Report 1

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Contents

1 I.1 Hilbert space and linear operator

1.1 Exercise 1

Let $\{f_n\}_{n=1}^{\infty} \subset \mathcal{H}$ and $f_\infty \in \mathcal{H}$. Then,

$$\begin{aligned} &\text{(i)} \quad s-\lim_{n\to\infty} f_n = f_\infty \implies w-\lim_{n\to\infty} f_n = f_\infty \\ &\text{(ii)} \quad w-\lim_{n\to\infty} f_n = f_\infty \quad \text{and} \quad \lim_{n\to\infty} \|f_n\|_{\mathcal{H}} = \|f_\infty\|_{\mathcal{H}} \quad \implies \quad s-\lim_{n\to\infty} f_n = f_\infty. \end{aligned}$$

【Proof】

(i) Suppose $s-\lim_{n\to\infty} f_n = f_\infty$. For all $g \in \mathcal{H}$, we have

$$|\langle g, f_n - f_\infty \rangle| \leqq \|g\|_{\mathcal{H}} \|f_n - f_\infty\|_{\mathcal{H}}.$$

RHS converges to 0 as $n \to \infty$ since $s-\lim_{n\to\infty} f_n = f_\infty$, and thus $\lim_{n\to\infty} \langle g, f_n - f_\infty \rangle = 0$.

(ii) Suppose $w-\lim_{n\to\infty} f_n = f_\infty$ and $\lim_{n\to\infty} \|f_n\|_{\mathcal{H}} = \|f_\infty\|_{\mathcal{H}}$. Note that $\lim_{n\to\infty} \|f_n\|^2_{\mathcal{H}} = \|f_\infty\|^2_{\mathcal{H}}$ holds from $\lim_{n\to\infty} \|f_n\|_{\mathcal{H}} = \|f_\infty\|_{\mathcal{H}}$. We have

$$\begin{aligned} \|f_n - f_\infty\|^2_{\mathcal{H}} &= |\langle f_n - f_\infty, f_n - f_\infty \rangle| \\ &= |\langle f_n, f_n - f_\infty \rangle - \langle f_\infty, f_n - f_\infty \rangle| \\ &\leqq |\langle f_n, f_n - f_\infty \rangle| + |\langle f_\infty, f_n - f_\infty \rangle|. \end{aligned}$$

Evaluating $|\langle f_n, f_n - f_\infty \rangle|$, we get

$$\begin{aligned} |\langle f_n, f_n - f_\infty \rangle| &= |\langle f_n, f_n \rangle - \langle f_n, f_\infty \rangle| \\ &= |\langle f_n, f_n \rangle - \langle f_\infty, f_\infty \rangle - \langle f_n - f_\infty, f_\infty \rangle| \\ &= |\|f_n\|^2_{\mathcal{H}} - \|f_\infty\|^2_{\mathcal{H}} - \langle f_n - f_\infty, f_\infty \rangle| \\ &\leqq |\|f_n\|^2_{\mathcal{H}} - \|f_\infty\|^2_{\mathcal{H}}| + |\langle f_n - f_\infty, f_\infty \rangle| \\ &= |\|f_n\|^2_{\mathcal{H}} - \|f_\infty\|^2_{\mathcal{H}}| + |\langle f_\infty, f_n - f_\infty \rangle|. \end{aligned}$$

Thus,

$$\|f_n - f_\infty\|^2_{\mathcal{H}} \leqq |\|f_n\|^2_{\mathcal{H}} - \|f_\infty\|^2_{\mathcal{H}}| + |\langle f_\infty, f_n - f_\infty \rangle| + |\langle f_\infty, f_n - f_\infty \rangle|.$$

The all three terms in RHS converges to 0 as $n \to \infty$ since $\lim_{n\to\infty} \|f_n\|^2_{\mathcal{H}} = \|f_\infty\|^2_{\mathcal{H}}$ and $w-\lim_{n\to\infty} f_n = f_\infty$. Thus we get $\lim_{n\to\infty} \|f_n - f_\infty\|^2_{\mathcal{H}} = 0$, i.e., $s-\lim_{n\to\infty} f_n = f_\infty$. □

1.2 Exercise 2

Let $\{B_n\}_{n=1}^{\infty} \subset \mathcal{B}(\mathcal{H})$, $B_\infty \in \mathcal{B}(\mathcal{H})$. Then,

$$\text{(i)} \quad u-\lim_{n\to\infty} B_n = B_\infty \implies s-\lim_{n\to\infty} B_n = B_\infty$$

$$\text{(ii)} \quad s-\lim_{n\to\infty} B_n = B_\infty \implies w-\lim_{n\to\infty} B_n = B_\infty.$$

【Proof】

(i) Suppose $u-\lim_{n\to\infty} B_n = B_\infty$, i.e., $\lim_{n\to\infty} \|B_n - B_\infty\| = 0$.

Let $f \in \mathcal{H}$. If $f = 0$, obviously $\|B_n f - B_\infty f\|_{\mathcal{H}} = 0 \to 0$ as $n \to \infty$. Consider the case $f \neq 0$. Then,

$$\begin{aligned}\|B_n f - B_\infty f\|_{\mathcal{H}} &= \|f\|_{\mathcal{H}} \frac{\|(B_n - B_\infty)f\|_{\mathcal{H}}}{\|f\|_{\mathcal{H}}} \\ &\leqq \|f\|_{\mathcal{H}} \sup_{\substack{f\in\mathcal{H}\\ f\neq 0}} \frac{\|(B_n - B_\infty)f\|_{\mathcal{H}}}{\|f\|_{\mathcal{H}}} \\ &= \|f\|_{\mathcal{H}} \|B_n - B_\infty\| \underset{n\to\infty}{\to} 0.\end{aligned}$$

Thus $s-\lim_{n\to\infty} B_n = B_\infty$.

(ii) Suppose $s-\lim_{n\to\infty} B_n = B_\infty$. Then, for all $f, g \in \mathcal{H}$,

$$|\langle f, (B_n - B_\infty)g\rangle| \leqq \|f\|_{\mathcal{H}} \|(B_n - B_\infty)g\|_{\mathcal{H}} = \|f\|_{\mathcal{H}} \|B_n g - B_\infty g\|_{\mathcal{H}} \underset{n\to\infty}{\to} 0.$$

Thus $w-\lim_{n\to\infty} B_n = B_\infty$.

□

2 I.2 Ideals in $\mathcal{B}(\mathcal{H})$

2.1 Exercise 3

For Hilbert space $\mathcal{H}$, let $\mathcal{F}(\mathcal{H})$ be a set of finite rank operators, $\mathcal{B}(\mathcal{H})$ be a set of bounded linear operators, and $\mathcal{K}(\mathcal{H})$ be a set of compact operators. Then,

$$\text{(i) } \mathcal{F}(\mathcal{H}) \subset \mathcal{B}(\mathcal{H}) \text{ ideal} \qquad \text{(ii) } \mathcal{K}(\mathcal{H}) \subset \mathcal{B}(\mathcal{H}) \text{ ideal}$$

【Proof】

(i) First, we have to check the inclusion $\mathcal{F}(\mathcal{H}) \subset \mathcal{B}(\mathcal{H})$. Let $T \in \mathcal{F}(\mathcal{H})$. There exists $\{f_j, g_j\}_{j=1}^N \subset \mathcal{H}$ s.t. $Tf = \sum_{j=1}^N \langle f_j, f\rangle g_j$ for all $f \in \mathcal{H}$. Then, for all $f \in \mathcal{H}$, we have

$$\|Tf\| = \left\|\sum_{j=1}^N \langle f_j, f\rangle g_j\right\| \leqq \sum_{j=1}^N |\langle f_j, f\rangle| \|g_j\| \leqq \sum_{j=1}^N \|f_j\|\|f\|\|g_j\| = \left[\sum_{j=1}^N \|f_j\|\|g_j\|\right] \|f\|.$$

Therefore $T \in \mathcal{B}(\mathcal{H})$ and $\mathcal{F}(\mathcal{H}) \subset \mathcal{B}(\mathcal{H})$.

Next, I'll show the ideality, i.e.,

$$T \in \mathcal{F}(\mathcal{H}), S \in \mathcal{B}(\mathcal{H}) \implies TS, ST \in \mathcal{F}(\mathcal{H}).$$

Let $T \in \mathcal{F}(\mathcal{H})$ and $S \in \mathcal{B}(\mathcal{H})$. Then, from $T \in \mathcal{F}(\mathcal{H})$, there exists $\{f_j, g_j\}_{j=1}^N \subset \mathcal{H}$ s.t. $Tf = \sum_{j=1}^N \langle f_j, f \rangle g_j$ for all $f \in \mathcal{H}$. we have, for all $f \in \mathcal{H}$,

$$TS(f) = T(S(f)) = \sum_{j=1}^N \langle f_j, Sf \rangle g_j = \sum_{j=1}^N \langle S^* f_j, f \rangle g_j.$$

Since $\{S^* f_j, g_j\}_{j=1}^N \subset \mathcal{H}$, we get $TS \in \mathcal{F}(\mathcal{H})$. Moreover,

$$ST(f) = S(T(f)) = S\left(\sum_{j=1}^N \langle f_j, f \rangle g_j\right) = \sum_{j=1}^N \langle f_j, f \rangle S g_j$$

and $\{f_j, Sg_j\}_{j=1}^N \subset \mathcal{H}$. Thus $ST \in \mathcal{F}(\mathcal{H})$.

(ii) $\mathcal{K}(\mathcal{H}) \subset \mathcal{B}(\mathcal{H})$ follows from the definition of $\mathcal{K}(\mathcal{H})$.

Let me show the ideality. Let $T \in \mathcal{K}(\mathcal{H})$ and $S \in \mathcal{B}(\mathcal{H})$. Since $T \in \mathcal{K}(\mathcal{H})$, there is $\{T_n\}_{n=1}^\infty \subset \mathcal{F}(\mathcal{H})$ s.t. $\|T_n - T\| \to 0$. From (i), we see $\{T_n S\}_{n=1}^\infty \subset \mathcal{F}(\mathcal{H})$. Moreover,

$$\|T_n S - TS\| = \|(T_n - T)S\| \leqq \|T_n - T\| \|S\| \to 0.$$

Thus $TS \in \mathcal{K}(\mathcal{H})$. Similarly, we get $\{ST_n\}_{n=1}^\infty \subset \mathcal{F}(\mathcal{H})$ from (i) and

$$\|ST_n - ST\| = \|S(T_n - T)\| \leqq \|S\| \|T_n - T\| \to 0$$

and thus $ST \in \mathcal{K}(\mathcal{H})$.

□

3 I.3 General linear operator

3.1 Exercise 4

Let $D := \left\{ f \in L^2 \;\middle|\; \int_{\mathbb{R}} |xf(x)|^2 \, dx < \infty \right\}$, and define $X : D \to L^2$ by

$$[Xf](x) = xf(x).$$

Then, (i) $D \subsetneq L^2$ (ii) D is dense in L^2 (iii) (X, D) is not bounded.

【Proof】

(i) Clearly, $D \subset L^2$ from the definition of D. To show $D \subsetneq L^2$, we have to find f s.t. $f \in L^2$ and $f \notin D$.

Consider $f : \mathbb{R} \to \mathbb{R}$ defined by $f(x) = \begin{cases} \dfrac{1}{x} & \text{if } x \geqq 1 \\ 0 & \text{otherwise} \end{cases}$.

Then, $f \in L^2$ but we have

$$\int_{\mathbb{R}} |xf(x)|^2 \, dx = \int_1^\infty \left| x \cdot \frac{1}{x} \right|^2 \, dx = \int_1^\infty 1 \, dx = \infty.$$

Thus $f \notin D$.

(ii) First, I'll check $C_c \subset D$, where C_c is the set of continuous functions with compact support.

For arbitrary $f \in C_c$, we can see

$$\int_{\mathbb{R}} |xf(x)|^2 \, dx = \int_{\mathrm{supp} f} |xf(x)|^2 \, dx + \int_{\mathbb{R} \setminus \mathrm{supp} f} |xf(x)|^2 \, dx = \int_{\mathrm{supp} f} |xf(x)|^2 \, dx,$$

and the mapping $x \mapsto |xf(x)|^2$ is continuous on $\mathrm{supp} f$, which is compact in $\mathbb{R}$, thus the integral is finite and therefore $f \in D$.

Now, we have $C_c \subset D \subset L^2$, and using the fact that C_c is dense in L^2, we can see D is dense in L^2.

(iii) Suppose (X, D) is bounded, i.e., suppose there exists $M > 0$ such that

$$\|Xf\|_2 \leqq M \|f\|_2 \text{ for all } f \in D$$

hence

$$\|Xf\|_2^2 \leqq M^2 \|f\|_2^2 \text{ for all } f \in D.$$

Let n be a natural number such that $n > M$ (e.g. $n := \lfloor M \rfloor + 1$, where $\lfloor \cdot \rfloor$ is the floor function), and define $f : \mathbb{R} \to \mathbb{C}$ by $f(x) = \begin{cases} 1 & \text{if } x \in [n, n+1] \\ 0 & \text{otherwise} \end{cases}$.

Then, $f \in D$ since $\int_{\mathbb{R}} |xf(x)|^2 \, dx = \int_n^{n+1} x^2 \, dx < \infty$ and we have

$$\|f\|_2^2 = \int_{\mathbb{R}} |f(x)|^2 \, dx = \int_n^{n+1} 1 \, dx = 1$$

and

$$\|Xf\|_2^2 = \int_{\mathbb{R}} |xf(x)|^2 \, dx = \int_n^{n+1} x^2 \, dx = n^2 + n + \frac{1}{3}.$$

Thus we get $n^2 + n + \frac{1}{3} \leqq M^2$. This is contradiction because $M^2 < n^2 < n^2 + n + \frac{1}{3}$. Therefore (X, D) is not bounded.

□

3.2 Exercise 5

Let (X, D) be the operator defined in Exercise 4. Then,

$$\text{(i) } \sigma_p(X) = \emptyset \qquad \text{(ii) } \sigma(X) = \mathbb{R}$$

【Proof】

(i) Suppose some $a \in \mathbb{C}$ is in $\sigma_p(X)$. Then, there is $f \in L^2$ s.t. $f \neq 0$ (in the sense of L^2) and $Xf = af$. Thus we have $(x-a)f(x) = 0$ a.e. $x \in \mathbb{R}$. This means that there exists $N \subset \mathbb{R}$, whose Lebesgue measure is zero, such that $(x-a)f(x) = 0$ for $x \in \mathbb{R} \setminus N$.

Now, assume $a \notin \mathbb{R}$. Dividing the equation above by $x - a$, we get $f(x) = 0$ for $x \in \mathbb{R} \setminus N$. This indicates that $f(x) = 0$ a.e. $x \in \mathbb{R}$, but this contradicts $f \neq 0$. Thus $a \in \mathbb{R}$.

Noting that $(x-a)f(x) = 0$ for $x \in \mathbb{R} \setminus N$, we can say

$$f(x) = 0 \text{ for } x \in (\mathbb{R} \setminus N) \cap (\mathbb{R} \setminus \{a\}).$$

The complement of $(\mathbb{R} \setminus N) \cap (\mathbb{R} \setminus \{a\})$ is $N \cup \{a\}$ and its Lebesgue measure is zero, so $f(x) = 0$ a.e. $x \in \mathbb{R}$. This contradicts $f \neq 0$.

Consequensely, such a doesn't exist, i.e., $\sigma_p(X) = \emptyset$.

(ii) According to [2] and [3], the resolvent set of X, say $\rho(X)$, can be written as

$$\rho(X) = \{\lambda \in \mathbb{C} \mid \mathrm{Ker}(X - \lambda \cdot 1) = \{0\} \text{ and } \mathrm{Ran}(X - \lambda \cdot 1) = L^2\}.$$

Let me use this fact.

First, I'll show $\mathbb{R} \subset \sigma(X)$. Let $\lambda \in \mathbb{R}$. Suppose $\mathrm{Ran}(X - \lambda \cdot 1) = L^2$. Define $g := \sqrt{\chi_{(\lambda,\lambda+1)}}$. Clearly $g \in L^2$. From $\mathrm{Ran}(X - \lambda \cdot 1) = L^2$, there exists $h \in L^2$ s.t. $(X - \lambda \cdot 1)h = g$. Then, $(x-\lambda)h(x) = g(x)$ a.e. $x \in \mathbb{R}$, and we get $h(x) = \dfrac{g(x)}{x-\lambda}$ a.e. $x \in \mathbb{R}$ because $\{\lambda\}$ is singleton in $\mathbb{R}$. Now, consider the square integral of h. We have

$$\int_{\mathbb{R}} |h(x)|^2 \, dx = \int_{\mathbb{R}} \left| \frac{g(x)}{x-\lambda} \right|^2 dx = \int_{\mathbb{R}} \frac{\chi_{(\lambda,\lambda+1)}(x)}{(x-\lambda)^2} \, dx = \int_{\lambda}^{\lambda+1} \frac{1}{(x-\lambda)^2} \, dx = \int_0^1 \frac{1}{x^2} \, dx = \infty.$$

This contradicts $h \in L^2$. Therefore $\mathrm{Ran}(X - \lambda \cdot 1) \neq L^2$, so $\lambda \notin \rho(X)$, i.e., $\lambda \in \sigma(X)$.

Conversely, let me show that $\sigma(X) \subset \mathbb{R}$. This is equivalent to $\mathbb{C} \setminus \mathbb{R} \subset \rho(X)$ so it suffices to show that $\lambda \in \mathbb{C} \setminus \mathbb{R}$ implies $\lambda \in \rho(X)$.

Let $\lambda \in \mathbb{C} \setminus \mathbb{R}$.

First, suppose $\mathrm{Ker}(X - \lambda \cdot 1) \neq \{0\}$. Then, there exists $g \neq 0$ s.t. $(X - \lambda \cdot 1)g = 0$. Thereupon $(x - \lambda)g(x) = 0$ a.e. $x \in \mathbb{R}$. Dividing the equation by $x - \lambda$ gives us $g(x) = 0$ a.e. $x \in \mathbb{R}$, but this contradicts $g \neq 0$. Thus $\mathrm{Ker}(X - \lambda \cdot 1) = \{0\}$.

Next, assume $\mathrm{Ran}(X - \lambda \cdot 1) \neq L^2$. From the definition of $X - \lambda \cdot 1$, the inclusion $\mathrm{Ran}(X - \lambda \cdot 1) \subset L^2$ must hold so it follows that $\mathrm{Ran}(X - \lambda \cdot 1) \subsetneq L^2$. Then, there is $g \in L^2$ s.t. $g \notin \mathrm{Ran}(X - \lambda \cdot 1)$. Define $h : \mathbb{R} \to \mathbb{C}$ as $h(x) = \dfrac{g(x)}{x-\lambda}$. Note that for all $x \in \mathbb{R}$, we have $|x-\lambda| = \sqrt{(x-\mathrm{Re}\lambda)^2 + (\mathrm{Im}\lambda)^2} \geqq |\mathrm{Im}\lambda| > 0$ since $\mathrm{Im}\lambda \neq 0$. Thus we get

$$\int_{\mathbb{R}} |h(x)|^2\, dx = \int_{\mathbb{R}} \left| \frac{g(x)}{x-\lambda} \right|^2 dx \leqq \frac{1}{|\mathrm{Im}\lambda|^2} \int_{\mathbb{R}} |g(x)|^2\, dx < \infty.$$

This shows $h \in L^2$. Moreover, $[(X-\lambda\cdot 1)h](x) = (x-\lambda)h(x) = g(x)$. This contradicts $g \notin \mathrm{Ran}(X-\lambda\cdot 1)$. Therefore $\mathrm{Ran}(X-\lambda\cdot 1) = L^2$.

Thus we get $\mathrm{Ker}(X-\lambda\cdot 1) = \{0\}$ and $\mathrm{Ran}(X-\lambda\cdot 1) = L^2$, i.e., $\lambda \in \rho(X)$.

We have shown that any $\lambda \in \mathbb{C}\setminus\mathbb{R}$ belongs to $\rho(X)$, and therefore $\sigma(X) \subset \mathbb{R}$.

Then we have finished the proof of $\sigma(X) \supset \mathbb{R}$ and $\sigma(X) \subset \mathbb{R}$. Eventually $\sigma(X) = \mathbb{R}$.

□

3.3 Exercise 6

Let $(A, D(A))$ is densely defined linear operator. Then,

(I) $(A^*, D(A^*))$ is closed   (II) $\mathrm{Ker}A^* = (\mathrm{Ran}A)^\perp$

【Proof】

Note that $\langle f, Ag\rangle = \langle A^*f, g\rangle$ for $f \in D(A^*)$ and $g \in D(A)$. This is because, for $f \in D(A^*)$ and $g \in D(A)$, there is $f^* \in \mathcal{H}$ which guarantees $\langle f, Ag\rangle = \langle f^*, g\rangle = \langle A^*f, g\rangle$.

(I) Let $\{f_n\} \subset D(A^*)$ with $f_n \to f \in \mathcal{H}$ and $\{A^*f_n\}$ is Cauchy sequence. We have to show $f \in D(A^*)$ and $\lim_{n\to\infty} A^*f_n = A^*f$.

(i) $f \in D(A^*)$.

Since $\{A^*f_n\}_{n=1}^\infty$ is Cauchy in Hilbert space $\mathcal{H}$, there exists $f^* \in \mathcal{H}$ s.t. $\lim_{n\to\infty} A^*f_n = f^*$. Then, for any $g \in D(A)$, we have

$$\langle f, Ag\rangle = \langle \lim_{n\to\infty} f_n, Ag\rangle = \lim_{n\to\infty}\langle f_n, Ag\rangle = \lim_{n\to\infty}\langle A^*f_n, g\rangle = \langle \lim_{n\to\infty} A^*f_n, g\rangle = \langle f^*, g\rangle,$$

and thus $f \in D(A^*)$.

(ii) $\lim_{n\to\infty} A^*f_n = A^*f$.

This follows from the definition of f^* and A^*. f^* has been defined as $\lim_{n\to\infty} A^*f_n = f^*$, and we have $A^*f = f^*$ from the definition of A^*. Thereupon $\lim_{n\to\infty} A^*f_n = f^* = A^*f$.

(II) Let $f \in \mathrm{Ker}A^*$. Then, for all $g \in \mathrm{Ran}A$, there exists $h \in D(A)$ s.t. $g = Ah$, and thus we have

$$\langle f, g\rangle = \langle f, Ah\rangle = \langle A^*f, h\rangle = \langle 0, h\rangle = 0.$$

Therefore $f \in (\mathrm{Ran}A)^\perp$ and we get $\mathrm{Ker}A^* \subset (\mathrm{Ran}A)^\perp$.

Conversely, let me show $(\mathrm{Ran}A)^\perp \subset \mathrm{Ker}A^*$. Assume $f \in (\mathrm{Ran}A)^\perp$. Set $f^* := 0 \in \mathcal{H}$. Then, for all $g \in D(A)$, we have $\langle f, Ag\rangle = 0$ from $f \in (\mathrm{Ran}A)^\perp$ so

$$\langle f, Ag\rangle = 0 = \langle 0, g\rangle = \langle f^*, g\rangle.$$

Thus, $A^* f = f^* = 0$ from the definiion of A^*. Hereupon $f \in \mathrm{Ker}A^*$.

Therefore we get $\mathrm{Ker}A^* \subset (\mathrm{Ran}A)^\perp$ and $(\mathrm{Ran}A)^\perp \subset \mathrm{Ker}A^*$, i.e., $\mathrm{Ker}A^* = (\mathrm{Ran}A)^\perp$.

□

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