \wedge \eta \leq t\} = A \cap (\{\tau \leq t\} \cup \{\eta \leq t\}) = (A \cap \{\tau \leq t\}) \cup (A \cap \{\eta \leq t\}) \in \mathcal{F}_t$$

therefore, $A \in \mathcal{F}_{\tau\wedge\eta}$ and the converse is shown. □

About stopping time, there is a result for martingales and is called stopping theorem. To prove it, we prepare a result in discrete time martingale first without proof.

**Theorem 3.4.** *Let $(\Omega, \mathcal{F}, P, (\mathcal{F}_n)_{n\in\mathbb{N}}, (M_n)_{n\in\mathbb{N}})$ be a martingale and $\tau_1, \tau_2$ two a.s. bounded stopping times of filtration $(\mathcal{F}_n)_{n\in\mathbb{N}}$ with $\tau_1 \leq \tau_2$ a.s. Then $E[M_{\tau_2}|\,\mathcal{F}_{\tau_1}] = M_{\tau_1}$ and $M_{\tau_1}, M_{\tau_2}$ are integrable. The same statement holds for supermartingale with $E[M_{\tau_2}|\,\mathcal{F}_{\tau_1}] \leq M_{\tau_1}$, and for submartingale with $E[M_{\tau_2}|\,\mathcal{F}_{\tau_1}] \geq M_{\tau_1}$.*

**Theorem 3.5** (Stopping theorem)**.** *Let $(\Omega, \mathcal{F}, P, (\mathcal{F}_t)_{t\in\mathcal{T}}, (M_t)_{t\in\mathcal{T}})$ be a right-continuous martingale and let two $\tau_1, \tau_2$ be two a.s. bounded stopping times with $\tau_1 \leq \tau_2$ a.s. Then $E[M_{\tau_2}| \mathcal{F}_{\tau_1}] = M_{\tau_1}$. The same statement holds for supermartingale with $E[M_{\tau_2}| \mathcal{F}_{\tau_1}] \leq M_{\tau_1}$, and for submartingale with $E[M_{\tau_2}| \mathcal{F}_{\tau_1}] \geq M_{\tau_1}$.*

*Proof.* Assume $(M_t)_{t\in\mathcal{T}}$ is martingale. Let $b > 0$ and $\tau$ be stopping time taking only finitely many values $t_1 < t_2 < \cdots < t_n$ and bounded above by $b$. We can discretize continuous time martingale into discrete time martingale $(M_{t_k})_{0\leq k\leq n}$ with respect to filtration $(\mathcal{F}_{t_k})_{0\leq k\leq n}$. We have $E[M_b| \mathcal{F}_\tau] = M_\tau$ and $(M_\tau)_\tau$ for $\tau$ ranging over the set of stopping times taking only finitely many values bounded above by $b$, hence is uniformly integrable.

Let $(\tau_n^1)_n$ and $(\tau_n^2)_n$ be sequences of stopping times taking only finitely many values and decreasing to $\tau_1$ and $\tau_2$ respectively, and $\tau_n^2 \geq \tau_n^1$ a.s. for every $n$. We have $E[M_{\tau_n^2}| \mathcal{F}_{\tau_n^1}] = M_{\tau_n^1}$ by $(M_{\tau_n^2})$ is martingale. Conditioning on both sides with respect to $\mathcal{F}_{\tau_1}$, since $\mathcal{F}_{\tau_1} \subset \mathcal{F}_{\tau_2}$, we get

$$E[M_{\tau_n^2}| \mathcal{F}_{\tau_1}] = E[M_{\tau_n^1}| \mathcal{F}_{\tau_1}] \tag{3.0.1}$$

Since $M_{\tau_n^1} \to M_{\tau_1}$ and $M_{\tau_n^2} \to M_{\tau_2}$ as $n \to \infty$, the convergence also takes place in $L^1$. So $(M_{\tau_n^1})_n$ and $(M_{\tau_n^2})_n$ are uniformly integrable. Therefore taking limit on $n$ leads to result $E[M_{\tau_2}| \mathcal{F}_{\tau_1}] = M_{\tau_1}$.

Now assume $(M_t)_{t\in\mathcal{T}}$ is supermartingale. Replace $=$ with $\leq$ in (3.0.1), we have $E[M_{\tau_n^2}| \mathcal{F}_{\tau_1}] \leq E[M_{\tau_n^1}| \mathcal{F}_{\tau_1}]$. For submartingale case, it is similar. □

# 4 Martingale transform

A discrete time stochastic process $(\Omega, \mathcal{F}, P, (\mathcal{F}_n)_{n\in\mathbb{N}}, (X_n)_{n\in\mathbb{N}})$ is **predictable** if $X_n$ is $\mathcal{F}_{n-1}$-measurable for all $n \geq 1$.

**Definition 4.1.** Let $M := (\Omega, \mathcal{F}, P, (\mathcal{F}_n)_{n\in\mathbb{N}}, (M_n)_{n\in\mathbb{N}})$ be a univariate stochastic process, and let $X := (\Omega, \mathcal{F}, P, (\mathcal{F}_n)_{n\in\mathbb{N}}, (X_n)_{n\in\mathbb{N}})$ be an adapted and predictable univariate stochastic process. For $n \in \mathbb{N}$, define

$$(X \cdot M)_n := X_0M_0 + \sum_{j=1}^{n} X_j(M_j - M_{j-1})$$

If $M$ is a martingale, the above is called **martingale transform**.

We have two simple results below.

**Theorem 4.2.** *Assume $X_n$ is bounded for $n \in \mathbb{N}$, that is,* $\sup_\omega |X_n(\omega)| < \infty$.

*(1) If $M$ is martingale, then $X \cdot M$ is also martingale.*

*(2) If $M$ is submartingale and $X_n \geq 0$, then $X \cdot M$ is also submartingale.*

*Proof.* We can see $(X \cdot M)_n$ is $\mathcal{F}_n$-measurable and integrable due to boundedness of $X$. Moreover, for $n \geq 1$,

$$E[(X \cdot M)_{n+1} - (X \cdot M)_n | \mathcal{F}_n] = E[X_{n+1}(M_{n+1} - M_n) | \mathcal{F}_n] = X_{n+1} E[M_{n+1} - M_n | \mathcal{F}_n]$$

The term $X_{n+1} E[M_{n+1} - M_n | \mathcal{F}_n]$ becomes 0 since $M$ is a martingale, from that we can know

$$E[(X \cdot M)_{n+1} | \mathcal{F}_n] = E[(X \cdot M)_n | \mathcal{F}_n] = (X \cdot M)_n$$

for every $n \in \mathbb{N}$. Hence when $M$ is martingale, $X \cdot M$ is also martingale. For submartingale case it is easy to get the conclusion from inequality. □