Gaussian process

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Summaries

- N-dimensional r.v. $X = (X_1, \ldots, X_N)^T$ on probability space $(\Omega, \mathcal{F}, \mathbb{P})$ is Gaussian vector if for any $a = (a_1, \ldots, a_N)^T \in \mathbb{R}^N$, the r.v. $a \cdot X$ is a Gaussian r.v. on $(\Omega, \mathcal{F}, \mathbb{P})$. The univariate random variables $X_1, \ldots, X_N$ are jointly Gaussian.

- $X = (X_1, \ldots, X_N)$ N-dimensional Gaussian vector with mean $\mathbb{E}(X) = 0 \in \mathbb{R}^N$, then X is degenerate if and only if the Gaussian random variables $X_1, \ldots, X_N$ are linearly independent, there exists $a \in \mathbb{R}^N, a \neq 0$ such that $a \cdot X = 0$.

- Simply consider $t \in \mathcal{T}$ where $\mathcal{T}$ is a subset of $\mathbb{R}$. The family $X := (X_t)_{t\in\mathcal{T}}$ with $X_t : \Omega \to \Lambda$ a random variable on probability space $(\Omega, \mathcal{F}, \mathbb{P})$ is a Gaussian process if and only if for any finite family $\{t_1, \ldots, t_N\} \subset \mathcal{T}$ with $t_j < t_{j+1}$, N-dimensional vector $(X_{t_1}, \ldots, X_{t_N})^T$ is Gaussian vector.

- Some famous Gaussian processes will be discussed in next chapter.

- A filtration $(\mathcal{F}_t)_{t\in\mathcal{T}}$ is a family of σ-subalgebras of $\mathcal{F}$ satisfying $\mathcal{F}_s \subset \mathcal{F}_t$ whenever $s \leq t$.

- A stochastic process consists of $X := (\Omega, \mathcal{F}, \mathbb{P}, (\mathcal{F}_t)_{t\in\mathcal{T}}, (X_t)_{t\in\mathcal{T}})$ with probability space $(\Omega, \mathcal{F}, \mathbb{P})$, filtration $(\mathcal{F}_t)_{t\in\mathcal{T}}$, and a family of random variables $(X_t)_{t\in\mathcal{T}}$ on Ω, taking values in measurable space $(\Lambda, \mathcal{E})$. X_t is measurable with respect to $\mathcal{F}_t$.

- Natural filtration, augmented natural filtration.

- For two stochastic processes X, Y with family of random variables $(X_t)_{t\in\mathcal{T}}$ and $(Y_t)_{t\in\mathcal{T}}$, they are indistinguishable if $\mathbb{P}(\cup_{t\in\mathcal{T}}\{\omega \in \Omega \mid X_t(\omega) \neq Y_t(\omega)\}) = 0$.

- Stochastic process X is progressively measurable if for any $t \in \mathcal{T}$ the map $\mathcal{T} \cap [0,t] \times \Omega \to \Lambda$ i.e. $(s,\omega) \mapsto X_s(\omega)$ is measurable.
- Stochastic process X taking values in standard space $(\Lambda, \mathcal{E})$ is continuous if for every $\omega \in \Omega$ the map $\mathcal{T} \to \Lambda$ i.e. $t \mapsto X_t(\omega)$ is continuous. In other words, X is a.s. continuous if $\mathbb{P}\left(\{\omega \in \Omega \mid \lim_{s \to t} X_s(\omega) = X_t(\omega)\}\right) = 1$.
- A stochastic process $B := (\Omega, \mathcal{F}, \mathbb{P}, (\mathcal{F}_t)_{t \in \mathbb{R}_+}, (B_t)_{t \in \mathbb{R}_+})$ taking values in $\mathbb{R}$ is 1-dimensional Brownian process if (1) $B_0 = 0$ a.s.; (2) For $\forall\ 0 \le s \le t$ the random variable $B_t - B_s$ is independent of $\mathcal{F}_t$; (3) For $\forall\ 0 \le s \le t$ the random variable $B_t - B_s$ is Gaussian random variable $N(0, t-s)$.
- For every $0 \le t_1 < \cdots < t_N$, the N-dimensional vector $B := (B_{t_1}, \ldots, B_{t_n})^T$ is Gaussian vector with $\mathbb{E}(B) = 0$. And $\mathbb{E}(B_t B_s) = t \wedge s$.
- For partition $\mathcal{P}_l = \{t_0^l, t_1^l, \ldots, t_{n_l}^l\}$ in interval $[a,b]$ with $t_0^l = a$ and $t_{n_l}^l = b$ and $t_j^l < t_{j+1}^l$, the variation of $f : [a,b] \to \mathbb{R}$ is $\mathrm{var}_{[a,b]}(f) := \sup_{\mathcal{P}_l} \sum_{j=1}^{n_l} |f(t_j^l) - f(t_{j-1}^l)|$. If $\mathrm{var}(f) < \infty$ then f is said to be of finite variation or bounded variation.

1 Gaussian vector

Exercise 1.1 (pp.12 Exercise 2.1.2.)**.** Check that if X_1, X_2 are independent and standard Gaussian random variables, then $(X_1, X_2)^T$ is a Gaussian vector. Show that the random variable $a_1 X_1 + a_2 X_2$ is a Gaussian random variable with mean 0 and variance $a_1^2 + a_2^2$. Generalize for N independent and standard Gaussian random variables.

Proof. Since X_1, X_2 are independent standard Gaussian r.v., $\mathbb{E}(X_i) = 0$ and $\mathrm{Var}(X_i) = 1$ for $i = 1, 2$. For $a = (a_1, a_2) \neq 0$, the product $a \cdot X = a_1 X_1 + a_2 X_2$. It is easily to see

$$\mathbb{E}(a \cdot X) = \mathbb{E}(a_1 X_1 + a_2 X_2) = a_1 \mathbb{E}(X_1) + a_2 \mathbb{E}(X_2) = 0$$

and

$$\mathrm{Var}(a \cdot X) = \mathrm{Var}(a_1 X_1 + a_2 X_2) = a_1^2 \mathrm{Var}(X_1) + a_2^2 \mathrm{Var}(X_2) = a_1^2 + a_2^2$$

Hence the random variable $a_1 X_1 + a_2 X_2$ is Gaussian r.v. with mean 0 and variance $a_1^2 + a_2^2$. Because it is linear combination of two standard Gaussian

random variables, it itself is again Gaussian random variable. From it we can know (X_1, X_2) is Gaussian vector. Generalize it to N independent standard Gaussian r.v., we will get

$$\mathbb{E}\left(\sum_{j=1}^{N} a_j X_j\right) = \sum_{j=1}^{N} a_j \,\mathbb{E}(X_j) = 0$$

and

$$\mathrm{Var}\left(\sum_{j=1}^{N} a_j X_j\right) = \sum_{j=1}^{N} a_j^2 \,\mathrm{Var}(X_j) = \sum_{j=1}^{N} a_j^2$$

It is $a \cdot X \sim N(0, \sum_{j=1}^{N} a_j^2)$ for N case. For N-dimensional vector $a = (a_1, \ldots, a_N)$, the linear combination $a \cdot X = a_1 X_1 + \cdots + a_N X_N$ is Gaussian random variable with mean 0 and variance $\sum_{j=1}^{n} a_j^2$. □

Lemma 1.2 (pp.12 Lemma 2.1.3.)**.** *If X is a N-dimensional Gaussian vector and if $M \in M_{N\times N}(\mathbb{R})$, check that the new vector MX is also an N-dimensional random vector.*

Proof. Let $Y = MX$. For N-dimensional vector $a \neq 0$, $a \cdot Y = a \cdot (MX) = a^T MX = (M^T a)^T X = b^T X = b \cdot X$, here $b = M^T a \in \mathbb{R}^N$. Suppose X has distribution of $N(\mu, \Sigma)$ where μ mean vector and Σ covariance matrix.

$$\mathbb{E}(Y) = \mathbb{E}(MX) = M\,\mathbb{E}(X) = M\mu$$

$$\begin{aligned}\mathrm{Cov}(Y) &= \mathbb{E}\left((Y - \mathbb{E}(Y))(Y - \mathbb{E}(Y))^T\right) = \mathbb{E}\left((MX - M\mu)(MX - M\mu)^T\right) \\ &= \mathbb{E}\left(M(X-\mu)(X-\mu)^T M^T\right) = M\,\mathbb{E}\left((X-\mu)(X-\mu)^T\right) M^T \\ &= M\Sigma M^T\end{aligned}$$

Hence MX is Gaussian vector with distribution $N(M\mu, M\Sigma M^T)$. □

Proposition 1.3 (pp.13 Proposition 2.1.4.)**.** *N-dimensional random vector X is Gaussian if and only if its moment generating function $\mathbb{E}(e^{a\cdot X})$ exists for all $a \in \mathbb{R}^N$ and satisfies*

$$\mathbb{E}(e^{a\cdot X}) = \exp\left(a \cdot \mathbb{E}(X) + \frac{1}{2} a^T \, Cov(X) a\right)$$

Proof. The moment generating function of N-dimensional random vector X is $\mathbb{E}(e^{a\cdot X})$ for $a \in \mathbb{R}^N$. Suppose Gaussian vector $X \sim N(\mu, \Sigma)$. We can rewrite $a \cdot X = a \cdot (\mu + (X - \mu)) = a \cdot \mu + a \cdot (X - \mu)$, thus $e^{a\cdot X} = e^{a\cdot\mu} e^{a\cdot(X-\mu)}$.

Since $e^{a\cdot\mu}$ is constant, $\mathbb{E}(e^{a\cdot X}) = e^{a\cdot\mu}\,\mathbb{E}(e^{a\cdot(X-\mu)})$. We can see $X-\mu \sim N(0,\Sigma)$ and $a\cdot(X-\mu) \sim N(0, a^T\Sigma a)$ due to property of Gaussian distribution. It only needs to calculate expectation of Gaussian r.v. whose distribution is $N(0, a^T\Sigma a)$. From elementary probability we knew it is $e^{\frac{1}{2}a^T\Sigma a}$.

Hence

$$\mathbb{E}(e^{a\cdot X}) = e^{a\cdot\mu}e^{\frac{1}{2}a^T\Sigma a} = \exp\left(a\cdot\mathbb{E}(X) + \frac{1}{2}a^T\,\mathrm{Cov}(X)a\right)$$

On the contrary, since the moment generating function defines uniqueness of distribution of r.v., from the form we can deduce X is Gaussian vector. □

Proposition 1.4 (pp.13 Proposition 2.1.5.)**.** *Let $X = (X_1,\dots,X_N)^T$ be a N-dimensional Gaussian vector. Its covariance matrix $Cov(X)$ is diagonal if and only if the Gaussian random variables $X_1,\dots,X_N$ are independent.*

Proof. If the covariance matrix of N-dimensional Gaussian vector X is diagonal matrix, it means for any $i \neq j$, $\mathrm{Cov}(X)_{i,j} = 0$. Hence X_i, X_j are independent, meaning $X_1,\dots,X_N$ are all independent.

For the inverse, it is obvious. □

2 Brownian process

Exercise 2.1 (pp.18 Exercise 2.4.2.)**.** The random variables X^1, X^2 on $(\Omega,\mathcal{F},\mathbb{P})$ are independent if and only if the σ-algebras $\sigma(X_1),\sigma(X_2)$ they generate are independent, and a random variable X on $(\Omega,\mathcal{F},\mathbb{P})$ is independent of a σ-subalgebra $\mathcal{G}$ of $\mathcal{F}$ if and only if X is independent of every $\mathcal{G}$-measurable random variables.

Proof. Let $A \in \sigma(X_1)$ and $B \in \sigma(X_2)$, then there exist Borel sets C, D such that $A = (X_1)^{-1}(C)$ and $B = (X_2)^{-1}(D)$. Since X_1 and X_2 are independent,

$$\mathbb{P}(A\cap B) = \mathbb{P}(A)\,\mathbb{P}(B) = \mathbb{P}(X_1\in C)\,\mathbb{P}(X_2\in D)$$

and

$$\mathbb{P}(A\cap B) = \mathbb{P}((X_1)^{-1}(C)\cap(X_2)^{-1}(D)) = \mathbb{P}((X_1)^{-1}(C))\,\mathbb{P}((X_2)^{-1}(D))$$

for every $A\in\sigma(X_1), B\in\sigma(X_2)$. Hence the two σ-algebras are independent. For the inverse, consider $(X_1)^{-1}(A)\in\sigma(X_1), (X_2)^{-1}(B)\in\sigma(X_2)$, since two σ-algebras are independent,

$$\mathbb{P}((X_1)^{-1}(A)\cap(X_2)^{-1}(B)) = \mathbb{P}((X_1)^{-1}(A))\,\mathbb{P}((X_2)^{-1}(B))$$

It is equivalent to

$$\mathbb{P}((X_1 \in A) \cap (X_2 \in B)) = \mathbb{P}(X_1 \in A)\,\mathbb{P}(X_2 \in B)$$

for every Borel subset A, B, meaning X_1 and X_2 are independent random variables.

For $A \in \mathcal{B}(\mathbb{R})$ and $B \in \mathcal{G}$, let Y be $\mathcal{G}$-measurable r.v., since X is independent of $\mathcal{G}$, it means

$$\mathbb{P}((X \in A) \cap (Y \in B)) = \mathbb{P}(X \in A)\,\mathbb{P}(Y \in B)$$

therefore X and Y are independent.

On the contrary, assume X is independent of every $\mathcal{G}$-measurable r.v., define indicator r.v. $Y = \mathbf{1}_B$ for every $B \in \mathcal{G}$, X is independent of it. For any Borel set $A \in \mathbb{R}$,

$$P((X \in A) \cap (Y \in B)) = \mathbb{P}(X \in A)\,\mathbb{P}(Y \in B) = \mathbb{P}(X \in A)\,\mathbb{P}(B)$$

and it means X is independent of σ-subalgebra $\mathcal{G}$. □

Exercise 2.2 (pp.18 Exercise 2.4.3.)**.** Show that the Brownian process is a Gaussian process.

Proof. For Brownian motion $(B_t)_{t\in\mathbb{R}_+}$, consider $\forall\ 0 \le t_1 < t_2 < \cdots < t_n$ and its corresponding random vector $B = (B_{t_1}, \ldots, B_{t_n})$. Define $Y_1 = B_{t_1} - B_0, Y_2 = B_{t_2} - B_{t_1}, \ldots, Y_n = B_{t_n} - B_{t_{n-1}}$.

From property of Brownian process, $Y_1 \sim N(0, t_1), Y_2 \sim N(0, t_2 - t_1), \ldots, Y_n \sim N(0, t_n - t_{n-1})$. Every Y_i is independent of $\mathcal{F}_{t_{i-1}}$ for $1 \le i \le n$. The random vector $(B_{t_1}, \ldots, B_{t_n})$ becomes $(Y_1, Y_1 + Y_2, \ldots, Y_1 + \cdots + Y_n)$. Since every Y_i is normally distributed, the linear combination of $Y_1, \ldots, Y_n$ is also normally distributed. Next compute mean and covariance matrix of this random vector.

$$\mathbb{E}(Y_i) = 0 \Rightarrow \mathbb{E}\left(\sum_{j=1}^{i} Y_j\right) = 0,\ 1 \le i \le n \Rightarrow \mathbb{E}(B) = 0$$

We will have a look at what elements make covariance matrix. For its (i, j)

entry,

$$
\begin{aligned}
(\mathrm{Cov}(B))_{i,j} &= \mathrm{Cov}(B_{t_i}, B_{t_j}) = \mathrm{Cov}\left(\sum_{p=1}^{i} Y_p, \sum_{q=1}^{j} Y_q\right) \\
&= \mathbb{E}\left(\left(\sum_{p=1}^{i} Y_p - \mathbb{E}\left(\sum_{p=1}^{i} Y_p\right)\right)\left(\sum_{q=1}^{j} Y_q - \mathbb{E}\left(\sum_{q=1}^{j} Y_q\right)\right)\right) \\
&= \mathbb{E}\left(\sum_{p=1}^{i} Y_p \sum_{q=1}^{j} Y_q\right) \\
&= \mathbb{E}\left(\sum_{k=1}^{i\wedge j} Y_k^2\right) \text{(since the term whose subscript difference is 0)} \\
&= \sum_{k=1}^{i\wedge j} \mathrm{Var}(Y_k) \\
&= t_i \wedge t_j
\end{aligned}
$$

Hence we got covariance matrix $\Sigma = ((t_i \wedge t_j)_{i,j})$. For every $a \neq 0$, it can still be shown that $a \cdot B = \sum_{i=1}^{n} a_i B_{t_i}$ is of Gaussian distribution. So B is Gaussian vector, Brownian process is indeed Gaussian process. □

Proposition 2.3 (pp.18 Proposition 2.4.6.)**.** *Let* $(\Omega, \mathcal{F}, \mathbb{P}, (\mathcal{F}_t)_{t\in\mathbb{R}_+}, (B_t)_{t\in\mathbb{R}_+})$ *be* 1*-dimensional Brownian process. Then*

(1) For any $s \geq 0$, $(\Omega, \mathcal{F}, \mathbb{P}, (\mathcal{F}_{t+s})_{t\in\mathbb{R}_+}, (B_{t+s} - B_s)_{t\in\mathbb{R}_+})$ *is* 1*-dimensional Brownian motion.*

(2) $(\Omega, \mathcal{F}, \mathbb{P}, (\mathcal{F}_t)_{t\in\mathbb{R}_+}, (-B_t)_{t\in\mathbb{R}_+})$ *is* 1*-dimensional Brownian process.*

(3) For any $c > 0$, $(\Omega, \mathcal{F}, \mathbb{P}, (\mathcal{F}_{\frac{t}{c^2}})_{t\in\mathbb{R}_+}, (cB_{\frac{t}{c^2}})_{t\in\mathbb{R}_+})$ *is* 1*-dimensional Brownian process.*

(4) The random variables defined by $Z_t := tB_{\frac{1}{t}}$ *for* $t > 0$ *and* $Z_0 = 0$ *define a natural Brownian process.*

Proof. (1) We can see $B_0 = B_s - B_s = 0$ a.s. For $\forall\ 0 \leq t_1 \leq t_2$, the random variable $(B_{t_2+s} - B_s) - (B_{t_1+s} - B_s) = B_{t_2+s} - B_{t_1+s}$ is independent of $\mathcal{F}_s$. Meanwhile it is Gaussian random variable $N(0, t_2 - t_1)$. So $(B_{t+s} - B_s)_{t\in\mathbb{R}_+}$ is Brownian process.

(2) $-B_0 = B_0 = 0$ a.s. For $\forall\ 0 \leq s \leq t$, $-B_t - (-B_s) = B_s - B_t$ is independent of $-\mathcal{F}_s$. And $B_s - B_t = -(B_t - B_s) = N(0, t-s)$ since Gaussian distribution is symmetric. Hence $(-B_t)_{t\in\mathbb{R}_+}$ is Brownian process.

(3) When $t = 0$, $cB_0 = 0$ a.s. For $\forall\ 0 \le s \le t$,

$$cB_{\frac{t}{c^2}} - cB_{\frac{s}{c^2}} = c(B_{\frac{t}{c^2}} - B_{\frac{s}{c^2}})$$

Since $B_{\frac{t}{c^2}} - B_{\frac{s}{c^2}}$ is independent of $\mathcal{F}_{\frac{s}{c^2}}$ from $(B_t)_{t\in\mathbb{R}_+}$ is Brownian process, we can see the above also independent of $\mathcal{F}_{\frac{s}{c^2}}$.

And since $B_{\frac{t}{c^2}} - B_{\frac{s}{c^2}} \sim N(0, \frac{t}{c^2} - \frac{s}{c^2})$, its multiplication with positive constant $c > 0$ is again of Gaussian distribution $N(0, t - s)$.

(4) We will use Proposition 2.4.4. in lecture note to prove it's Brownian process.

Obviously $Z_0 = 0$ a.s. For every $0 < t_1 < t_2 < \cdots < t_n$, $(Z_{t_1}, \ldots, Z_{t_n}) = (t_1 B_{\frac{1}{t_1}}, \ldots, t_n B_{\frac{1}{t_n}})$. Recall the definition of Gaussian vector and Brownian process is Gaussian process, we could know $(Z_{t_1}, \ldots, Z_{t_n})$ is again Gaussian vector.

For $0 < s < t$, then $\frac{1}{t} < \frac{1}{s}$ and $\frac{1}{s} \wedge \frac{1}{t} = \frac{1}{t}$,

$$\mathbb{E}(Z_s Z_t) = st\,\mathbb{E}(B_{\frac{1}{s}} B_{\frac{1}{t}}) = st\left(\frac{1}{s} \wedge \frac{1}{t}\right) = st\frac{1}{t} = s = s \wedge t$$

meaning $(Z_t)_{t\in\mathbb{R}_+}$ is Brownian process. □

3 Bounded variation

Exercise 3.1 (pp.19 Exercise 2.4.7.)**.** Let $f : [a, b] \to \mathbb{R}$. (1) If f is increasing, check that $\mathrm{var}_{[a,b]}(f) = f(b) - f(a)$. (2) If $f \in C^1([a, b])$, check that $\mathrm{var}_{[a,b]}(f) = \int_a^b |f'(t)|\mathrm{d}t$.

Proof. (1) Since f is increasing, for every $t^l_{j-1} < t^l_j$ where $1 \le j \le n$, $f(t^l_{j-1}) \le f(t^l_j)$. Hence $|f(t^l_j) - f(t^l_{j-1})| = f(t^l_j) - f(t^l_{j-1})$. For partition $\mathcal{P}_l$ on $[a, b]$,

$$\mathrm{var}_{[a,b]}(f) = \sup_{\mathcal{P}_l} \sum_{j=1}^{n_l} |f(t^l_j) - f(t^l_{j-1})| = \sup_{\mathcal{P}_l} \sum_{j=1}^{n_l} (f(t^l_j) - f(t^l_{j-1}))$$

It is $\sup_{\mathcal{P}_l}(f(b) - f(a)) = f(b) - f(a)$.

(2)

$$\begin{aligned}
\mathrm{var}_{[a,b]}(f) &= \sup_{\mathcal{P}_l} \sum_{j=1}^{n_l} |f(t_j^l) - f(t_{j-1}^l)| \\
&= \sup_{\mathcal{P}_l} \sum_{j=1}^{n_l} |f'(\theta_j^l)(t_j^l - t_{j-1}^l)| \quad (\theta_j^l \in [t_{j-1}^l, t_j^l]) \\
&= \int_a^b |f'(t)| \mathrm{d}t
\end{aligned}$$

due to definition of Riemann integral as $|\mathcal{P}_l| \to 0$ and f is C^1 function. $\square$